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A model of fluid mud transport

By

Christopher Lawrence Hall

A Dissertation
Submitted to the Faculty of
Mississippi State University
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
in Civil and Environmental Engineering
in the Department of Civil and Environmental Engineering

Mississippi State, Mississippi

May 2014
A model of fluid mud transport

By

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Ports and waterways are vital to the economy of the United States. In the contiguous United States, there are some 25,000 miles of channels and over 300 ports. Together, this system carries 2 billion tons of freight with a value of over $700 billion annually. Ninety percent of all United States imports and exports travel through these ports and waterways.

Dredging of these waterways in the United States costs over $1 billion annually. As ship draft increases, more dredging would be required to keep these ports and waterways open. Fine sediments are very common in these systems and have properties that can reduce dredging efficiency, including easy resuspension into the water column and cohesion among individual particles. Fluid mud is a high concentration aqueous solution of fine sediments that exhibits unique properties, including movement under gravity. A numerical model of fluid mud could be used to predict sediment fate as well as evaluating potential channel modifications to reduce dredging.

The goal of this research is to test the flow of fluid mud under shear from the water column and develop a numerical model to simulate the transport of fluid mud.
First, laboratory experiments are conducted to ascertain the effects of shear from the water column on the fluid mud layer. Next, a finite element numerical model is developed to simulate the physics of fluid mud, including any effects from shear over the mud layer. Results from the numerical model are compared to laboratory experiments, and the fluid mud model is developed for easy linkage to existing hydrodynamic models for forcing information.
DEDICATION

I would like to dedicate this work to my wife, Jacqueline. You are everything to me, and I love you with all my heart. Thank you for your love and support, I could not have done this without you.
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I cannot thank Dr. McAnally enough for his tireless help and mentoring during these years of study. His wisdom, experience, and advice have been invaluable to me. I would like to thank Dr. Berger for his extensive one-on-one work with me, and Dr. Martin and Dr. Hayter for their support and guidance on this work. Also, I would like to thank Mr. Joe Ivy and his student workers for their help on the flume construction.

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CHAPTER I
INTRODUCTION

1.1 Need for Research

Ports and waterways are vital to the economy of the United States. In the contiguous United States, there are some 25,000 miles of channels and over 300 ports. Together, this system carries 2 billion tons of freight with a value of over $700 billion annually. Ninety percent of all United States imports and exports travel through these ports and waterways. As Figure 1 shows, waterborne transportation is more efficient than either trucks or trains, with a single barge carrying the equivalent amount of cargo as fifty-eight large semi trucks and more than thirteen railcars (IDOT, 2006).

Waterborne transportation requires certain channel characteristics for continued access. One of these characteristics is depth. Often, channels are dug deeper than the surrounding area. Most inland channels follow existing river courses, and these natural systems are in constant change, depending on rainfall, land use, and other factors. Over time, channels can fill in with sediment carried by the rivers. Coastal channels can also fill in over time as tides and currents carry sediment into the channels. These sediments need to be removed for ships or barges to continue to access the port or channel.

Dredging is the removal of sediments from underwater locations. This process can be very expensive, and in 2012, the United States spent over $1.2 billion dredging over 200 million cubic yards of sediments (Center, 2013).
There are several different types of dredges, including the hopper dredge, cutterhead dredge, and mechanical dredge. The hopper dredge removes sediment from the bottom, places it in a hopper on board, and then moves to a different location to discharge the sediment before returning to the dredging site to continue dredging. Cutterhead dredges stir up the sediment at the dredge site and pump the slurry through a series of pipes to a discharge site. Mechanical dredges often use large baskets to remove sediments one scoop at a time and place them nearby or on a barge for transport to a discharge site.
A common concern among all types of dredging is the fate of the dredged material. Near the dredge site, some sediment would be resuspended by the dredge, and some sediment may spill from a barge or hopper. At the dredged material placement site, the sediment is often discharged into the water where the local conditions can dictate the ultimate location of the sediment. The physical properties of the sediment also determine its final position.

There are two general types of sediments that are commonly dredged: coarse-grained sediments and fine-grained sediments. Coarse-grained sediments are those that are larger than 62.5 microns (0.0625 mm) (Vanoni, 2006b). This category contains sands, gravels, and larger particles such as cobbles and boulders. These sediments tend to settle quickly, have a density of approximately 2,650 kg/m$^3$ (Vanoni, 2006b), and exhibit no cohesive properties. These sediments are often transported near or in frequent contact with the bed, and this is known as the bed load.

Fine-grained sediments are those sediments smaller than 62.5 microns and contain silts and clays. The physical properties of this class of sediments are more complicated than those of the coarse-grained sediments. The density of individual clay and silt particles is near 2,650 kg/m$^3$; however, these sediments are rarely found as individual particles. An important property these smaller sediments have is cohesion, which causes the particles to stick together or flocculate. As these particles stick together, they incorporate water in between them, reducing the aggregate’s overall density. These aggregates can also floc together, further reducing the overall density while creating much larger units. This flocculation can ultimately lead to an aggregate with a density just above that of water. Since settling velocity is directly proportional to density, the
closer an aggregate’s density is to water, the lower its settling velocity. As the aggregates begin to slowly settle, the particles become so closely packed that they limit the movement of water between them, reducing their ability to settle further. This is known as hindered settling and can lead to the formation of fluid mud (Garcia, 2006).

Fluid mud is “a high concentration aqueous suspension of fine-grained sediment in which settling is substantially hindered by the proximity of sediment grains and flocs, but which has not formed an interconnected matrix of bonds strong enough to eliminate the potential for mobility (McAnally et al., 2007)”\textsuperscript{1}. The sediment concentrations in fluid mud can range from tens to hundreds of grams per liter, resulting in bulk densities from 1,080 to 1,200 kg/m\textsuperscript{3} (McAnally et al., 2007).

Fluid mud can severely affect navigation. The sharp increase in sediment concentration at the top of the fluid layer, known as a lutocline, can return a false bottom to ship sonar systems. This would indicate a bottom above the actual hard bottom, signifying to the ship’s captain that the channel may not be deep enough to sail through. Figure 2 shows a dual frequency fathometer with the higher frequency showing a bottom approximately a meter above the sampling equipment bottom (Alexander et al., 1997).
Fluid mud can also fill in channels faster than it can be removed through dredging, restricting port access and limiting navigation. Fluid mud can form or flow into navigation channels, where it remains until it settles out and consolidates into a firm bottom. Fluid mud occurs in many places around the world and the United States, including San Francisco, CA; Gulfport, MS; Mobile, AL; and Savannah, GA. Fluid mud can greatly increase dredging costs. One way is by indicating a shallower bottom than the actual bottom, (see Figure 1.2) requiring dredging more frequently. In addition, fluid mud can be resuspended during dredging and reform once dredging is completed without being removed from the channel. Fluid mud may be removed from the channel during dredging and reform at the discharge site before flowing back into the channel and refilling it.

A numerical model of fluid mud is needed to help predict fluid mud behavior. Existing fluid mud models are limited in their scope and usefulness. A model developed
by Teeter and Johnson considers flow on a slope, but not under shear (Teeter and Johnson, 2005). Formation of fluid mud in this model only allows for creation from bed failure on a slope, not from hindered settling or liquefaction from waves. Other models, such as one developed by Le Hir et al, are limited to the vertical dimension and one horizontal dimension to attempt to capture the vertical profile of the mud and water (Le Hir et al., 2001). This model also assumes steady flow conditions, which are unlikely to occur in real systems. A two-dimensional fluid mud model could indicate where fluid mud is likely to form, how it moves once formed, and where it eventually settles. The model could also predict how fluid mud fills in channels or how it behaves once it is pumped to a discharge site after dredging. This could reduce dredging costs by showing where to place fine sediments to limit fluid mud formation and flow back into the channel. Ports and waterways affected by fluid mud could also use this model to develop strategies to reduce fluid mud and reduce the need for dredging.

1.2 Objective

The objective of this research is to evaluate the movement of fluid mud under shear in a laboratory and to develop computer code module that models the formation, dissipation, flow, and early consolidation of fluid mud based on the results of the laboratory experiments.

1.3 Approach

The approach followed in this research consists of combining existing equations that describe fluid mud formation, dissipation, flow, and consolidation to create a computer code to model fluid mud transport. This code derives all necessary
hydrodynamic forcing data, including salinity and velocities, from existing numerical hydraulic models. Laboratory studies of fluid mud behavior are used to verify and add to the overall knowledge of fluid mud flow. These experiments test fluid mud flow under shear stress, as previous studies have demonstrated fluid mud flow on a slope (van Kessel and Kranenburg, 1996).

1.4 Scope

This code is developed to be able to obtain necessary hydrodynamic information from existing models and to use an existing interface, such as the Surface-Water Modeling System (SMS) (Aquaveo, 2014) to display the results from the fluid mud code. This dissertation will be presented in the following order:

1. Introduction
2. Modeling Physics of Fluid Mud
3. Fluid Mud Flow Under Shear Stress Experiments
4. Fluid Mud Flow Experiments and Results
5. Numerical Model for Fluid Mud Transport
6. Fluid Mud Model Results
7. Discussion of Experiments
8. Conclusions and Recommendations
CHAPTER II
MODELING PHYSICS OF FLUID MUD

The following sections describe the physical behavior of fluid mud. These physical processes form the basis for the fluid mud model. The main processes include formation, movement, dissipation, and consolidation.

2.1 Formation

Fluid mud is mainly formed through two separate processes, settling and bed liquefaction. The settling velocity of a single spherical particle is described by Stokes law and is shown in Equation 2.1 (Vanoni, 2006a). As seen in this equation, the settling velocity is dependent on viscosity, specific gravities of the fluid and particle, diameter of the particle, and gravity. This equation is valid for Reynolds numbers <0.1, which is nearly a still fluid (Vanoni, 2006a). This equation also is only valid for a spherical particle, which natural sediment particles are not.

\[ w = \frac{gd^2}{18\nu_w} \left( \frac{\gamma_s - \gamma}{\gamma} \right) \]  

(2.1)

Where:

- \( w \) = Settling velocity (m/s)
- \( g \) = Gravitational acceleration (9.81 m/s\(^2\))
- \( d \) = Diameter of the spherical particle (m)
\( \gamma_s \) = Specific weight of the sediment particle (2.65)

\( \gamma \) = Specific weight of the fluid (1 for water)

\( \nu_w \) = Kinematic viscosity of water (1×10\(^{-6}\) m\(^2\)/s)

The equation for the fall velocity of a spherical particle has been expanded throughout the entire range of Reynolds numbers and can be seen in Equation 2.2 (Vanoni, 2006a). This equation includes a drag coefficient, \( C_D \), which is dependent on the Reynolds number.

\[
W^2 = \frac{4}{3} \frac{g \rho_d}{C_d} \left( \frac{\gamma_s - \gamma}{\gamma} \right) 
\]

(2.2)

For Reynolds numbers \( (R) \) less than 0.1, \( C_D = 24/R \), and Equation 2.2 reduces to Equation 2.1 (Vanoni, 1975). For larger Reynolds numbers, \( C_D \) has been determined experimentally and can be taken from a chart developed by Rouse as seen in Figure 2.1 (Rouse, 1937). Again, this equation is limited to spherical particles but now applies to all Reynolds numbers. This graph contains a secondary scale of \( F/\left( \rho_w \nu_w^2 \right) \), where \( F \) is the submerged weight of a sphere, given in Equation 2.3, \( \rho_w \) is the density of water (kg/m\(^3\)), and \( \nu_w \) is the kinematic viscosity of water (m\(^2\)/s).

\[
F = \frac{\pi d^3}{6} \left( \gamma_s - \gamma \right)
\]

(2.3)
Once \( F/(\rho_w v_w^2) \) is known, using the secondary scale on the above graph, Figure 2.1, the Reynolds number is obtained from the \( C_D-R \) curve. Then, the fall velocity can be calculated from the Reynolds number for the particle, \( R_p = \frac{wd}{\nu_w} \).

Figure 2.1 Drag coefficient of spheres as a function of Reynolds number (Rouse, 1937)

The next step in the development of an equation for the settling of natural sediments includes the effects of the shape of the particle. The equation for the submerged weight, \( F \), of a non-spherical geometric particle is shown in Equation 2.4 (Vanoni, 2006a). This equation calculates the submerged weight of the particle to be
used on the previously mentioned secondary scale in Figure 2.1. This equation is described by (McNown et al., 1951).

\[ F = K(3\pi\mu_w w d_n) \quad (2.4) \]

Where

- \( F \) = Submerged weight of the particle
- \( K \) = Resistance factor, which is related to the shape factor \((SF)\)
- \( SF \) = Shape factor, \( \frac{a_l}{\sqrt{b_l c_l}} \)
- \( a_l, b_l, c_l \) = Lengths of the three perpendicular axes of the particle (m)
- \( \mu_w \) = Dynamic viscosity of the fluid (Pa s)
- \( d_n \) = Nominal diameter of the particle (m)

\( K \) can be obtained from Figure 2.2 (McNown et al., 1951). Once \( K \) and \( F \) are known, they can be applied to Figure 2.1 to calculate the drag coefficient, \( C_D \), and then the settling velocity. While effects from shape on fall velocity have now been included, Equation 2.4 assumes clear, quiescent fluid and a single particle.
Figure 2.2  Comparison of $K$ for multiple shapes for Reynolds numbers less than 0.1 (McNown et al., 1951)
As sediment concentration increases, neighboring sediment particles begin to interfere with the settling of each other. McNown and Lin showed that even for small concentrations (6-10%) of quartz spheres, the fall velocity of the particles could be slowed by up to 30% (McNown and Lin, 1952). This type of particle interference during settling is very apparent in fine sediments.

Fine sediments demonstrate significant cohesive properties. These cohesive properties become more apparent as particle size decreases (Mehta and McAnally, 2009). As fine sediment particles in a fluid impact each other, the cohesive properties cause them to adhere to each other and form larger aggregations, or flocs. Initially, as the floc sizes increase, settling velocity also increases. As the floc size continues to increase, some water is incorporated into the floc in the spaces between the grains, decreasing the overall specific gravity of the floc (Garcia, 2006). This lowers the settling velocity of the aggregate. As they continue to settle, they begin to interfere with each other by restricting the flow of water through the pore spaces between the flocs, which is hindered settling. This pattern can be seen in Figure 2.3 (Mehta and McAnally, 2009). Floc formation is dependent on sediment concentration, so sediment concentration can be used as a general parameter for estimating settling velocity of the flocs (Krone, 1962). McAnally et al. define the concentration levels mentioned above as $C_1$, $C_2$, and $C_3$ (McAnally et al., 2007; Mehta and McAnally, 2009). $C_1$ is the concentration where flocs begin to form and settling velocity increases over the free settling velocity of the particles. $C_2$ is the concentration where the maximum settling velocity occurs. Between $C_2$ and $C_3$ is hindered settling, and above $C_3$ consolidation dominates. This is shown in Figure 2.3.
Figure 2.3  A representative plot of settling velocity and flux variation with suspension concentration (Mehta and McAnally, 2009)

Hwang shows the development of a settling velocity equation for the zones where concentration is less than $C_1$, between $C_1$ and $C_3$, and greater than $C_3$ (Hwang, 1989).

This can be seen in Equation 2.5. Typical value ranges are taken from Sedimentation Engineering: Processes, Measurements, Modeling, and Practice (Garcia, 2006).

\[
\begin{align*}
  w &= w_{sf} \quad \text{if } C < C_1 \\
  w &= \frac{a_w C_{nw}}{(C^2 + b_w^2)^{m_w}} \quad \text{if } C_1 < C < C_2 \\
  \text{~negligible} & \quad \text{if } C_3 < C
\end{align*}
\]

(2.5)

Where

- $w_{sf} =$ Free settling velocity (Equation 2.1) (m/s)
- $C =$ Suspension concentration (kg/m$^3$)
- $a_w =$ Velocity scaling coefficient (Range from 0.001-0.230)
$n_w$ = Flocculation settling exponent (Range from 0.40-2.80)

$b_w$ = Hindered settling coefficient (Range from 1.30-25.0)

$m_w$ = Hindered settling exponent (Range from 1.00-2.80)

$C_1$ = Concentration where flocs form and settling velocity increases

(0.1-0.30 kg/m$^3$)

$C_2$ = Concentration where the maximum settling velocity occurs (1-15 kg/m$^3$)

$C_3$ = Concentration above which consolidation dominates (~75 kg/m$^3$)

Based on Equation 2.5, the concentrations at the peak settling velocity, $C_2$, and at maximum flux, $C'_2$, can be found and are given in Equations 2.6 and 2.7 (Mehta and McAnally, 2009).

$$C_2 = \frac{b_w}{\sqrt{\frac{2m_w}{n_w} - 1}}$$  \hspace{1cm} (2.6)

$$C'_2 = \frac{b_w}{\sqrt{\frac{2m_w}{n_w + 1} - 1}}$$  \hspace{1cm} (2.7)

The concentration at maximum flux, $C'_2$, occurs at the top of the fluid mud layer. Once the concentration of a suspension reaches $C'_2$, fluid mud has formed at the depth below that concentration.

Winterwerp shows the development of a hindered settling formula described by Equation 2.8 (Winterwerp, 2002)

$$w = \frac{w_s (1 - \phi_s) (1 - \phi_p)}{1 + 2.5 \phi}$$  \hspace{1cm} (2.8)
Where

\[ w_{s,r} = \text{Settling velocity (Equation 2.10) (m/s)} \]

\[ \phi^* = \text{Minimum of 1 or } \phi \]

\[ \phi_p = \text{Volumetric concentration of the primary particles, } C/\rho_s \]

\[ \rho_s = \text{Density of the sediment (kg/m}^3\text{)} \]

\[ \phi = \text{Volumetric concentration (Equation 2.9)} \]

The volumetric concentration, \( \phi \), is defined in Equation 2.9 (Winterwerp, 2002).

\[ \phi = \left( \frac{\rho_s - \rho_w}{\rho_f - \rho_w} \right) \frac{C}{\rho_s} \quad (2.9) \]

Where

\[ \rho_s = \text{Density of the sediment (kg/m}^3\text{)} \]

\[ \rho_f = \text{Density of the flocs (kg/m}^3\text{)} \]

\[ \rho_w = \text{Density of the water (kg/m}^3\text{)} \]

The settling velocity for Equation 2.8 is expressed as the following equation.

\[ w_{s,r} = \frac{(\rho_s - \rho_w)g d_f^2}{18 \mu_w} \quad (2.10) \]

Where

\[ \mu_w = \text{Viscosity of water (0.001 Pa s)} \]

\[ d_f = \text{Floc diameter (m)} \]
The first term in the numerator of Equation 2.8, \((1 - \phi_s)\), accounts for return flow from the settling of the particles; the second term of the numerator, \((1 - \phi_p)\), accounts for the reduced gravity due to the decreased settling velocity; and the denominator, \((1 + 2.5\phi)\), accounts for the increased viscosity due to the increased concentration in the method of Einstein (Einstein, 1906).

Dankers shows the development of a similar equation to Winterwerp (Winterwerp 2002), with the inclusion of an exponent on the return flow term, to account for possible nonlinear effects (Dankers and Winterwerp, 2007). This equation is seen below in Equation 2.11.

\[
\mathbf{w} = \mathbf{w}_{s,r} \frac{(1-\phi)^m(1-\phi_p)}{1+2.5\phi}
\]  
(2.11)

The exponent, \(m\), can be found if \(\phi_cr\) is known. \(\phi_cr\) is the concentration where the settling mode changes from settling with two interfaces to settling with one interface. Settling with two interfaces involves settling both at the interface between the water and mud and a second interface within the mud layer where the concentration increases sharply, while settling with one interface involves settling only at the interface between the water and mud layer. This relationship is described below in Equation 2.12 (Dankers and Winterwerp, 2007).

\[
m = \frac{1}{2} \frac{5\phi_{cr}^2 - 2\phi_{cr} + 4 + \sqrt{25\phi_{cr}^4 + 60\phi_{cr}^3 - 116\phi_{cr}^2 + 64\phi_{cr} + 16}}{\phi_{cr}(2+5\phi_{cr})}
\]  
(2.12)

The above equations demonstrate the effect of increased concentration on reducing settling velocity. This reduced settling velocity can lead to the formation of
fluid mud. Fluid mud can also be formed through liquefaction caused by waves. As waves pass over the bed, the increased pressure from the wave can increase pore pressure in the bed and overcome the yield strength of the bed. Equation 2.13 describes pressure under a wave based on linear wave theory (Eagleson and Dean, 1966).

\[ p = -\rho_w g z + \rho_w g \frac{\cosh k (h_w + z)}{\cosh kh_w} a \sin(kx - \omega t) \]  

(2.13)

Where

\[ z \] = Vertical dimension, negative downward from mean water level (m)

\[ k \] = Wave number, \( \left( \frac{2\pi}{L} \right) \)

\[ L \] = Wavelength (m)

\[ h \] = Mean water depth (m)

\[ a \] = Amplitude of the wave (m)

\[ x \] = Horizontal position (m)

\[ \omega \] = Wave frequency (Hz), \( \left( \frac{2\pi}{T} \right) \)

\[ T \] = Wave period (s)

\[ t \] = Time (s)

At the bottom of the water column, where \( z = -h \), Equation 2.8 becomes the following equation.

\[ p = \rho_w g h_w + \rho_w g \left[ \frac{1}{\cosh kh_w} \right] a \sin(kx - \omega t) \]  

(2.14)
Equation 2.14 shows that the primary pressure increase is due to wave amplitude, $a$, over surface of the water. This pressure increase over a cohesive bed can exceed the yield stress of the bed over time. Van Kessel and Kranenburg show the development of a deviatoric shear stress in the bed described by Equations 2.15-2.23 (van Kessel and Kranenburg, 1998).

$$
\tau_{dev}(z,t) = \sqrt{\tau_{xz}^2 + \frac{1}{4} (\sigma_x - \sigma_z)^2}
$$

$$
\tau_{xz} = i(c + b_0 k z + 2( b_1 \exp(kz) - b_2 \exp(-kz))) \exp(i(kx - \omega t)) + \tau_{xz}^s
$$

$$
\sigma_x = -(2b_0 + b_1 \exp(kz) + b_2 \exp(-kz)) \exp(i(kx - \omega t)) + \sigma_x^s
$$

$$
\sigma_z = -(2b_0 + 3(b_1 \exp(kz) + b_2 \exp(-kz))) \exp(i(kx - \omega t)) + \sigma_z^s
$$

$$
b_0 = -\left( \frac{\exp(kD) + a_1 \exp(-kD)}{a_1 + 1} \right) p_0
$$

$$
b_1 = \frac{1}{a_1 + 1} p_0
$$

$$
b_2 = \frac{a_1}{a_1 + 1} p_0
$$

$$
c = 2 \left( \frac{a_1 - 1}{a_1 + 1} \right) p_0
$$

$$
a_1 = \frac{1}{2} \frac{\exp(kD)(1 - kD) - 1}{\exp(-kD)(1 + kD) - 1}
$$

(2.15 – 2.23)

Where

- $k$ = Wave number
- $D$ = Thickness of the sediment bed
- $p_0$ = Pore water pressure amplitude
- $\omega$ = Wave frequency
Shear stress on the slope

\[ \tau = \text{Shear stress on the slope} \]

\[ \sigma_x, \sigma_z = \text{Stresses generated from the slope} \]

Experiments conducted by van Kessel and Kranenburg demonstrate that when \( \tau_{dev} \) exceeds the yield stress in the bed, the bed liquefies (van Kessel and Kranenburg, 1998).

Li and Mehta developed an equation for the liquefaction depth of fluid mud, assuming the bed is a linear viscoelastic material (Li and Mehta, 2001). The lift forces in the bed from the waves must overcome both cohesion and reduced gravity, so the acceleration of a particle, given below, must be exceeded for fluid mud formation to occur (Li and Mehta, 2001).

\[ \zeta_{max}^Y = \alpha_g g' \]  \hspace{1cm} (2.24)

Where

\[ \zeta_{max}^Y = \text{Acceleration of the particle} \]

\[ \alpha_g = \text{Mud specific coefficient to modulate gravity} \]

\[ g' = \text{Reduced gravity}, \quad g' = \frac{g[\rho_s(z_c) - \rho_w]}{\rho_w} \]

\[ z_c = \text{Depth of liquefaction (m)} \]

By assuming a shear-Voigt model for the bed and a pressure increase from the linear wave theory as described above in Equation 2.13, the maximum deflection of the surface is given below (Li and Mehta, 2001).

\[ \zeta_{max} = \frac{p_{wave}}{k} \frac{1}{k[(1 - \beta^2)^2 + (2\xi \beta)^2]^\frac{1}{2}} \]  \hspace{1cm} (2.25)
Where

\[ p_{\text{wave}} = \text{Pressure amplitude from the wave} \]

\[ \beta = \dfrac{\omega}{\omega_0} \]

\[ \omega_0 = (k/m_a)^{1/2} \]

\[ m_a = \text{Equivalent mass term} \]

\[ \xi = \dfrac{c_{\text{vis}}}{2m\omega} \]

\[ c_{\text{vis}} = \text{Equivalent viscous coefficient} \]

Using a shape function \( \Gamma = (1 - z'/h_{\text{bed}}) \), where \( h_{\text{bed}} \) is the thickness of the bed, the depth of the fluid mud, \( z_c \), can be found by iterating the following equation (Li and Mehta, 2001).

\[ \zeta_{\text{max}}^\prime = \omega^2 \zeta_{\text{max}} \] (2.26)

\[ \omega^2 \zeta_{\text{max}} \Gamma(z_c) = \alpha g \dfrac{\rho_{\text{bed}}(z_c) - \rho_w}{\rho_w} \] (2.27)

Utilizing the above equations can determine the depth of fluid mud formation due to wave stresses.

The preceding section has described how fluid mud can form due to the lowered settling velocity from the higher concentration of sediment as well as from the cyclical pressure increases due to waves. Once the mud is formed, the next physical process of interest is transport.
2.2 Movement

The initiation of fluid mud movement can occur through two separate methods. The first method is gravity induced flow on a slope. The second method is shear-induced flow from movement of the overlying fluid. Both methods need to overcome the yield strength of the mud, if one is present, before flow starts to occur.

Fluid mud behavior under shear from either gravity or flow has been described as a non-Newtonian fluid, specifically a Bingham plastic. Non-Newtonian fluids do not exhibit a linear shear-strain curve. Bingham plastics exhibit a yield stress before movement initiates. This relationship is described by Liu and Mei as seen in Equation 2.28 (Liu and Mei, 1990).

\[
\mu_{fm} \frac{\partial u_{fm}}{\partial z} = \begin{cases} 
0, & \text{if } |\tau| < \tau_0 \\
\tau - \tau_0 \text{sgn} \frac{\partial u_{fm}}{\partial z}, & \text{if } |\tau| > \tau_0
\end{cases}
\]  

(2.28)

Where

\( \tau \) = Shear stress

\( \tau_0 \) = Yield stress

\( u_{fm} \) = Velocity in the mud layer (m/s)

\( \mu_{fm} \) = Viscosity of the mud layer (Pa s)

Liu and Mei as well as Le Hir note that fluid mud and other similar materials are likely not true Bingham plastics but have some large viscosity at low shear (Le Hir, 1997). Liu and Mei present this as Equation 2.29.
\[
\rho_{fm} \frac{\partial u_{fm}}{\partial z} = \begin{cases} 
\frac{\tau}{\nu_{fm,1}}, & \text{if } |\tau| < \tau_0 \\
\frac{\tau}{\nu_{fm}} + \frac{\tau_0}{\nu_{fm}} \left(1 - \frac{\nu_{fm}}{\nu_{fm,1}}\right) \text{sgn} \frac{\partial u_{fm}}{\partial z}, & \text{if } |\tau| > \tau_0 
\end{cases}
\]  

(2.29)

Where

\( \nu_{fm,1} \) = Viscosity for low shear rates (m\(^2\)/s)

\( \nu_{fm} \) = Viscosity for high shear rates (m\(^2\)/s), and \( \nu_i >> \nu \)

\( \rho_{fm} \) = Density of the mud layer (kg/m\(^3\))

The shear stress within the fluid mud layer can be written as Equation 2.30 (Liu and Mei, 1990).

\[
\tau = \rho g(h - z) \cos\Theta (\tan\Theta - \frac{\partial h}{\partial x})
\]

\[
\tau = \rho_{fm} g(h_{fm} - z) \cos(\theta) \left(\tan \theta - \frac{\partial h_{fm}}{\partial x}\right)
\]  

(2.30)

Where

\( h_{fm} \) = Thickness of the fluid mud layer (m)

\( z \) = Elevation above the bed (m)

\( \theta \) = Slope of the bed

The shear stress on the fluid mud (at \( z=0 \)) must be greater than \( \tau_0 \) for significant motion to begin. The velocity profile between the fluid mud and the yield surface is parabolic, as described in Equation 2.31, and the velocity between the yield surface and free surface is constant as plug flow, as described in Equation 2.32 (Liu and Mei, 1990).
\[ u_{fm} = \frac{1}{\mu_{fm}} \rho_{fm} g \cos \theta \left( \tan \theta - \frac{\partial h_{fm}}{\partial x} \right) \left( \frac{1}{2} z^2 - h_0 z \right), \quad 0 < z < h_{fm,0} \]  
\[ (2.31) \]

\[ u_{fm,p} = \frac{h_{fm,0}^2}{2 \mu_{fm}} \rho_{fm} g \cos \theta \left( \tan \theta - \frac{\partial h}{\partial x} \right), \quad h_{fm,0} < z < h \]  
\[ (2.32) \]

Where

\[ u \quad = \text{Velocity profile (m/s)} \]
\[ u_{fm,p} \quad = \text{Plug flow velocity (m/s)} \]
\[ h_{fm,0} \quad = \text{Height of the yield surface (m)} \]

To solve for the yield surface \( h_{fm,0} \), free surface \( h_{fm} \), and local discharge \( q_{fm} \), the three following equations need to be solved for the three above unknowns (Liu and Mei, 1990).

\[ \pm \tau_0 = \tau_0 s n u_{fm,p} = \rho_{fm} g \left( h_{fm} - h_{fm,0} \right) \cos \theta \left( \tan \theta - \frac{\partial h_{fm}}{\partial x} \right) \]  
\[ (2.33) \]

\[ q_{fm} = \int_0^h u_{fm} dz + u_{fm,p} \left( h_{fm} - h_{fm,0} \right) = \frac{1}{6\mu_{fm}} \rho_{fm} g \cos \theta \left( \tan \theta - \frac{\partial h_{fm}}{\partial x} \right) \left( 3h_{fm} - h_{fm,0} \right) \]  
\[ (2.34) \]

\[ \frac{\partial h_{fm}}{\partial t} + \frac{\partial q_{fm}}{\partial x} = 0 \]  
\[ (2.35) \]

Van Kessel and Kranenburg show the development of similar equations to Liu and Mei (Liu and Mei, 1990), specifically Equation 2.33, for laminar mud flow (van Kessel and Kranenburg, 1996). Van Kessel and Kranenburg also have conducted laboratory experiments of fluid mud on a sloping bed. Results from these experiments indicate that fluid mud would flow on a sloping bed under gravity. These experiments also illustrate “the agreement [between model and experimental results] is satisfactory
(van Kessel and Kranenburg, 1996),” demonstrating that the above equations are valid for fluid mud flow on a slope.

A fluid mud model developed by Teeter and Johnson also specifies fluid mud flow on a slope, without shear from the water column (Teeter and Johnson, 2005). The model evaluates the yield stress for the fluid mud against the gravitational force produced on the slope throughout the fluid mud layer. The yield stress in this model is defined as

$$\tau_0 = TUY_1 \left( \frac{C}{\rho_s} \right)^{TUY_2}$$  \hspace{1cm} (2.36)

Where

$$TUY_1 = \text{Yield stress constant}$$
$$TUY_2 = \text{Yield stress constant}$$

The shear stress developed by gravity is given as

$$\tau_g = G_1 g \Delta \rho h_{fm} \sin \theta$$  \hspace{1cm} (2.37)

Where

$$G_1 = \text{Constant assumed to be one (1),}$$
$$\Delta \rho = \text{Density difference between the fluid mud and water (kg/m}^3)$$

If $\tau_g$ exceeds $\tau_0$, the model evaluates two conditions. The first condition involves gravitational forces exceeding the yield strength at a single point, and plug flow occurring above that point. The second case involves gravitational forces exceeding the yield stress over a certain thickness of fluid mud. In this case, plug flow occurs above the layer, and
layer flow occurs over the thickness, where the yield stress is exceeded. The average flow speed of the layer is defined, assuming the bulk Richardson number for the flow equal to 2. The above-mentioned form of the Richardson number is given in Equation 2.38 and the average layer flow speed is given in Equation 2.39 (Teeter and Johnson, 2005).

\[
Ri = \frac{g \Delta \rho f_m \cos \Theta}{\rho f_m U_{fm}^2}
\]  
\hspace{1cm} (2.38)

\[
U_{fm} = \left(\frac{1}{2} gh_{fm} \frac{\Delta \rho}{\rho f_m} \cos \Theta\right)^{\frac{1}{2}}
\]  
\hspace{1.7cm} (2.39)

The plug flow velocity is given as

\[
U_{fm,p} = \frac{2h_{fm}U_{fm}}{h_{fm} + h_{fm,p}}
\]  
\hspace{1.7cm} (2.40)

Where

\[
h_{fm,p} \text{ is the plug flow layer thickness (m)}
\]

This equation assumes a vertical velocity profile over the layer flow thickness and velocity returns to zero when the yield stress exceeds the gravitational stress (Teeter and Johnson, 2005).

The second method of fluid mud flow, shear-induced flow, is theoretically possible; however, “there is no evidence that shear flows over fluid mud cause it to flow while retaining the characteristics of fluid mud (McAnally et al., 2007).” Experiments are conducted as described in Chapter IV regarding this method of movement.
The preceding section has described the development of the equations for the flow of fluid mud. This has included flow as a Bingham plastic with a yield stress under the force of gravity. The next section describes the dissipation of fluid mud.

2.3 Dissipation

Fluid mud is primarily dissipated through entrainment into the overlying water column. Shear stress in excess of that required to initiate or maintain motion of the fluid mud layer creates mixing at the interface of the fluid mud and water layers. A form of the Richardson number, a ratio of potential to kinetic energy, has been used to describe mixing between stratified water layers of fresh and salt water. This is similar to the process of fluid mud entrainment, and the Richardson number has been used in several fluid mud entrainment models (Mehta and Srinivas, 1993; Teeter and Johnson, 2005).

Mehta and Srinivas started by defining the nondimensional entrainment rate, $E$, as Equation 2.41.

$$E = \frac{u_e}{U_w} = c_e R_i^{-n_e}$$

(2.41)

Where

$u_e$ = Entrainment velocity (m/s)

$U_w$ = Mean water velocity (m/s)

$c_e, n_e$ = Constants

$Ri$ = Richardson number (Equation 2.42)

Here, the Richardson number is defined in terms of buoyancy, $\Delta b$, with $h$ defined as the height of the fluid above the fluid mud layer as in Equation 2.42.
\[ Ri = \frac{h_w \Delta b}{U_w} \]  

(2.42)

Evaluating Equation 2.42 for \( \Delta b = g \frac{\rho_{fm} - \rho_w}{\rho_{fm}} \), Equation 2.42 can be restated as Equation 2.43.

\[ Ri = \frac{h_w g (\rho_{fm} - \rho_w)}{\rho_{fm} U_w^2} \]  

(2.43)

Some sources (Kranenburg, 1994; Mehta and Srinivas, 1993) begin with the turbulent kinetic energy equation (TKE) to develop an entrainment model. Mehta and Srinivas began with “the non-stationary, turbulent energy balance for horizontally homogeneous boundary layer above the fluid mud layer with mean flow in the longitudinal direction (Mehta and Srinivas, 1993)” as seen in Equation 2.44.

\[ \frac{\partial q}{\partial t} = -u'w' \frac{\partial U_w}{\partial z} - g \frac{\partial}{\rho_w} \frac{W' \rho'}{W' \rho'} + \epsilon \]  

(2.44)

Where

- \( q \) = Turbulent kinetic energy per unit mass
- \( u' \) = Turbulent fluctuation of \( U_w \) (m/s)
- \( v', w' \) = Velocity fluctuations (m/s)
- \( \rho' \) = Turbulent fluctuation of density (kg/m\(^3\))
- \( \epsilon \) = Rate of energy dissipation per unit mass

After evaluating fluxes and scaling factors and substituting terms, a generalized nondimensional entrainment equation is developed (Mehta and Srinivas, 1993).

\[ E = A_e R_i^{-1} + B_e P_e^{-\frac{1}{2}} - D_e R_i \]  

(2.45)
Where

\[ A_e = \text{Constant} \]
\[ B_e = \text{Constant} \]
\[ D_e = \text{Constant} \]
\[ Pe = \text{Peclet number, defined as } Pe = U_w h / \kappa \]
\[ \kappa = \text{Molecular diffusivity} \]

On the right hand side of Equation 2.45, the first term is Equation 2.41 with \( n_e = 1 \). This term is the entrainment portion due to shear. The second term represents the diffusion portion of total entrainment. The last term on the right describes settling and cohesion, which works against the upward entrainment. Results from experiments conducted in a racetrack flume with various sediment types as well as comparison with previous experimental data direct Mehta to determine the best-fit parameters for \( A_e \) and \( D_e \). The constant \( B_e \) was assumed to be zero, as diffusion is negligible at the lower Richardson numbers (higher velocities) included in this paper. The other constants were determined to be 0.0052 and 0.000016 for \( A_e \) and \( D_e \), respectively (Mehta and Srinivas, 1993).

Kranenburg also developed an entrainment model for fluid mud (Kranenburg, 1994). The model is developed starting from a form of the TKE (Equation 2.46). This work was further developed by Kranenburg and Winterwerp (Kranenburg and Winterwerp, 1997).

\[
\frac{\partial \bar{q}}{\partial t} + \bar{u} \bar{W} \frac{\partial \bar{q}}{\partial z} - \frac{2}{\bar{W}} \frac{\tau_w}{\rho_r} \bar{u} - \alpha_d \frac{g}{\rho_r} \bar{W}' \bar{s}' + \frac{\partial D_r}{\partial z} + \epsilon = 0
\]  

(2.46)
Where

\( \overline{q} \) = Mean turbulent kinetic energy

\( u, w \) = Velocity components (m/s)

\( W \) = Width of the channel (m)

\( \rho_r \) = Reference density (kg/m\(^3\))

\( \tau_w \) = Shear stress from sidewalls (Pa)

\( s \) = Dry density of the sediment concentration (kg/m\(^3\))

\( D_r \) = Redistribution term

\( \epsilon \) = Dissipation rate

\( \alpha_d \) = Fractional density difference

\( z \) = Vertical direction, positive downward and zero at the free surface

\( \overline{u'w'} \) = Turbulent transport term

\( \overline{w's'} \) = Turbulent transport term

The following boundary conditions are described for the free surface, Equation 2.47, and the base of the fluid mud layer, Equation 2.48.

\[
\overline{w's'} + w_s \bar{s} = 0, \quad \overline{u'w'} = u_*^2, \quad D_r = 0
\]  

(2.47)

\[
\overline{w's'} + w_s \bar{s} = 0, \quad \overline{u'w'} = u_H^2, \quad D_r = 0, \quad \bar{q} = 0, \quad \bar{s} = s_H, \quad \bar{u} = U_H
\]  

(2.48)

Where

\( u_* \) = Friction velocity at the free surface (m/s)

\( u_H \) = Friction velocity at the bed, \( z=H \) (m/s)

\( U_H \) = Water Velocity at \( z=H \) (m/s)
\[ s_H = \text{Sediment concentration at the bed (kg/m}^3) \]

Integrating and substitution of boundary conditions into Equation 2.46 yields

Equation 2.49.

\[ \frac{d}{dt} \left( \int_0^H q \, dz \right) + \int_0^H u' w' \frac{\bar{H}}{\bar{z}} \, dz - 2 \frac{\lambda}{W} U^3 H - \alpha \frac{g}{\rho_r} \int_0^H s' w' \, dz + \int_0^H \epsilon \, dz = 0 \]

\[ I \quad II \quad III \quad IV \quad V \]  

(2.49)

Term I is the turbulent kinetic energy storage term, II and III are the shear production terms of TKE, IV is turbulent sediment transport, and V is the TKE dissipation term. Integrating the equation and combining the terms gives Equation 2.50.

\[ c_q \frac{d}{dt} (u_i^2 H) - 2 c_H u_H (U - U_H) - c_* u_i^2 (U - U_H) - c_H (U - U_H)^2 \frac{dH}{dt} + \\
-2 c_w \frac{H}{W} U^3 + B \frac{dH}{dt} + (U - U_H) u_b^2 + c_b u_b^2 \frac{dH}{dt} + 2 \frac{\alpha_s}{\rho_r} \tilde{g} \tilde{w}_s H = 0 \]

(2.50)

Where

\[ c_q \quad = \text{Empirical constant} \]

\[ c_H \quad = \text{Empirical constant} \]

\[ c_* \quad = \text{Empirical constant} \]

\[ c_w \quad = \text{Empirical constant} \]

\[ c_b \quad = \text{Empirical constant} \]

\[ B \quad = \text{Total buoyancy} \]

\[ u_b^2 \quad = \text{Work per unit mass needed to destroy the bed structure} \]

\[ \lambda \quad = \text{Friction factor}. \]
Teeter and Johnson used a similar equation to Equation 2.41 to describe the entrainment of fluid mud (Teeter and Johnson, 2005). This variation included the value of the exponent, $n_e$, of 3/2, as shown in Equation 2.51. The -3/2 power law is described by Linden and supported through laboratory experiments of Long as well as Xuequan and Hopfinger (Linden, 1973; Long, 1975; Xuequan and Hopfinger, 1986).

\[
Em = \frac{u_e}{u_s} = KRi_s^{-3/2}
\]  

(2.51)

Values of $K$ in Equation 2.51 are found to be a median value of 2.8 (Teeter, 2002), which reasonably compares to the value of 3.8 found by Xuequan and Hopfinger (Xuequan and Hopfinger, 1986). For Equation 2.51, the interfacial Richardson number, $Ri_s$, is described in the following equation (Equation 2.52).

\[
Ri_s = \frac{g \Delta \bar{\rho} h}{\rho u_{*}^2}
\]  

(2.52)

Where

\[\bar{\rho} = \text{Average density between the layers (kg/m}^3)\]

In summary, the above equations describe the entrainment of fluid mud into the overlying water column. The following section describes another mechanism of removing material from the fluid mud layer, known as consolidation.

### 2.4 Consolidation

Consolidation removes material from the fluid layer mud from the bottom of the layer by creating new firm bed material. Through this process, the bottom of the fluid
mud layer becomes part of the top bed layer, which involves the last type of settling described in Equation 2.5 when local concentrations exceed $C_3$. Toorman and Berlamont presents a sediment mass balance from Kynch in terms of excess density, seen in Equation 2.53 (Kynch, 1952; Toorman and Berlamont, 1993).

$$\frac{\partial \Delta \rho}{\partial t} = -\frac{\partial}{\partial z} (w_s \Delta \rho)$$

(2.53)

Tiller describes the force balance in a saturated soil as seen in Equation 2.54 (Tiller, 1981).

$$\frac{1}{g} \frac{\partial \sigma'}{\partial z} = -\Delta \rho + \frac{\rho_w}{k} w_s$$

$$\frac{1}{g} \frac{\partial \sigma}{\partial z} = -\Delta \rho + \frac{\rho_w}{k_p} w_s$$

(2.54)

Where

$\sigma'$ = Effective stress

$k_p$ = Permeability

 Been relates settling rate to permeability during hindered settling as Equation 2.55 (Been, 1980).

$$w_{s0} = k_p \frac{\Delta \rho}{\rho_w} + k_p \frac{\Delta \rho}{1 + e \rho_w}$$

(2.55)

Where

$w_{s0}$ = Settling velocity

$e$ = Void ratio.
By substituting Equation 2.54 into Equation 2.53, and substituting Equation 2.55 into the resulting equation produces Equation 2.56 (Toorman and Berlamont, 1993).

\[
\frac{\partial \Delta \rho}{\partial t} + \frac{\partial}{\partial z} \left[ w_s \Delta \rho^2 + \frac{w_s g \partial \sigma'}{\partial \Delta \rho} \frac{\partial \Delta \rho}{\partial z} \right] = 0
\]

(2.56)

The above equation requires two relationships: (1) settling rate to excess density and (2) effective stress to excess density. Toorman and Berlamont then develop two equations for two modes of consolidation, loose soil consolidation and compact soil consolidation (Toorman and Berlamont, 1993). The combined form of these equations for the full range of excess densities is

\[
w_s = w_{s1} e^{-\left(\frac{\Delta \rho}{\Delta \rho_1}\right) F_t} + w_{s2} \left(1 - \frac{\Delta \rho}{\Delta \rho_2}\right)^3 (1 - F_t)
\]

\[
F_t = e^{-\left(\frac{\Delta \rho}{\Delta \rho t}\right)^{n_s}}
\]

(2.57)

Where

- \(w_{s1}\) = Settling rates for \(\Delta \rho=0\) for a loose soil or concentrated suspension (m/s)
- \(w_{s2}\) = Settling rates for \(\Delta \rho=0\) compact soil (m/s)
- \(\Delta \rho\) = Excess density (kg/m³)
- \(\Delta \rho_1\) = Excess density at the maximum settling flux (kg/m³)
- \(\Delta \rho_2\) = Maximum compaction excess density (kg/m³)
- \(F_t\) = Transition function
- \(\Delta \rho_t\) = Transitional excess density (kg/m³)
- \(n_s\) = Constant
Mehta and McAnally present this equation in terms of concentrations as Equation 2.58 (Mehta and McAnally, 2009).

\[ w_{sc} = w_{sc1} e^{-\left(\frac{C}{C_{s1}}\right) F_t} + w_{sc2} \left(1 - \frac{C}{C_{s2}}\right)^{m_t} (1 - F_t) \]

\[ F_t = e^{-\left(\frac{C}{C_{t}}\right)^{n_t}} \quad (2.58) \]

Mehta and McAnally presents a table of experimental and field data for the above constants (Mehta and McAnally, 2009). The ranges of these values are listed in the table below.

<table>
<thead>
<tr>
<th>( w_{sc1} ) (m/s)</th>
<th>( C_{s1} ) (kg/m(^3))</th>
<th>( w_{sc2} ) (m/s)</th>
<th>( C_{s2} ) (kg/m(^3))</th>
<th>( m_t )</th>
<th>( n_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6–5( \times )10(^{-4})</td>
<td>15–31</td>
<td>3–7( \times )10(^{-6})</td>
<td>205–1000</td>
<td>15–210</td>
<td>3–6</td>
</tr>
</tbody>
</table>

This section has described the consolidation of fluid mud both as a change in density over time as well as a settling velocity. The settling velocity based on concentration, Equation 2.58, is utilized as the consolidation routine within the computer code developed in Chapter V.

### 2.5 Existing Fluid Mud Models

Previous models have been developed to describe fluid mud transport and processes. This section details the existing models and their limitations.

Odd and Cooper (Odd and Cooper, 1989) developed a two-dimensional depth averaged model to describe fluid mud movement, including an interfacial shear term from the water column. The authors neglected both the convective acceleration terms and eddy diffusion terms. They also used a yield stress to resist fluid mud motion, by
preventing movement at low shear rates and acting opposite the flow as a viscosity during mud motion. Output from this model was compared to field data and revealed some mixed agreement.

Le Hir (Le Hir, 1997) developed a one-dimensional vertical model and a two-dimensional vertical model to describe mud movement. These models captured the change in concentration through the water column, and the two-dimensional model was tested using cohesive material discharge on a slope. This adequately captured the movement of the fluid mud down slope, in agreement with other laboratory studies (van Kessel and Kranenburg, 1996). Le Hir (Le Hir et al., 2001) extended his earlier work and used a one dimensional vertical model to capture behavior of the “continuous” sediment and water column as a whole, without using erosion or settling parameters. This treated the water column as a continuum, and adequately captured the sediment concentrations at an estuarine turbidity maximum. This model is limited in that it focuses on the vertical variation and not on the lateral distribution and movement of the mud.

Winterwerp et al. (Winterwerp et al., 2002) developed a two-layer, two-dimensional depth averaged model for fluid mud flow. This included both shear and pressure effects from the water column on the mud layer as well as supercritical flow in the mud. This work was done in support of water injection dredging and as such may not be applicable to naturally forming fluid mud.

Guan et al. (Guan et al., 2005) modeled fluid mud using the three-dimensional Princeton Ocean Model (POM). This model was refined near the bed to capture the sedimentation processes and fluid mud formation. This model did not treat fluid mud as
a separate layer but as a high concentration area within the vertical continuum of sediment-water.

Teeter and Johnson (Teeter and Johnson, 2005) used a fluid mud module to describe the movement of fluid mud in the Atchafalaya Bay area. This model was limited, as movement was only initiated through gravitational forces and not shear from the overlying water column. This model treated failure of the bed on a slope as the method of fluid mud creation, with plug flow above the failure and a velocity profile below the failure level.

Merckelbach and Winterwerp (Merckelbach and Winterwerp, 2007) extended the work of (Winterwerp et al., 2002) to create a quick estimate for mud flow in estuaries. This model used numerous simplifications, such as a constant depth throughout the estuary, during development. These assumptions limit the accuracy of the model, but allow for a rapid approximation of mud movement from water injection dredging.

Based on these modeling studies, a two-dimensional, depth-averaged flow model to describe fluid mud is appropriate under some conditions and has been used in the past (Odd and Cooper, 1989; Teeter and Johnson, 2005; Winterwerp et al., 2002). This two-layer approach is used in Chapter V.

2.6 Conclusion

The above sections describe the equations relating to the four main physical processes of fluid mud – formation, movement, dissipation, and consolidation. Chapter V develops these equations into a combined fluid mud model that describes these physical behaviors.
CHAPTER III
FLUID MUD FLOW UNDER SHEAR STRESS EXPERIMENTS

The following sections describe the setup for the fluid mud flow under shear stress experiments. As mentioned previously in Section 2.2, no evidence is found to support the movement of fluid mud under shear flows. The following experiments are designed to determine if fluid mud flow occurs under shear stress. The specific details of each experiment are shown in Chapter IV.

3.1 Equipment

A 50-foot long, 1-foot wide, and 1-foot deep titling flume is used as the testing apparatus. The flume is fed by a 15-horsepower pump from an 1800-gallon reserve basin. The water is fed into a 3-foot deep stilling basin at the head of the flume. The flume is tilted to have a uniform slope of 0.001. A valve at the upstream inflow pipe is used to adjust the flow from fully closed with no flow to fully open with maximum flow. A recirculating pipe allows water to return to the reserve basin when the upstream valve is closed. An adjustable gate at the downstream end of the flume is used to adjust water depth in the flume. Water is contained behind the gate by installing a clear plastic sheet to prevent leakage around the gate. After flowing over the gate, the water spills into an L-shaped stilling basin. To minimize water loss, a wooden U-shaped channel is installed underneath the end of the flume to the beginning of the stilling basin. This directs water
into the basin and minimizes spilling due to splashing. A V-notch weir with a point
gauge at the end of the stilling basin allows for flow measurement through the flume.
The V-notch weir is calibrated to allow for accurate determination of the flow. The
calibration values are included in Appendix A. A diagram of the pump and pipe system
is shown in Figure 3.1. Photos of the flume and pump apparatus are shown in Figures 3.2
through 3.7.
Figure 3.1  Pump and pipe circulation schematic
Figure 3.3  Pump with intake and recirculating piping. Basin is under green plates.
Figure 3.4  View of flume from upstream end. Control valve and stilling basin can be seen on the left.
Figure 3.5  Plywood channel at end of flume
Figure 3.6  Calibrated weir at end of stilling basin.
Figure 3.7  Weir set up for measurements.
A raised bed is created out of LEXAN and installed in the flume. The bed is 0.5 feet high and 1 foot wide to fit the entire width of the flume. The raised bed is constructed out of ¼” LEXAN in 8-foot sections. Each rectangular section is supported with 3 vertical 6” pieces of LEXAN running in the direction of the flow, with caps on each end. Each cap has twelve 1” holes to allow for water to fill the sections, removing the air spaces and preventing the sections from floating. The angled sections are supported with caps running across the width of the section every 2 feet. The LEXAN pieces are attached using ⅛” hex screws into counter sunk and tapped holes. A depression is built into the elevated bed to contain the fluid mud during the testing. The depression is 0.5 feet deep, with 1V:16H slopes on the upstream and downstream ends of the depression. An 8-foot section of flat, elevated bed precedes the depression. A total of 12 feet of flat elevated bed are located downstream of the depression. A flow straightener built of 1-foot sections of 1-inch PVC pipe is constructed over the first foot of the upstream elevated bed to minimize entrance turbulence in the incoming water flow. After installation of the raised bed into the flume, silicone caulk is applied at junction of the LEXAN and wall, as well as the LEXAN sections to provide a smooth surface and minimize turbulence. The counter sunk screws are also caulked over to provide a smooth floor. The raised bed sections are siliconed together, with a bead of silicone attaching the sections to the floor of the weir. In addition, one small piece of LEXAN approximately 2”x6” is affixed to the floor of the flume at either end of the raised bed to prevent any movement of the bed. A drawing of the set up is included in Figure 3.8, and photos of the installed bed can be seen in Figures 3.9 through 3.15.
Figure 3.8  Design of raised bed in flume
Figure 3.9  View of LEXAN bed from upstream end of flume, no flow straightener.
Figure 3.10  Alternate view of LEXAN bed looking downstream.
Figure 3.11  View of bed from above flume looking downstream
Figure 3.13  Slope of depression in raised bed looking downstream
Figure 3.14 Alternate view of depression in raised bed
Figure 3.15  Flow straightener and raised bed in flume.
The fluid mud is created from commercially available sodium bentonite, obtained from the Oktibbeha County Co-op. The mud is mixed in a drum next to the flume to create a slurry with a known density and concentration, with a target concentration of 20 g/L. The mud is pumped into the flume using a submersible sump pump with a “Y” valve installed to modulate the inflow of fluid mud to minimize entrainment and turbulence. After each experimental run, the flume, reserve basin, and stilling basins are cleaned of sediments to begin each run similarly with clean water. Figures 3.16 through 3.18 show the mixing of the fluid mud and how it is pumped into the flume.
Figure 3.16  Creation of fluid mud
Figure 3.17 Mud creation using hand drill and grout mixer
Figure 3.18 Controlled addition of fluid mud to filled flume
Two miniature pressure sensors from GE/Druck (PCDR 81) are used to determine if fluid mud or a solid bed existed in the containment area. The working pressure range of the sensors is 350 mbar or 5 psi. Complete specifications for the sensors can be seen in Appendix A (GE, 2008). The pressure sensors are attached to a point gauge to measure depth below the water surface and to check the sensor results. This arrangement can be seen in Figure 3.19. The sensors are attached to a laptop through a LabVIEW interface NI 9205, as seen in Figures 3.20 through 3.22. The computer is set up with a LabVIEW code to view their output. One sensor is equipped with a porous stone to measure pore water pressure, and the second sensor is without the stone to measure the total pressure. This allows for determination of the effective stress in the mud and the thickness of the fluid mud and firm bed, if present.
Figure 3.19  Pressure sensors. Pore pressure sensor with stone in upper right and total pressure sensor without stone at bottom.
Figure 3.20 LabView interface
Figure 3.21  LabView interface block diagram
Figure 3.22  LabView interface, NI 9205, between pressure sensors and computer
3.2 Fluid Mud Presence

The pressure sensors are set up to determine the effective stress in the mud. Effective stress is the difference between the pore pressure and total pressure in the mud. In the water or fluid mud, if present, the effective stress is practically zero. In a bed, particle-to-particle contact carries some of the pressure, and the total pressure is greater than the pore pressure, producing a positive effective stress. The change between zero and positive effective stress indicates the transition from fluid mud to a bed condition.

3.3 Fluid Mud Movement

The test for movement of fluid mud is initiated by opening the inflow pipe to super-elevate the upstream water surface of the flume. This started water flow through the flume and into the weir. The fluid mud is watched for movement by two observers as well as one or more video cameras for documentation. The flow rate required to initiate motion is measured by the point gauge over the calibrated downstream weir as shown in Figure 3.8. Experimental details and results are presented in Chapter IV.
CHAPTER IV
FLUID MUD FLOW EXPERIMENTS AND RESULTS

The following sections describe the experiments conducted on the flow of fluid mud and the results from those experiments.

4.1 General Experimental Setup

The general experimental setup is used for all experiments. To begin the fluid mud flow under shear stress experiments, the flume is filled, and the fluid mud is added to the bottom of the filled flume to minimize mixing and entrainment.

The water depth for the flume is set with an adjustable tailwater flap at the end of the flume. The flap does not leak due to the installation of the polyethylene sheet, as described in Chapter III. The flume is filled using the main recirculating pump. Once the flume is full, the valve at the headwater of the flume is closed, and the water level is allowed to stabilize and any waves to dissipate before the introduction of the fluid mud.

The fluid mud is created from commercially available sodium bentonite. A set mass of the bentonite is added to a set volume of water as discussed below to produce a mud of known density. The mud is mixed using a hand drill with an extended grout mixer to reach the bottom of the barrel. When running, the mixer creates a vortex in the middle of the barrel that mixes the entire depth of mud.
The mud is pumped into the center of the sloped portion of the LEXAN bed, as seen in Figure 3.18. By pumping the mud under the water, the higher density mud is allowed to flow along the bottom under the lower density water, with minimal mixing of the two fluids. The pump for the introduction of the mud is a sump pump, equipped with a garden hose “Y” valve that allowed for adjustment of the inflow rate to reduce the velocity of the incoming mud and reduce mixing. The second outflow of the “Y” valve is left unrestricted and creates a current in the barrel of fluid mud which allows for continuous mixing and a well mixed mud.

After the fluid mud is added, the two miniature pressure sensors are lowered into the fluid mud, to check for effective stress. Output from the sensors is sampled at 101 Hz. The output from the pressure sensors are fed through a LabView interface NI-9205 and into a LabView code that exported the raw and filtered results to a file as well as a graph on screen. The filter uses a 101 point (1 second) running average to remove some of the minute fluctuations from the output of the sensors.

Once the fluid layer are determined to be fluid mud from the pressure sensors, the valve at the headwater of the flume is slightly opened to introduce a flow of water into the flume and produce shear stress over the fluid mud layer. Two observers and a video camera are watching the flume to check for movement as well as recording the height over the calibrated v-notch weir at the end of the flume to determine the flow rate through the flume. The next sections describe the individual details of each experiment, and the results from those experiments.
4.2 Initial Fluid Mud Flow Test

The first experiment for fluid mud flow under shear is set up as described above. The initial concentration chosen for the fluid mud is 20 g/L. To achieve this concentration, 3 kg of sodium bentonite is added to 151 liters of water (approximately 40 gallons). The volume of the void in the LEXAN floor is 113 liters, so the 151 liter mixture provided more than the required volume of fluid mud. However, once pumping began, some mixing did occur, with the result that only 38 liters of mud is pumped into the 113 liter void. This produces an actual mud concentration of 6.7 g/L. Once the fluid mud is pumped into the flume, the heights of the water surface and mud surface are recorded. The depth of water for this experiment is 0.23 meters (9.04 inches) above the bottom of the flume and a maximum mud depth of 0.13 m (5.27 in). The sensors are lowered through the water and mud layer to check for effective stress. Figure 4.1 shows the results from this test. The sensors do not have identical zero points (as shown in Figure 4.1 “Pore pres” and “Total Pres”). A constant value is added to the pore pressure results to account for this offset. The resulting value is included in the following figures as “Pore pres (const).” Negative values on the y-axis of the chart only indicate that the voltage offset from zero for the sensors is accounted for. This value is within the specified ±10mV offset (GE, 2008).

As seen in Figure 4.1, both total pressure and pore pressure are the same, so there is no effective stress, and particle to particle structural support is not present. For the first 30 seconds, the sensors are allowed to equalize. From 30 to 45 seconds, the sensors are lowered to the top of the fluid mud layer. At 64 seconds, the sensors are lowered into the mud layer, and reach the bottom at 90 seconds. At 109 seconds, the sensors are raised
from the bottom, leaving the fluid mud layer at 137 seconds and the water column at 149 seconds.

![Figure 4.1 Sensor output from Experiment 1](image)

The results from the sensor test indicate there is no bed formation within the flume. Visual inspection of the mud layer reveals a fluid-like material and a definite interface between the water layer above and the mud layer below, indicating a density change, or lutocline, between the layers. When raising the sensors from the mud layer, a small visible void is seen, and the mud slowly flowed back into the void, indicating a density greater than water. These observations indicate that fluid mud is present, and the experiment is started.
To begin the experiment, the valve at the head of the flume is slightly opened and water began to flow. Movement is first noticed at a flow rate of $1.5 \times 10^{-4}$ m$^3$/s. The flow is slowly increased, and continued observations are made as the flow increased. The maximum flow rate is $1.2 \times 10^{-3}$ m$^3$/s. At this flow rate, a second sensor test is conducted to determine if any bed had formed. The water rises to a depth of 0.25 m, and the mud layer thins to 0.067 m, which is a reduction of approximately half its initial thickness. At 0 seconds, the sensor is lowered to the top of the fluid mud layer, which is reached at approximately 24 seconds. At roughly 43 seconds, the sensors are lowered to the bottom, which is reached at 50 seconds. At 60 seconds, the sensors are raised, leaving the mud layer by 78 seconds and reaching the initial depth at 94 seconds. The second sensor test demonstrates that mud is still present and no bed had formed since there is no effective stress, as shown in Figure 4.2.

Figure 4.2 Sensor results from the end of Experiment 1
4.2.1 Observations

The primary purpose of these experiments is to observe if fluid mud flowed under shear stress without entrainment. This initial experiment demonstrates that fluid mud did flow under shear, with a small amount of entrainment. Entrainment does not appreciably increase with the observed flow rates, although the flow rate of fluid mud does increase. The mud is observed to travel upslope on the downstream side of the valley, further showing that the shear is moving the mud and that it overcame the effects of gravity.

Images are taken from one of the videos during the test, seen in Figure 4.3, to show the movement of fluid mud indicated by the pointer showing the peak of an interfacial wave. Interfacial waves are plainly observed, as seen in Figure 4.4. An apparent velocity profile is present in the mud layer, as indicated by movement along the top of the layer and a nonmoving bottom. The mud layer was not thick enough to determine if the velocity profile was linear or nonlinear. Once the fluid mud passes the top of the slope, it continues to flow as a layer without substantial entraining into the water flow. At this concentration of mud, the smallest flow rate begins to move the mud. A yield strength is not observed, or is too small to observe with the limits of the equipment, specifically the control of the valve at the head of the flume did not allow for very minute adjustments.

Figure 4.5 shows the time series of the fluid mud thickness and total water surface elevation over the experiment. Figure 4.6 shows video frames taken perpendicular to the flume. The change in mud thickness is plainly visible. Measurements of the thickness of the mud layer and water surface were taken from the video frames with the program ImageJ, which is based on a program by the National Institute of Health (Rasband, 2014).
Figure 4.3  Fluid mud flowing

(a)-(d) Progression of interfacial wave moving from right to left, indicated with arrow
Figure 4.3(Continued)
Figure 4.4  Interfacial breaking wave in fluid mud layer

Figure 4.5  Time series of Mud and Water Elevations during Experiment 1

Data gap is due to camera movement during experiment
Figure 4.6   Fluid mud movement

(a)-(f) Time series photos of mud thickness
Figure 4.6 (Continued)
Figure 4.6 (Continued)
4.3 Effects of Increased Concentration

The next parameter to test is the effect of concentration on the fluid mud flow under shear. Based on the initial experiment, a concentration of 20 g/L is chosen for this experiment. From the mixing that occurred during the filling of the flume with mud during Experiment 1, a concentration of 80 g/L is created by adding 12 kg of sodium bentonite to 151 liters of water. If a similar quantity of mixing, as observed in Experiment 1, occurs then the concentration of the mud would still be above the target of 20 g/L. After filling the flume with water and following the steps outlined in Section 4.1, the 80 g/L concentration of mud is pumped into the flume. A similar amount of mixing occurs, with 49 liters of mud mixing to fill the 113 liter space. This mixing reduces the concentration of the mud to 35 g/L. Once the mud is pumped into the flume, the pressure sensors are lowered into the water column. For this test, the depth of water is 0.24 m, and the depth of mud is 0.15 m. The sensors are lowered to the mud-water interface, and the pressures are recorded. At 30 seconds, the sensors are lowered to the bottom of the flume, which is reached at 38 seconds. The sensors are then immediately removed to a point above the mud layer. At 70 seconds, the sensors are lowered into the fluid mud layer, reaching the bottom of the flume at 100 seconds. At 127 seconds, the sensors are raised, reaching the top of the water column at 165 seconds. The sensor results can be seen in Figure 4.7. These results show that the total pressure and pore pressure are equal, no effective stress is present, and no bed has formed. Visual inspection of the mud layer reveals similar behavior to Experiment 1. A small void is seen after removing the pressure sensors, which the mud flowed in to fill, indicating a viscosity greater than water. The clear and abrupt change between the water layer and mud layer is present,
indicating a large density change. Based on these observations, the presence of fluid mud is confirmed, and the experiment is started.

![Figure 4.7 Sensor results from the beginning of Experiment 2](image)

To begin Experiment 2, the flow valve is opened, and water begins flowing through the system. Movement is first noticed at a flow rate of $1.6 \times 10^{-4}$ m$^3$/s. The flow was slowly increased, and continued observations were made as the flow increased. The maximum flow rate was $7 \times 10^{-4}$ m$^3$/s. At this flow rate, an Acoustic Doppler Velocimeter (ADV) is inserted into the flow and measures the velocity at 4 different depths. The first measurement is conducted midway between the surface and the mud layer. The second measurement is conducted at the mud-water interface, and the third measurement is just below the surface of the mud. The fourth measurement is just above the bottom while the
fifth measurement is back at the interface, and the sixth measurement is back to the middle of the water column. The results are summarized in Table 4.1.

Table 4.1   ADV results from Experiment 2

<table>
<thead>
<tr>
<th>Test File Name</th>
<th>Location</th>
<th>Average Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEFAULT0005.ADV</td>
<td>Midwater</td>
<td>2.49 cm/s</td>
</tr>
<tr>
<td>DEFAULT0006.ADV</td>
<td>Interface</td>
<td>0.18 cm/s</td>
</tr>
<tr>
<td>DEFAULT0007.ADV</td>
<td>Top of mud</td>
<td>0.23 cm/s</td>
</tr>
<tr>
<td>DEFAULT0008.ADV</td>
<td>Bottom of mud</td>
<td>4.05 cm/s</td>
</tr>
<tr>
<td>DEFAULT0009.ADV</td>
<td>Interface</td>
<td>0.25 cm/s</td>
</tr>
<tr>
<td>DEFAULT0010.ADV</td>
<td>Midwater</td>
<td>3.77 cm/s</td>
</tr>
</tbody>
</table>

The velocity measurement at the bottom of the mud is likely a measurement error, which is possibly due to the high concentration of sediment in the bottom layer of the mud affected the ADV. It is possible that some circulation pattern is present, with mud flowing back down the slope to create an eddy-like pattern; however, this behavior is not seen visually. Based on this, the actual velocity near the bed appears close to zero. The other measurements agree with similar values between the interface and top of the mud layer. During the measurement of the velocities, the flow rate continues to increase as seen in the midwater velocities. The flow rate at the end of the test is $1 \times 10^{-3}$ m$^3$/s.

At this point, another sensor test is run to see if any bed has formed during the experiment. The sensors are initially located above the fluid mud layer, which is now a thickness of 0.13 m. At 60 seconds, the sensors are lowered to above the fluid mud layer, which is reached by 75 seconds. At 100 seconds, the sensors are lowered into the fluid mud and reach the bottom of the flume at 150 seconds. At 200 seconds, the sensors are
raised from the bottom and removed from the water column at 244 seconds. The shape of the total pressure curve in Figure 4.8 shows that a bed layer did form during this experiment. The delay in the increase of pressure is probably due to the disturbance of the bed from the intrusion of the sensors. The insertion of the sensors most likely pushes some of the bed away, which then reforms during the period of 154 to 200 seconds around the sensors, increasing the total pressure. Based on physical measurements outside the flume, a 0.036 m of bed forms, as seen in Figure 4.8.

![Figure 4.8](image.png)

Figure 4.8 Sensor results from the end of Experiment 2, demonstrating the presence of a bed structure
4.3.1 Observations

The primary goal of this experiment is to observe the effects of increased concentration on fluid mud flow under shear. During this experiment, the valve is opened extremely slowly, and at very low flow rates, no movement of the fluid mud is apparent. These flow rates are too small to measure over the weir, amounting to slightly more than a trickle. The lowest measureable flow rate of $1.6 \times 10^{-4} \text{ m}^3/\text{s}$ does have associated mud movement, indicating a very small yield strength of the mud. Other observations are similar to the lower concentration experiment with visible interfacial waves and small levels of entrainment. Figure 4.9 shows the movement of an interfacial wavefront as seen from above the flume. These images demonstrate the interfacial waves as well as a small amount of entrainment. A velocity profile is present in the mud layer, demonstrated by the ADV (Table 4.1). As the mud crossed the top of the slope, it continues to flow as a layer without fully mixing with the water. Entrainment is more prevalent in this portion of the flume, which is likely due to increased velocities from the decreased cross-sectional area.
Figure 4.9  Movement of wavefront under shear in fluid mud layer as seen from above. (a)-(e) Progression of interfacial wave moving from right to left, indicated with arrow.
Figure 4.9 (Continued)
4.4 Settling Experiment

Following the two flow experiments, a settling experiment is conducted to observe the behavior of a high concentration of fluid mud as it settles for a long period of time. In this experiment, the mud is allowed to settle for a period of 12 hours with no flow. The flume is drained and flushed of most residual mud. It is then refilled with clean water to prepare for the settling test. Once the flume is full, the mud is pumped into the depression. The mud used is the same 80 g/L concentration that is used for the previous experiment. Since only 49 liters of the 151 liters prepared is used for the flow experiment, the same mud created earlier in the day is used for the settling experiment. Filling the depression with mud takes much longer with much less mixing, and the filling is stopped when the depth of the mud was only 0.1 m (4 inches). At this point, the “Y”-
valve on the sump pump is beginning to spray the mud out of the water as opposed to mixing it when the mixing barrel was deeper. The total volume pumped into the valley is 57 liters. The volume occupied by the mud is 50 liters when pumping was stopped. This indicates the possibility of consolidation during filling.

After filling the flume, a sensor test is conducted on the mud. The water depth is 0.22 m, and the mud depth is 0.10 m. The sensors are initially at the top of the water layer, and after 13 seconds, are lowered into the water approximately an inch. After 60 seconds, the sensors are lowered to the top of the mud layer. At 134 seconds, the sensors are lowered to the bottom of the flume. At 168 seconds, the sensors are raised back to the top of the mud layer and into the water column, and the test ends at approximately 250 seconds. Figure 4.10 shows the results from the sensor test. This test demonstrates that consolidation did occur during filling and some bed has formed, as there is effective stress present. The delay is likely due to the disruption of the bed during the lowering of the sensors.
The bed is allowed to consolidate for ten minutes and another sensor test is conducted to check for any bed formation. During the ten minutes, the mud layer consolidates an additional 0.017 m. The sensors are initially placed 0.03 m below the water surface, and lowered to the bottom after 30 seconds. The sensors reach the bottom at 76 seconds, and are raised beginning at 106 seconds, returning to the initial depth at 160 seconds. Figure 4.11 shows the results from this test, which indicate that a cohesive bed has formed, and that during the ten-minute period, more consolidation occurred and the bed increased in thickness.
The mud layer is then allowed to consolidate for 12 hours. At the end of the 12 hours, the mud layer consolidates to approximately 0.038 m (1.5 in), as seen in Figure 4.12. A sensor test is initially started at the beginning of the twelve hours with the sensors in the mud; however, the mud consolidates below the level of the sensors so a new sensor test begins after the 12 hour consolidation period. This test starts with the sensors just above the bed, and then the sensors are raised higher into the water column to attempt to dislodge any stuck sediment particles. The sensors are then lowered through the bed to the bottom and raised back up. The results from the sensor test are shown in Figure 4.13. These results indicate that there is no effective stress as pore pressure and total pressure are the same (near time 125 seconds). However, looking at times 75 and
175 seconds, total pressure is greater than pore pressure in the water column when they should be equal. These apparent erroneous results are likely due to leaving the sensors in the mud during the 12 hour test. The mud could have clogged the stone in the pore pressure sensor during their extended time in contact.

Figure 4.12  Depth of bed after overnight settling
4.4.1 Observations

This experiment is conducted to observe the consolidation properties of the fluid mud layer. The fluid mud layer does consolidate, and the formation of a cohesive bed is observed during the high concentration flow experiment. During the settling experiments, the fluid mud consolidates much faster even during the filling of the flume, with measurable bed formation in the ten minute settling period. Settling is very prevalent when the water in the flume is quiescent compared to flowing, and the higher concentration fluid mud settles more readily than the lower concentration fluid mud.

4.5 Summary

Experimental results indicate that fluid mud did flow under shear stress in these experiments. Pressure sensor results and the presence of effective stress indicate the presence of fluid mud during flow experiments as well as formation of a cohesive bed.
during the high concentration flow experiment and the settling experiment. These results and observations are examined further in Chapter VII.
CHAPTER V

NUMERICAL MODEL FOR FLUID MUD

In this chapter, the equations that will be used in the fluid mud model, the conceptual model, and the finite element development will be presented.

5.1 Model Description

The fluid mud model uses equations described in Chapter II. These equations track both momentum in the fluid mud layer and thickness of the layer. Each equation describes the physical processes affecting fluid mud, which include formation, movement, erosion/entrainment, and consolidation, which is expressed as a rate of change of each of these parameters.

The primary assumption in this fluid mud model is that the fluid mud layer always behaves as a fluid and not as a solid with plug flow, as other models have used (Teeter and Johnson, 2005). This assumption is supported by the experiments presented in previous chapters as well as experimental results from other studies. Kusuda et al observed that the “velocity profiles are unlike those of plug flow” and “mobile fluid mud behaves as a Newtonian fluid (Kusuda et al., 1993).” Mehta and Srinivas also noted that fluid mud moves at very low velocities and a Bingham Plastic model is not suitable (Mehta and Srinivas, 1993).
Based on these observations, the Navier-Stokes equations for motion along with the continuity equation provide a logical starting point for the development of a fluid mud model. Existing hydrodynamic models provide the driving forces (settling from water column and shear from water velocity) for the fluid mud model.

5.2 Conceptual Model

The following box model (Figure 5.1) provides a basic outline for the proposed fluid mud model and the linkage to existing hydrodynamic model.

![Proposed Conceptual Fluid Mud Model](image-url)
The necessary hydrodynamic information, including depth, velocity, sediment concentration, etc., act as input to the fluid mud model. If fluid mud is not present in the model domain, the code skips the fluid mud calculations. If fluid mud is present or the conditions within the hydrodynamic model indicate fluid mud formation, the fluid mud routine begins the calculations. The fluid mud model then feeds back into the hydrodynamic model entrainment values, changes to the bed, and changes to the fluid mud layer.

5.3 Model Equation Development

The continuity equation and the Navier-Stokes equations are presented below in Equations 5.1 through 5.4.

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0 \\
\rho_{fm} \left( \frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial uw}{\partial z} \right) &= \left( \frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right) - \frac{\partial p}{\partial x} + \rho_{fm} g_x \\
\rho_{fm} \left( \frac{\partial v}{\partial t} + \frac{\partial vu}{\partial x} + \frac{\partial vv}{\partial y} + \frac{\partial vw}{\partial z} \right) &= \left( \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right) - \frac{\partial p}{\partial y} + \rho_{fm} g_y \\
\rho_{fm} \left( \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial vw}{\partial y} + \frac{\partial ww}{\partial z} \right) &= \left( \frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \tau_{zz} \right) - \frac{\partial p}{\partial z} + \rho_{fm} g_z
\end{align*}
\]

Where

\( u, v, w \) = x-, y-, and z-direction velocities (m/s)
\( \tau_{xx}, \tau_{yx}, \tau_{zx}, \tau_{yy}, \tau_{zy}, \tau_{zz} \) = Viscous Stresses (Pa)
\( p \) = Pressure (Pa)
\( g_x, g_y, g_z \) = Gravitational acceleration in the x, y, and z directions
As described previously in Chapter I, fluid mud is a thick fluid which occurs typically in areas with low velocities since high velocities will often keep the fine sediments suspended in the water column and away from the bed. The full 3-dimensional equations are computationally intensive and physically complicated, and certain simplifications are used to reduce the complication. One simplification is assuming a constant density and concentration within the mud layer. Another simplification is neglecting vertical accelerations. Another one is depth-averaging.

Depth-averaging the Navier-Stokes equations and neglecting the vertical momentum equation simplifies the numerics of the equations. This simplification emphasizes the area of interest, the horizontal directions, while assuming certain behavior in the vertical direction.

The following assumptions are made during the development of the depth-averaged equations:

1. The flow is incompressible, where the density of an individual particle is assumed to be constant;
2. The density of the fluid is constant;
3. The pressure is hydrostatic, where pressure is only a function of depth;
4. Velocity distribution in the fluid is uniform;
5. The Coriolis force can be neglected;
6. The bed is not changing;
7. The bed is impenetrable;
8. Bed shear stresses can be modeled using empirical formulations;
9. Top surface stresses can be modeled using empirical formulations; and
10. The bed slope is mild.

To depth average the equations, the Leibniz Integral Rule is applied to each term in the equations. The Leibniz Rule is given as Equation 5.5:

\[
\int_a^b \frac{\partial f}{\partial t} \, dx = \frac{\partial}{\partial t} \int_a^b f(x, t) \, dx - f(b, t) \frac{\partial b}{\partial t} + f(a, t) \frac{\partial a}{\partial t}
\]  

(5.5)

The following notation, Equation 5.6, is used to represent the integration over the fluid depth, where \( h \) represents fluid thickness, \( z_0 \) is the bottom of the fluid mud layer and top of the bed, and \( z_0 + h \) is the surface of the fluid mud layer. This surface variable, \( z_0 + h \), is referred to as \( \eta \). Capital letters represent the depth-averaged values of the variables – \( U \) is the depth-averaged value of \( u \), the x-direction velocity.

\[
F_h = \int_{z_0}^{z_0 + h} f \, dz
\]  

(5.6)

5.3.1 Continuity Equation

By applying the Leibniz Integral Rule to each term in the continuity equation, the following equations are developed (Equations 5.7, 5.8, and 5.9):

\[
\int_{z_0}^{\eta} \frac{\partial u}{\partial x} \, dz = \frac{\partial}{\partial x} \int_{z_0}^{\eta} u \, dz - u(\eta) \frac{\partial \eta}{\partial x} + u(z_0) \frac{\partial z_0}{\partial x} = \frac{\partial U_h}{\partial x} - u(\eta) \frac{\partial \eta}{\partial x} + u(z_0) \frac{\partial z_0}{\partial x}
\]  

(5.7)

\[
\int_{z_0}^{\eta} \frac{\partial v}{\partial y} \, dz = \frac{\partial}{\partial y} \int_{z_0}^{\eta} v \, dz - v(\eta) \frac{\partial \eta}{\partial y} + v(z_0) \frac{\partial z_0}{\partial y} = \frac{\partial V_h}{\partial y} - v(\eta) \frac{\partial \eta}{\partial y} + v(z_0) \frac{\partial z_0}{\partial y}
\]  

(5.8)

\[
\int_{z_0}^{\eta} \frac{\partial w}{\partial z} \, dz = w(\eta) - w(z_0)
\]  

(5.9)
Rearranging the terms and grouping surface and bed terms together results in Equation 5.10.

\[
\frac{\partial U_h}{\partial x} + \frac{\partial V_h}{\partial y} - [u(\eta) \frac{\partial \eta}{\partial x} + v(\eta) \frac{\partial \eta}{\partial y} - w(\eta)] + [u(z_0) \frac{\partial z_0}{\partial x} + v(z_0) \frac{\partial z_0}{\partial y} - w(z_0)] = 0 \quad (5.10)
\]

The kinematic boundary conditions for the top and bottom of the fluid are specified as Equations 5.11 and 5.12, respectively.

\[
\frac{\partial \eta}{\partial t} + u(\eta) \frac{\partial \eta}{\partial x} + v(\eta) \frac{\partial \eta}{\partial y} - w(\eta) = 0 \quad (5.11)
\]

\[
\frac{\partial z_0}{\partial t} + u(z_0) \frac{\partial z_0}{\partial x} + v(z_0) \frac{\partial z_0}{\partial y} - w(z_0) = 0 \quad (5.12)
\]

Substituting 5.11 and 5.12 into 5.10 results in the depth averaged continuity equation, Equation 5.13, which also includes sources and sinks.

\[
\frac{\partial h}{\partial t} + \frac{\partial (U_h h)}{\partial x} + \frac{\partial (V_h h)}{\partial y} \pm \text{Sources & Sinks} = 0 \quad (5.13)
\]

5.3.2 Horizontal Momentum Equations

The same method of depth-averaging is applied to the momentum equations, starting with the x-direction momentum equation, given in Equation 5.2. Gravity acts only in the z-direction and is neglected, which results in Equation 5.14 for the x direction momentum.

\[
\rho_f \left( \frac{\partial u}{\partial t} + \frac{\partial uu}{\partial x} + \frac{\partial uw}{\partial y} + \frac{\partial uw}{\partial z} \right) = \left( \frac{\partial}{\partial x} \tau_{x} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right) - \frac{\partial p}{\partial x} \quad (5.14)
\]
The process of depth-averaging each term inside the brackets on the left hand side (LHS) of Equation 5.14 is given in Equations 5.15 through 5.17. After which, the terms are combined, and bed and surface terms are grouped together. The resulting equation is presented in Equation 5.18.

\[ \int_{z_0}^{\eta} \frac{\partial u}{\partial t} \, dz = \frac{\partial}{\partial x} \int_{z_0}^{\eta} u \, dz - u(\eta) \frac{\partial \eta}{\partial t} + u(z_0) \frac{\partial z_0}{\partial t} = \frac{\partial u_h}{\partial t} - u(\eta) \frac{\partial \eta}{\partial t} + u(z_0) \frac{\partial z_0}{\partial t} \] (5.15)

\[ \int_{z_0}^{\eta} \frac{\partial uu}{\partial x} \, dz = \frac{\partial}{\partial x} \int_{z_0}^{\eta} u \, uu \, dz - (u(\eta))^2 \frac{\partial \eta}{\partial x} + (u(z_0))^2 \frac{\partial z_0}{\partial x} = \frac{\partial u_h u}{\partial x} - (u(\eta))^2 \frac{\partial \eta}{\partial x} + (u(z_0))^2 \frac{\partial z_0}{\partial x} \] (5.16)

\[ \int_{z_0}^{\eta} \frac{\partial uv}{\partial y} \, dz = \frac{\partial}{\partial y} \int_{z_0}^{\eta} u \, uv \, dz - u(\eta) v(\eta) \frac{\partial \eta}{\partial y} + u(z_0) v(z_0) \frac{\partial z_0}{\partial y} = \frac{\partial u_h v}{\partial y} - u(\eta) v(\eta) \frac{\partial \eta}{\partial y} + u(z_0) v(z_0) \frac{\partial z_0}{\partial y} \] (5.17)

\[ \int_{z_0}^{\eta} \frac{\partial uw}{\partial z} \, dz = u(\eta) w(\eta) - u(z_0) w(z_0) \] (5.18)

\[ \frac{\partial u_h}{\partial t} + \frac{\partial u_h u}{\partial x} + \frac{\partial u_h v}{\partial y} - u(\eta) \left( \frac{\partial \eta}{\partial t} + u(\eta) \frac{\partial \eta}{\partial x} + v(\eta) \frac{\partial \eta}{\partial y} - w(\eta) \right) + u(z_0) \left( \frac{\partial z_0}{\partial t} + u(z_0) \frac{\partial z_0}{\partial x} + v(z_0) \frac{\partial z_0}{\partial y} - w(z_0) \right) \] (5.19)

By substituting in the kinematic boundary conditions (Equations 5.11 and 5.12), the LHS of the x-direction momentum equation reduces to Equation 5.20.
\[
0.5 \left( \frac{\partial U_t}{\partial t} + \frac{\partial U_t U_t}{\partial x} + \frac{\partial U_t U_x}{\partial y} \right) = \text{LHS}
\] (5.20)

For the right hand side of Equation 5.14, start with pressure term in Equation 5.21.

\[
\int_{z_0}^{\eta} \frac{\partial p}{\partial x} dz = \frac{\partial}{\partial x} \int_{z_0}^{\eta} p dz - p(\eta) \frac{\partial \eta}{\partial x} + p(z_0) \frac{\partial z_0}{\partial x}
\] (5.21)

Pressure at the fluid mud-water interface, \(\eta\), is a function of the overlying water column, where \(\lambda\) is the water surface, as shown in Equation 5.22.

\[
p(\eta) \frac{\partial \eta}{\partial x} = \rho_{\text{water}} g (\lambda - \eta) \frac{\partial \eta}{\partial x}
\] (5.22)

By using the relationships of \(\lambda - \eta = h_w\) and \(\eta - z_0 = h\), the pressure term can be restated in terms of the layer thickness. Solving the derivative in terms of \(h_w\) and \(h\) leads to Equation 5.23.

\[
p(\eta) \frac{\partial \eta}{\partial x} = \rho_{\text{water}} g h_w \frac{\partial h}{\partial x} + \rho_{\text{water}} g h_w \frac{\partial z_0}{\partial x}
\] (5.23)

Pressure at the bed is a function of the water column and the mud layer, as shown in Equation 5.24.

\[
p(z_0) \frac{\partial z_0}{\partial x} = \rho_{\text{water}} g h_w \frac{\partial z_0}{\partial x} + \rho_{fm} g h \frac{\partial z_0}{\partial x}
\] (5.24)
Equation 5.25 shows the development of the first term from Equation 5.21. Then, regrouping terms of Equation 5.25 results in Equation 5.26.

\[
\frac{\partial}{\partial x} \int_{z_0}^{\eta} p \, dz = \frac{\partial}{\partial x} \int_{z_0}^{\eta} \rho_{fm} g (\eta - z) \, dz = \frac{\partial}{\partial x} \left[ \rho_{fm} g \eta z \bigg|_{z_0}^{\eta} - \rho_{fm} g \frac{z^2}{2} \bigg|_{z_0}^{\eta} \right] \tag{5.25}
\]

\[
\frac{\partial}{\partial x} \int_{z_0}^{\eta} p \, dz = \frac{\partial}{\partial x} \left[ \rho_{fm} g \left( \eta^2 - \eta z_0 - \frac{\eta^2}{2} - \frac{z_0^2}{2} \right) \right] \tag{5.26}
\]

By substituting the new relationship for water surface, \( \eta \), into Equation 5.26, Equation 5.27 is developed.

\[
\frac{\partial}{\partial x} \int_{z_0}^{\eta} p \, dz = \frac{\partial}{\partial x} \left[ \rho_{fm} g \left( \eta^2 - \eta z_0 - \frac{\eta^2}{2} - \frac{z_0^2}{2} \right) \right] = \rho_{fm} g \frac{\partial}{\partial x} \left( \frac{\eta^2}{2} \right) \tag{5.27}
\]

Combining like terms in Equation 5.27 results in the Equation 5.28.

\[
\frac{\partial}{\partial x} \int_{z_0}^{\eta} p \, dz = \frac{\partial}{\partial x} \left[ \rho_{fm} g \left( \frac{\eta^2}{2} \right) \right] = \rho_{fm} g \frac{\partial}{\partial x} \left( \frac{\eta^2}{2} \right) \tag{5.28}
\]

By utilizing the product rule on Equation 5.28, Equation 5.29 is developed.

\[
\frac{\partial}{\partial x} \int_{z_0}^{\eta} p \, dz = \rho_{fm} g \frac{\partial}{\partial x} \left( \frac{h^2}{2} \right) = \rho_{fm} g h \frac{\partial h}{\partial x} \tag{5.29}
\]

Substituting Equations 5.29, 5.23, and 5.22 into the pressure term, Equation 5.20, solving in terms of \( h \), and reducing terms results in the depth-averaged pressure term for
the bottom layer, Equation 5.30. This equation includes the pressure effects from the upper layer of fluid on the lower layer.

\[
\int_{z_0}^{η} \frac{∂p}{∂x} \, dz = \frac{1}{2} \rho_f m g \frac{∂h^z}{∂x} + \rho_f m g h \frac{∂z_0}{∂x} + \rho_{\text{water}} gh \frac{∂h_w}{∂x}
\] (5.30)

Finally, the shear stress terms from the right hand side of the momentum equation are depth-averaged, as seen in Equations 5.31 - 5.33

\[
\int_{z_0}^{η} \frac{∂τ_{xx}}{∂x} \, dz = \frac{∂}{∂x} \int_{z_0}^{η} τ_{xx} \, dz - τ_{xx}(η) \frac{∂η}{∂x} + τ_{xx}(z_0) \frac{∂z_0}{∂x} = \frac{∂τ_{xx} h}{∂x} - τ_{xx}(η) \frac{∂η}{∂x} + τ_{xx}(z_0) \frac{∂z_0}{∂x}
\] (5.31)

\[
\int_{z_0}^{η} \frac{∂τ_{yx}}{∂y} \, dz = \frac{∂}{∂y} \int_{z_0}^{η} τ_{yx} \, dz - τ_{yx}(η) \frac{∂η}{∂y} + τ_{yx}(z_0) \frac{∂z_0}{∂y} = \frac{∂τ_{yx} h}{∂y} - τ_{yx}(η) \frac{∂η}{∂y} + τ_{yx}(z_0) \frac{∂z_0}{∂y}
\] (5.32)

\[
\int_{z_0}^{η} \frac{∂τ_{zx}}{∂z} \, dz = τ_{zx}(η) - τ_{zx}(z_0)
\] (5.33)

Combining terms from Equations 5.31, 5.32, and 5.33, and group surface and bed terms together results in Equation 5.34. Substituting in lumped term for both surface and bed shear stress for the grouped terms results in Equation 5.35. The surface and bed shear stress terms are modeled using empirical equations as mentioned previously. Here, surface refers to the mud-water interface, \(h\) refers to the depth of the bottom layer and \(h_w\) refers to the depth of the surface layer.

\[
\frac{∂τ_{xx}}{∂x} + \frac{∂τ_{yx}}{∂y} = \left( τ_{xx}(η) \frac{∂η}{∂x} + τ_{yx}(η) \frac{∂η}{∂y} - τ_{zx}(η) \right) + \left( τ_{xx}(z_0) \frac{∂z_0}{∂x} + τ_{yx}(z_0) \frac{∂z_0}{∂y} - τ_{zx}(z_0) \right)
\] (5.34)
\[ \frac{\partial \tau_{xx} h}{\partial x} + \frac{\partial \tau_{yx} h}{\partial y} - \left( \tau_{\text{interface}} \right) \pm \left( \tau_{\text{bed}} \right) \]  

(5.35)

Combine terms from Equations 5.19, 5.30, and 5.35 together to get the x-direction depth averaged momentum equation, Equation 5.36.

\[ \rho_{fm} \left( \frac{\partial U h}{\partial t} + \frac{\partial U U h}{\partial x} + \frac{\partial U V h}{\partial y} \right) = -\frac{1}{2} \rho_{fm} g \frac{\partial h^2}{\partial x} - \rho_{fm} g h \frac{\partial z_0}{\partial x} - \rho_{\text{water}} g h \frac{\partial h_w}{\partial x} + \frac{\partial \tau_{xx} h}{\partial x} + \frac{\partial \tau_{xy} h}{\partial y} - \left( \tau_{\text{interface}} \right) \pm \left( \tau_{\text{bed}} \right) \]  

(5.36)

Dividing through by \( \rho_{fm} \) results in Equation 5.37.

\[ \frac{\partial U h}{\partial t} + \frac{\partial U U h}{\partial x} + \frac{\partial U V h}{\partial y} = -\frac{1}{2} g \frac{\partial h^2}{\partial x} - g h \frac{\partial z_0}{\partial x} - \rho_{\text{water}} \frac{g h}{\rho_{fm}} \frac{\partial h_w}{\partial x} + \frac{1}{\rho_{fm}} \frac{\partial \tau_{xx} h}{\partial x} + \frac{1}{\rho_{fm}} \frac{\partial \tau_{xy} h}{\partial y} - \frac{1}{\rho_{fm}} \left( \tau_{\text{interface}} \right) + \frac{1}{\rho_{fm}} \left( \tau_{\text{bed}} \right) \]  

(5.37)

By following the same procedure for the y-direction momentum, the set of depth-averaged equations can be seen in Equations 5.37, 5.38, and 5.39

\[ \frac{\partial V h}{\partial t} + \frac{\partial V U h}{\partial x} + \frac{\partial V V h}{\partial y} = -\frac{1}{2} g \frac{\partial h^2}{\partial y} - g h \frac{\partial z_0}{\partial y} - \rho_{\text{water}} \frac{g h}{\rho_{fm}} \frac{\partial h_w}{\partial y} + \frac{1}{\rho_{fm}} \frac{\partial \tau_{yx} h}{\partial x} + \frac{1}{\rho_{fm}} \frac{\partial \tau_{yy} h}{\partial y} - \frac{1}{\rho_{fm}} \left( \tau_{\text{interface}} \right) + \frac{1}{\rho_{fm}} \left( \tau_{\text{bed}} \right) \]  

(5.38)

\[ \frac{\partial h}{\partial t} + \frac{\partial (U h)}{\partial x} + \frac{\partial (V h)}{\partial y} = \text{Sources & Sinks} = 0 \]  

(5.39)
This is different than but equivalent to the more commonly seen version of the equations with a $g'$ term, as the following derivations show.

Beginning with the existing pressure terms from Equation 5.37,

$$-\frac{1}{2} \rho_{fm} g \frac{\partial h^2}{\partial x} - \rho_{fm} gh \frac{\partial z_0}{\partial x} - \rho_{water} gh \frac{\partial h_w}{\partial x}$$

After applying the product rule to Term 1, as in Equation 5.29,

$$-\rho_{fm} gh \frac{\partial h}{\partial x} - \rho_{fm} gh \frac{\partial z_0}{\partial x} - \rho_{water} gh \frac{\partial h_w}{\partial x}$$

Using the relationship for water surface as used in Equation 5.21 and substituting into the above equation

$$\rho_{water} gh \frac{\partial h_w}{\partial x} = \rho_{water} gh \left( \frac{\partial \lambda}{\partial x} - \frac{\partial h}{\partial x} - \frac{\partial z_0}{\partial x} \right) =$$

$$\left( \rho_{water} gh \frac{\partial \lambda}{\partial x} - \rho_{water} gh \frac{\partial h}{\partial x} - \rho_{water} gh \frac{\partial z_0}{\partial x} \right)$$

Substituting this expansion back into the pressure equation

$$-\rho_{fm} gh \frac{\partial h}{\partial x} - \rho_{fm} gh \frac{\partial z_0}{\partial x} - \rho_{water} gh \frac{\partial h}{\partial x} + \rho_{water} gh \frac{\partial \lambda}{\partial x} + \rho_{water} gh \frac{\partial z_0}{\partial x}$$

Regrouping terms
\[-\rho_{fm} g h \frac{\partial h}{\partial x} + \rho_{water} g h \frac{\partial h}{\partial x} - \rho_{fm} g h \frac{\partial z_0}{\partial x} + \rho_{water} g h \frac{\partial z_0}{\partial x} - \rho_{water} g h \frac{\partial \lambda}{\partial x} =
\]

\[-(\rho_{fm} g h \frac{\partial h}{\partial x} - \rho_{water} g h \frac{\partial h}{\partial x}) - (\rho_{fm} g h \frac{\partial z_0}{\partial x} - \rho_{water} g h \frac{\partial z_0}{\partial x}) - \rho_{water} g h \frac{\partial \lambda}{\partial x} =
\]

\[-(\rho_{fm} - \rho_{water}) g h \frac{\partial h}{\partial x} - (\rho_{fm} - \rho_{water}) g h \frac{\partial z_0}{\partial x} - \rho_{water} g h \frac{\partial \lambda}{\partial x} \]

Dividing through by \(\rho_{fm}\)

\[-\frac{(\rho_{fm} - \rho_{water})}{\rho_{fm}} g h \frac{\partial h}{\partial x} - \frac{(\rho_{fm} - \rho_{water})}{\rho_{fm}} g h \frac{\partial z_0}{\partial x} - \frac{\rho_{water}}{\rho_{fm}} g h \frac{\partial \lambda}{\partial x} \]

Replacing \(\frac{(\rho_{fm} - \rho_{water})}{\rho_{fm}} g\) with \(g'\) results in the more commonly seen version of the two layer pressure terms, Equation 5.40

\[-g' h \frac{\partial h}{\partial x} - g' h \frac{\partial z_0}{\partial x} - \frac{\rho_{water}}{\rho_{fm}} g h \frac{\partial \lambda}{\partial x} = -\frac{g' \partial h^2}{2} - g' h \frac{\partial z_0}{\partial x} - \frac{\rho_{water}}{\rho_{fm}} g h \frac{\partial \lambda}{\partial x} \quad (5.40)\]

The version of the pressure terms in Equations 5.37 and 5.38 has the advantage of using the typical model output variable of depth instead of having to recalculate water surface elevation as in Equation 5.40.

Initial conditions provide \(h, U,\) and \(V\), the depth, x-direction depth-averaged velocity, and y-direction depth-averaged velocity, respectively, for the above equations. However, the internal, bed, and interfacial shear stress terms still need to be defined. By using definitions from Panton, the internal shear stress terms are seen in Equations 5.41, 5.42, and 5.43 (Panton, 2005).
\[
\tau_{xx} = \mu_{fn} \left[ 2 \frac{\partial u}{\partial x} - \frac{2}{3} (\nabla \cdot V) \right] = \mu_{fn} \left[ 2 \frac{\partial u}{\partial x} \right]
\]

(5.41)

\[
\tau_{xy} = \mu_{fn} \left[ 2 \frac{\partial v}{\partial y} - \frac{2}{3} (\nabla \cdot V) \right] = \mu_{fn} \left[ 2 \frac{\partial v}{\partial y} \right]
\]

(5.42)

\[
\tau_{yx} = \mu_{fn} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)
\]

(5.43)

The interfacial shear stress is the linkage for the drag imposed on the mud by the overlying water column. The formulation of the interfacial shear term was taken from Dermissis and Partheniades (Dermissis and Partheniades, 1982) and is presented in Equations 5.44 and 5.45. The formulation of the interfacial shear stress as seen in Equation 5.44 was used in (Odd and Cooper, 1989; Winterwerp, 2002) and presented in (Mehta, 2013). The interfacial friction coefficient, \(f_i\), depends both on the density difference and velocity difference between the fluids.

\[
\tau_{ix} = \rho_w \frac{f_i \Delta u}{8} \| u \|
\]

(5.44)

\[
\tau_{iy} = \rho_w \frac{f_i \Delta v}{8} \| u \|
\]

(5.45)

\[
\| u \| = \sqrt{\Delta u^2 + \Delta v^2}
\]

(5.46)

Bed shear stress can be expressed through the Manning formulation or a formulation by Soulsby and Clarke for smooth mud beds (Soulsby and Clarke, 2005). The two formulations are user selectable within the fluid mud code. These are seen in Equations 5.47 and 5.48, respectively.
\[ \tau_{_{\text{bed}x}} = \frac{g \rho_{fm} n^2 U}{c_0 h^3} \sqrt{U^2 + V^2} \]

\[ \tau_{_{\text{bed}y}} = \frac{g \rho_{fm} n^2 V}{c_0 h^3} \sqrt{U^2 + V^2} \]  
(5.47)

\[ \text{Re}_c = \frac{\sqrt{U^2 + V^2} h}{\nu} \]

\[ C_{ds} = (0.0001615) \exp \left[ 6 \left( \text{Re}_c \right)^{-0.08} \right] \]

\[ \tau_{_{\text{bed}x}} = \rho_{fm} C_{ds} \sqrt{U^2 + V^2} \ U \]

\[ \tau_{_{\text{bed}y}} = \rho_{fm} C_{ds} \sqrt{U^2 + V^2} \ V \]  
(5.48)

Where

\[ n \quad \text{Manning coefficient} \]

\[ C_0 \quad \text{Unit constant, 1.0 for SI and 1.486 for English units} \]

\[ \nu \quad \text{Kinematic viscosity of mud (\(\mu/\rho\)).} \]

### 5.4 Sources and Sinks

The continuity equation described above (Equation 5.40) includes sources and sinks to account for addition and subtraction of mud to the fluid mud layer. The primary physical processes that add to the mud layer are settling from the water column and liquefaction of the bed by waves. Liquefaction is outside the scope of this model and is not included. The primary physical processes that remove material from the fluid mud layer are consolidation of the mud layer and entrainment into the overlying water column.

Settling from the overlying water column is how this model accounts for addition to the mud layer. The concentration of the water column is provided by the existing
Settling velocity is calculated and provided to the model by the user by using the Stokes settling velocity (Equation 2.1), or the settling velocity is calculated in the model using the settling continuum method provided by Equation 2.5, as seen in Chapter II. The settling velocity multiplied by the concentration provides the mass flux per area. Dividing the mass flux by the concentration of the fluid mud layer provides the depth increase of the fluid mud.

To calculate the addition to the mud layer, Equation 5.49 is used.

\[
\text{Source} = w_z \left( \frac{C_{\text{water}}}{C_{fm}} \right)
\]

(5.49)

Where

\[ C_{\text{water}} = \text{Concentration of the water column} \]
\[ C_{fm} = \text{Concentration of the mud layer} \]

Consolidation is the first way this model removes height from the mud layer. The model removes thickness from the mud layer and adds it to the bed below. This is calculated using Equation 2.59, with \( C_{fm} \) as the fluid mud layer concentration, which is related to the density of the layer through Equation 5.50 (Mehta and McAnally, 2009).

\[
C_{fm} = \frac{(\rho_{fm}-\rho_w)}{(\rho_s-\rho_w)} \rho_s
\]

(5.50)

Where

\[ \rho_s = \text{Density of sediment (2650 kg/m}^3) \]
Entrainment is the second method for removing mud from the fluid mud layer in this model. Equation 2.41 with $c_e=0.0052$ as presented by Mehta and Srinivas is used in this model to calculate entrainment into the overlying water column (Mehta and Srinivas, 1993). The velocity, $U$, in the entrainment equation and in the Richardson number calculation (Equation 2.43) is evaluated as the difference between the velocity of the mud layer and the water column, as demonstrated by (Mehta and McAnally, 2009).

5.5 Numerical Solutions

The preceding sections demonstrate the development of the differential equations describing fluid mud movement and the associated empirical relationships for surface shear, bed shear, and sources and sinks. To solve these equations, a numerical method is applied to evaluate these equations over a domain. Two primary methods are used to solve the 2-dimensional depth averaged equations – finite differences and finite elements. Finite difference methods replace the partial differential equations with finite differences, replacing $dx$ with $\Delta x$, for instance.

The finite element method (FEM) is “a general technique for constructing approximate solutions to boundary-value problems (Becker et al., 1981).” FEM is a numerical analysis tool used to solve differential equations over a domain divided into elements linked by nodes. Triangular elements are common choices, which can more easily approximate the complex geometry of natural systems. The differential equations are then integrated and solved in matrix form over the entire domain. FEM is used in one, two, and three dimensions in many types of engineering analysis and has recently been applied by the USACE into the multidimensional Adaptive Hydraulics Modeling System (ADH) (Berger et al., 2013).
The finite element method is a powerful tool and is used for this model. This allows for easier incorporation to the ADH modeling system to drive the fluid mud model with both hydrodynamic and sediment transport interactions but can be added to any model with appropriately matching the meshes.

5.6 Finite Element Method Application to the Fluid Mud Equations

The following section demonstrates the development of the finite element method for the continuity equation.

The FEM method uses a test or weight function times the differential equation, which is then integrated over the domain. The Galerkin weighed residuals method uses the same functions as the test functions to interpolate between the nodal values(Becker et al., 1981). This linear combination of variables is represented as Equation 5.51.

\[
F(x) \approx \sum \Phi_n F_n
\]  (5.51)

In Equation 5.51, \(F(x)\) is some function, \(\Phi_n\) is the basis function, and \(F_n\) are the nodal values of \(F(x)\).

An explicit solution scheme is used for these equations. For this, the time derivatives are approximated with a forward difference, and the variables at the new time are the only unknowns. By using an explicit solution technique, the computational burden is much lower than with an implicit solution, if the time step is not limited by very small grid spacing. This reduces the runtime for the fluid mud code and does not add significant load if added to other existing models. Using an explicit technique limits time
steps based on the Courant-Friedrichs-Lewy (CFL) stability criteria equal to a value of
less than a maximum of one, given in Equation 5.52.

\[
CFL = 1 \geq \frac{u \Delta t}{\Delta x}
\]

\[
\Delta t \leq \frac{\Delta x}{u}
\]  
(5.52)

In Equation 5.52, \(\Delta x\) is the minimum length of an element side, \(u\) is the maximum
speed of a gravity wave, and \(\Delta t\) is the time step.

Applied to the continuity equation, Equation 5.40, the forward difference is
shown in Equation 5.53, where \(H\) is the depth at the new time, \(t+\Delta t\), and \(h\) is the depth at
the current time step, \(t\). For the rest of the FEM derivation, capital letters (\(H, U, V\))
represent the variables at the new time step and lowercase letters (\(h, u, v\)) are the
variables at the old time step.

\[
\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} \pm \text{Sources & Sinks} = 0
\]

\[
\frac{H-h}{\Delta t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} \pm \text{Sources & Sinks} = 0
\]

\[
H = h - \Delta t \left( \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} \pm \text{Sources & Sinks} \right)
\]  
(5.53)

In the FEM, the variables are represented as linear combinations of shape
functions multiplied by the nodal values. The functions are linear and are presented in
Equation 5.54. Figure 5.2 shows a two-dimensional example of the linear shape
functions.
\[
\Phi_i = \begin{pmatrix}
1 - \xi - \eta \\
\xi \\
\eta
\end{pmatrix}
\]

(5.54)

Figure 5.2  Linear finite element shape functions in two-dimensions (From (Becker et al., 1981))
The linear approximations for the unknowns in Equation 5.53 are shown below in Equation 5.55. The sources and sinks terms has been dropped for brevity and clarity during this example.

\[ H = \sum \Phi_j H_j \]
\[ h = \sum \Phi_j h_j \]
\[ uh = \sum \Phi_j u_j \sum \Phi_j h_j \]
\[ vh = \sum \Phi_j v_j \sum \Phi_j h_j \]  

(5.55)

Multiplying Equation 5.53 by the test function and integrating over the domain gives Equation 5.56.

\[ \int_\Omega \Phi_i \sum \Phi_j H_j dA = \int_\Omega \Phi_i \sum \Phi_j h_j dA - \Delta t \left( \int_\Omega \Phi_i \frac{\partial}{\partial x} [\sum \Phi_j u_j \sum \Phi_j h_j] dA + \right. \]
\[ \left. \int_\Omega \Phi_i \frac{\partial}{\partial x} [\sum \Phi_j v_j \sum \Phi_j h_j] dA \right) \]  

(5.56)

In Equation 5.56, \( \Omega \) is the domain or element, \( A \) is the area, \( \Phi_i \) is the test function, and \( \Phi_j \) is the basis function.

Integrating by parts is used to solve the derivatives. This enforces conservation across interfaces throughout the domain. Equation 5.57 is the end result of integrating by parts on the \( uh \) term.

\[ \int_\Omega \Phi_i \frac{\partial}{\partial x} [\sum \Phi_j u_j \sum \Phi_j h_j] dA = - \int_\Omega \frac{\partial \Phi_i}{\partial x} [\sum \Phi_j u_j \sum \Phi_j h_j] dA + \Phi_i \sum \Phi_j u_j \sum \Phi_j h_j \cdot \vec{n_x} dl \]  

(5.57)
In Equation 5.57, \(d\Omega\) is the surface of the domain, \(l\) is the length, and \(n\) is the outward normal in the x-direction.

The surface integral on the right in Equation 5.57 is the flux through the edge of the domain, which for this case is zero and can be neglected. This also prevents mass from leaving through the edge of the model domain. The outward normals of a side between two adjacent elements point opposite to each other, so the edge integrals cancel out and are only counted along the outside of the model domain.

After integrating by parts and neglecting the surface integrals, restating the continuity equation with the linear approximations of the known values and the test function results in the weak statement of the continuity equation, Equation 5.58.

Find \(H(x,y)\) such that:

\[
\int_{\Omega} \Phi_i \sum \Phi_j H_j dA = \int_{\Omega} \Phi_i \sum \Phi_j h_j dA + \Delta t \left( \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \sum \Phi_j h_j \sum \Phi_j u_j dA + \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \sum \Phi_j h_j \sum \Phi_j v_j dA \right) \tag{5.58}
\]

The above process is repeated on the x- and y-direction momentum equations. The surface integrals are dropped for all terms except the gravity term in the momentum equations. When the surface integral for the \(\frac{1}{2}gh^2\) term is included, the edge of the domain is calculated as an impermeable reflecting boundary.

After repeating the integration by parts on the x- and y-direction momentum equations, the individual weak statements are given as follows in Equations 5.59 and 5.60, respectively.
Find \( UH(x,y) \) such that:

\[
\int_{\Omega} \Phi_i \Sigma \Phi_j U_j \Sigma \Phi_j H_j dA = \int_{\Omega} \Phi_i \Sigma \Phi_j u_j \Sigma \Phi_j h_j dA +
\]

\[
\Delta t \left( \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \Sigma \Phi_j u_j \Sigma \Phi_j H_j dA + \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j v_j \Sigma \Phi_j H_j dA +
\]

\[
\frac{1}{2} g \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \Sigma \Phi_j h_j \Sigma \Phi_j H_j dA - \frac{1}{2} g \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j h_j \Sigma \Phi_j H_j \cdot \vec{n}_x \ dl - g \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \Sigma \Phi_j \Sigma \Phi_j z_j dA -
\]

\[
g \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \Sigma \Phi_j h_j \Sigma \Phi_j h_{wj} dA - \frac{1}{\rho_{\text{fm}}} \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \Sigma \Phi_j \tau_{xxj} \Sigma \Phi_j H_j dA -
\]

\[
\frac{1}{\rho_{\text{fm}}} \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j \tau_{xyj} \Sigma \Phi_j H_j dA - \frac{1}{\rho_{\text{fm}}} \int_{\Omega} \Phi_i \Sigma \Phi_j \tau_{bed} dA + \frac{1}{\rho_{\text{fm}}} \int_{\Omega} \Phi_i \Sigma \Phi_j \tau_{lx} dA \right) (5.59)
\]

Find \( VH(x,y) \) such that:

\[
\int_{\Omega} \Phi_i \Sigma \Phi_j V_j \Sigma \Phi_j H_j dA = \int_{\Omega} \Phi_i \Sigma \Phi_j v_j \Sigma \Phi_j H_j dA +
\]

\[
\Delta t \left( \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \Sigma \Phi_j u_j \Sigma \Phi_j v_j H_j dA + \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j v_j \Sigma \Phi_j v_j H_j dA +
\]

\[
\frac{1}{2} g \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j h_j \Sigma \Phi_j H_j dA - \frac{1}{2} g \int_{\Omega} \frac{\partial \Phi_i}{\partial x} \Sigma \Phi_j h_j \Sigma \Phi_j H_j \cdot \vec{n}_y \ dl - g \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j \Sigma \Phi_j z_j dA -
\]

\[
g \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j h_j \Sigma \Phi_j h_{wj} dA - \frac{1}{\rho_{\text{fm}}} \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j \tau_{xyj} \Sigma \Phi_j H_j dA -
\]

\[
\frac{1}{\rho_{\text{fm}}} \int_{\Omega} \frac{\partial \Phi_i}{\partial y} \Sigma \Phi_j \tau_{xyj} \Sigma \Phi_j H_j dA - \frac{1}{\rho_{\text{fm}}} \int_{\Omega} \Phi_i \Sigma \Phi_j \tau_{bed} dA + \frac{1}{\rho_{\text{fm}}} \int_{\Omega} \Phi_i \Sigma \Phi_j \tau_{ly} dA \right) (5.60)
\]

After establishing the weak formulation of the equations, the model domain is next subdivided into triangular elements. To solve the equations over the individual elements, the equations must be transformed into local coordinates to match the test and basis functions, as seen in Equation 5.54. The linear transformation from global to local
coordinates must be completed on each element. Figure 5.3 shows the global to local mapping over an element. As the equations are written in terms of $dx$ and $dy$, the derivatives are transformed, using the following transformations, Equations 5.61 and 5.62.

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{d\xi}{dx} + \frac{\partial}{\partial \eta} \frac{d\eta}{dx} \tag{5.61}
\]

\[
\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{d\xi}{dx} + \frac{\partial}{\partial \eta} \frac{d\eta}{dx} \tag{5.62}
\]

In order to calculate $d\xi/dx$, $d\eta/dx$, $d\xi/dy$, and $d\eta/dy$ in the known terms of $dx/d\xi$, $dx/d\eta$, $dy/d\xi$, and $dy/d\eta$, the following Jacobian matrix (Equation 5.63) is used and inverted using the determinant of the Jacobian.
\[
\begin{bmatrix}
\frac{dx}{dy}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \xi}{\partial \eta} \\
\frac{\partial \eta}{\partial \xi}
\end{bmatrix}
\]

(5.63)

By using the determinant of the Jacobian, the inverted matrix is seen in Equation

5.64

\[
\begin{bmatrix}
\frac{\partial x}{\partial \xi} & -\frac{\partial x}{\partial \eta} \\
-\frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial \xi}{\partial \eta} \\
\frac{\partial \eta}{\partial \xi}
\end{bmatrix}
= \begin{bmatrix}
\frac{dx}{dy}
\end{bmatrix}
\]

(5.64)

The determinant of the Jacobian is given by 5.65.

\[
|J| = \left( \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \right)
\]

(5.65)

The unknown derivative terms are now given in terms of known values and seen in Equations 5.66 through 5.69.

\[
\frac{\partial \xi}{\partial x} = \frac{1}{|J|} \frac{\partial y}{\partial \eta}
\]

(5.66)

\[
\frac{\partial \xi}{\partial y} = -\frac{1}{|J|} \frac{\partial x}{\partial \eta}
\]

(5.67)

\[
\frac{\partial \eta}{\partial x} = -\frac{1}{|J|} \frac{\partial y}{\partial \xi}
\]

(5.68)
\[
\frac{\partial \eta}{\partial y} = \frac{1}{J} \frac{\partial \xi}{\partial \eta}
\]

(5.69)

By applying the above equations to the integration over the area of the element, the following transformation, Equation 5.70, is used to change the variables.

\[
\int_{\Omega} F(x, y)dA = \int_{y_1, y_2} \int_{x_1, x_2} F(x, y)dxdy = \int_{0}^{1} \int_{0}^{1-n} F(\xi, \eta)|J|d\xi d\eta
\]

(5.70)

This reduces the integration over an element to an easily solved set of definite integrals. This transformation is applied to each integral in the previous equations (Equations 5.58, 5.59, and 5.60). The derivatives of the shape functions must also be calculated using the determinant of the Jacobian to calculate the x- and y-direction derivatives. Equations 5.71 and 5.72 show the derivates of the shape functions.

\[
\begin{align*}
\Phi_x &= \begin{pmatrix}
\frac{\partial y}{\partial \xi} - \frac{\partial y}{\partial \eta} \\
\frac{\partial y}{\partial \eta}
\end{pmatrix}
\end{align*}
\]

(5.71)

\[
\begin{align*}
\Phi_y &= \begin{pmatrix}
-\frac{\partial y}{\partial \xi} \\
\frac{\partial y}{\partial \eta}
\end{pmatrix}
\end{align*}
\]

(5.72)
Next to be defined are the terms within the equations. The \(uh\), \(uuh\), \(uvh\), \(h^2\) terms are the product of the linear combinations of each term over the element. Equation 5.73 is the example for the product of two terms \((uh, vh, h^2)\), and Equation 5.74 is the product of three terms \((uuh, uvh)\). By using triangular elements, each combination is also made of three terms. The subscripts here refer to the nodal values.

\[
\sum \Phi_j u_j \sum \Phi_j h_j = (\Phi_1 u_1 + \Phi_2 u_2 + \Phi_3 u_3) \ast (\Phi_1 h_1 + \Phi_2 h_2 + \Phi_3 h_3)
\]

\[
\sum \Phi_j u_j \sum \Phi_j u_j \sum \Phi_j h_j = (\Phi_1 u_1 + \Phi_2 u_2 + \Phi_3 u_3) \ast (\Phi_1 u_1 + \Phi_2 u_2 + \Phi_3 u_3) \ast (\Phi_1 h_1 + \Phi_2 h_2 + \Phi_3 h_3)
\]

The surface integral on the gravitational term in the momentum equations is solved as a 1-dimensional finite element over the edge term. The outward normal is calculated internally during the computations for each element. This is accomplished by using a standard nodal numbering of each element counting counterclockwise from the bottom of the element.
The bed slope term, \( \frac{dz}{dx} \), is calculated as a constant over an element. The term is calculated as in Equation 5.75.

\[
\frac{dz}{dx} = \frac{d\Phi_1}{dx} z_1 + \frac{d\Phi_2}{dx} z_2 + \frac{d\Phi_3}{dx} z_3
\]

This same method is applied to the \( \frac{dz}{dy} \) term and a similar method is applied to the viscous terms, \( \tau_{xx}, \tau_{xy}, \) and \( \tau_{yy} \). The viscous stresses are given in Equations 5.41 through 5.43, and the velocity derivatives are calculated as in Equation 5.75.

The bed and surface stress terms, Equations 5.46 and 5.47 respectively, are calculated using the average of the nodal values over an element. For Equation 5.47, the average depth is calculated as Equation 5.76, where the subscript e is elemental average and the subscripts 1, 2, and 3 are the nodal values over the element.

\[
h_e = \frac{1}{3} \left( h_1 + h_2 + h_3 \right)
\]

This same process is repeated for the x- and y-direction velocities as well as the overlying water layer velocity in the surface stress terms.

5.7 Coding Overview

Now that all the integration terms are defined, the next step is the calculation of each element over the entire domain. The following outlines the coding and solution methods of the fluid mud model.
The computer program chosen to write the initial fluid mud code is MATLAB, a “high-level language and interactive environment for numerical computation, visualization, and programming” (Matlab, 2013a). This product is chosen for its ability to store multiple variables during iterations, easy visualization of variables during debugging, and its built-in matrix solvers. These built in solvers evaluate the matrices to determine the most efficient solver to apply to the system (Matlab, 2013b). The programming language in MATLAB is similar to the C programming language, which makes conversion to other computer codes somewhat easier. The full computer code is given in Appendix B.

The outline for the fluid mud code is presented below.

1. Read in all required variables and initial conditions;
2. Calculate CFL condition and time step value;
3. Loop over element calculations for continuity and momentum equations;
4. Solve for depth and velocity at new step; and
5. Loop back to Step 2.

### 5.8 Stabilization Routine

The Streamline Upwind/Petrov Galerkin formulation is included within the fluid mud routine for stabilization. The Petrov-Galerkin formulation differs from the typical Galerkin formulation in that the Petrov-Galerkin test function is different from the basis function (Berger and Stockstill, 1995). This formulation helps to detect and damp “wiggles” in the solution (Brooks and Hughes, 1982). The Streamline Upwind/Petrov-Galerkin formulation (SUPG) is presented and tested by Brooks and Hughes for a variety of different test cases. The SUPG formulation is successfully applied to numerical
models, including ADH (Berger et al., 2013) and HIVE2D (Berger and Stockstill, 1995).

The SUPG is essentially the Galerkin test function plus some value multiplied by the derivative of the test function. This essentially changes Equation 5.53 after the application of the test function into 5.77, as seen below for the continuity equation.

\[ \int \Phi_i \left( \frac{\partial h}{\partial t} + \frac{\partial u h}{\partial x} + \frac{\partial v h}{\partial y} \right) dA \rightarrow \int \left( \Phi_i + \alpha \Phi_i' \right) \left( \frac{\partial h}{\partial t} + \frac{\partial u h}{\partial x} + \frac{\partial v h}{\partial y} \right) dA \]  

(5.77)

The SUPG portion is applied to the momentum and continuity equations except for the temporal and diffusion terms. The diffusion terms are not included, as the second derivative of the velocity terms are not known. The temporal terms are small compared to the remainder of the equations and are not included. It is only applied over the interior of an element. It is not applied to breaks in the domain or over discontinuities in the solution (Berger, 2008). The full development of these terms is beyond the scope of this work, but the method is demonstrated in Berger and Stockstill (Berger and Stockstill, 1995). The SUPG terms are calculated from the eigenvector analysis of the nonconservative shallow water equations. The following equations (Equations 5.78 to 5.85) show the development of the SUPG terms.

\[
\hat{A} = \begin{bmatrix} \frac{\bar{u}}{\bar{a}} & \frac{1}{\bar{a}} & 0 \\ \frac{\bar{v}}{\bar{c}^2} & \frac{\bar{u}}{\bar{a}} & 0 \\ \frac{\bar{u}}{\bar{a}} & \frac{\bar{v}}{\bar{a}} & 0 \end{bmatrix}
\]  

(5.78)
\[
\hat{B} = \begin{bmatrix}
\frac{\bar{v}}{\bar{a}} & 0 & \frac{1}{\bar{a}} \\
0 & \frac{\bar{v}}{\bar{a}} & 0 \\
\bar{c}^2 & 0 & \frac{\bar{v}}{\bar{a}}
\end{bmatrix}
\]

(5.79)

Where

\( \bar{u} \) = Elemental average of the x-direction velocity, \( \frac{1}{3}(u_1 + u_2 + u_3) \)

\( \bar{v} \) = Elemental average of the y-direction velocity

\( \bar{c} \) = Elemental average of the wave celerity for the layer, for the bottom layer given by

\[
c = \sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} \left( \frac{h_1 h_2}{h_1 + h_2} \right)}
\]

\( \bar{a} = \sqrt{\bar{u}^2 + \bar{v}^2 + \bar{c}^2} \)

\[
SUPG = \alpha l \left\{ \frac{\partial \Phi}{\partial x} \hat{A} + \frac{\partial \Phi}{\partial y} \hat{B} \right\} \begin{bmatrix} C \\ X \end{bmatrix}
\]

(5.80)

\[
C = \frac{\partial u h}{\partial x} + \frac{\partial v h}{\partial y}
\]

(5.81)

\[
X = hu \frac{\partial u}{\partial x} + hv \frac{\partial u}{\partial y} + gh \frac{\partial h}{\partial x} + \frac{\rho_w}{\rho_{fm}} gh \frac{\partial h}{\partial x} + \frac{1}{\rho_{fm}} \tau_{bedx} \quad - \frac{1}{\rho_{fm}} \tau_{ix}
\]

(5.82)

\[
Y = hu \frac{\partial v}{\partial x} + hv \frac{\partial v}{\partial y} + gh \frac{\partial h}{\partial y} + \frac{\rho_w}{\rho_{fm}} gh \frac{\partial h}{\partial y} + \frac{1}{\rho_{fm}} \tau_{bedy} \quad - \frac{1}{\rho_{fm}} \tau_{iy}
\]

(5.83)

\[
\bar{a} = \sqrt{\bar{u}^2 + \bar{v}^2 + \bar{c}^2}
\]

(5.84)

\[
l = \sqrt{\text{Area}_\Omega}
\]

(5.85)

The time derivative terms are dropped from Equations 5.81, 5.82, and 5.83 compared to those used by Berger and Stockstill (Berger and Stockstill, 1995). This is
due to the explicit formulation of the fluid mud code compared to the implicit formulation of their work. The SUPG weight factor, $\alpha$, is a constant between 0.0 and 0.5 that impacts the effect the SUPG terms have on the equation calculations.

The integration of the SUPG terms is similar to the previously discussed Galerkin method, but without the integration by parts. Since the SUPG portion uses the derivative of the shape function and the shape functions are linear, the second derivative would be zero. Due to this, integration by parts is not an option. However, the derivatives are a constant over each element and may be pulled out of the integration, as in the viscosity and bottom elevation terms mentioned previously (Equation 5.75). The derivatives in the continuity equation are expanded and integrated as below.

$$C = \frac{\partial uh}{\partial x} + \frac{\partial vh}{\partial y} = \left(u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x}\right) + \left(v \frac{\partial h}{\partial y} + h \frac{\partial v}{\partial y}\right)$$

$$\int_{\Omega} \left(u \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x}\right) + \left(v \frac{\partial h}{\partial y} + h \frac{\partial v}{\partial y}\right) dA =$$

$$\left(\frac{\partial h}{\partial x} \int_{\Omega} \sum \Phi_j u_j dA + \frac{\partial u}{\partial x} \int_{\Omega} \sum \Phi_j h_j dA\right) + \left(\frac{\partial h}{\partial y} \int_{\Omega} \sum \Phi_j v_j dA + \frac{\partial v}{\partial y} \int_{\Omega} \sum \Phi_j h_j dA\right)$$

(5.86)

Likewise, for the x-direction momentum equation,

$$X = hu \frac{\partial u}{\partial x} + hv \frac{\partial u}{\partial y} + gh \frac{\partial h}{\partial x} + gh \frac{\partial z}{\partial x} + \rho_w g h \frac{\partial h_w}{\partial x} + \tau_{bed_x} - \tau_{ix}$$

$$\int_{\Omega} \left(hu \frac{\partial u}{\partial x} + hv \frac{\partial u}{\partial y} + gh \frac{\partial h}{\partial x} + gh \frac{\partial z}{\partial x} + \rho_w g h \frac{\partial h_w}{\partial x} + \tau_{bed_x} - \tau_{ix}\right) dA =$$

$$\frac{\partial u}{\partial x} \int_{\Omega} \left(\sum \Phi_j u_j \sum \Phi_j h_j\right) dA + \frac{\partial u}{\partial y} \int_{\Omega} \left(\sum \Phi_j v_j \sum \Phi_j v_j\right) dA + g \frac{\partial h}{\partial x} \int_{\Omega} \left(\sum \Phi_j h_j\right) dA +$$
\[
g \frac{\partial z}{\partial x} \int _{\Omega} (\sum \Phi_j h_j) \, dA + g \frac{\partial \rho w}{\partial x} \int _{\Omega} (\sum \Phi_j h_j) \, dA + \int _{\Omega} \tau_{bedx} \, dA - \int _{\Omega} \tau_{lx} \, dA \tag{5.87}
\]

Integration of \( C, X, \) and \( Y \) are done prior to applying them to the existing equations within the code. As \( \alpha \) can vary from 0 to 0.5, in the code these terms are added after the Galerkin part is calculated for each element. As the SUPG terms were derived from the matrix of the continuity, x-, and y-direction momentum equations, the above parameters are applied to all of the equations. As these terms are also after the finite difference approximation to the time derivative, they are all multiplied by \( \Delta t \), as shown below, and subtracted from the right hand side of the equations of continuity, x-direction momentum, and y-direction momentum, respectively.

\[
\text{continuity} = |J| \Delta t \left( \alpha l \left( \frac{\partial \phi_i}{\partial x} \left( \hat{A}_{i,1} C + \hat{A}_{i,2} X + \hat{A}_{i,3} Y \right) + \frac{\partial \phi_i}{\partial y} \left( \hat{B}_{i,1} C + \hat{B}_{i,2} X + \hat{B}_{i,3} Y \right) \right) \right) \tag{5.88}
\]

\[
\text{x momentum} = |J| \Delta t \left( \alpha l \left( \frac{\partial \phi_i}{\partial x} \left( \hat{A}_{2,1} C + \hat{A}_{2,2} X + \hat{A}_{2,3} Y \right) + \frac{\partial \phi_i}{\partial y} \left( \hat{B}_{2,1} C + \hat{B}_{2,2} X + \hat{B}_{2,3} Y \right) \right) \right) \tag{5.89}
\]

\[
\text{y momentum} = |J| \Delta t \left( \alpha l \left( \frac{\partial \phi_i}{\partial x} \left( \hat{A}_{3,1} C + \hat{A}_{3,2} X + \hat{A}_{3,3} Y \right) + \frac{\partial \phi_i}{\partial y} \left( \hat{B}_{3,1} C + \hat{B}_{3,2} X + \hat{B}_{3,3} Y \right) \right) \right) \tag{5.90}
\]

Where

\[
\hat{A}_{i,j} \hat{B}_{i,j} = \text{Value of } \hat{A} \text{ and } \hat{B} \text{ matrices (Equations 5.78 and 5.79) in row } i, \text{ column } j
\]
5.9 Other Code Modifications and Assumptions

Mass lumping of the terms from the time derivative is done during the code development. This concentrates the values on the diagonal of the matrix, simplifying the computations needed for the solution. This method is applied a number of times. For a full example applied to the Navier-Stokes equations, see (Niclasen and Blackburn, 1995).

During the testing phase of the fluid mud model, wetting and drying at the leading edge of the fluid mud was tested. This is needed to validate the model to the observed laboratory data presented in Chapters III and IV. Initial testing reveals the instability of the model during wetting and drying. To increase the stability of the model, a very thin depth (1 mm) of fluid mud is assumed to exist at all points in the domain. This allows the fluid mud to flow over “dry” areas while keeping the model stable. The code checks each element for depth and makes “dry” areas, where the depth is less than the minimum depth, wet by setting the depth to the minimum depth value. To prevent large errors in continuity, at minimum depth locations the velocity is assumed to be zero. This prevents continued mud flow down a slope, for instance. If velocity is not assumed to be zero at the minimum depth, the mud on the slope flows down the slope, is replaced during the minimum depth check, and repeated.

For the settling and consolidation equations, Equations 2.5 and 2.58 respectively, numerous parameters must be included. Based on the range of parameters given in Mehta and McAnally (Mehta and McAnally, 2009), the following are used as the default values. These values are likely needed to be modified depending on site specific sediment characteristics, but the following tables provide typical values if site specific
data are not available. Table 5.1 provides parameters for Equation 2.5, and Table 5.2 provides parameters for Equation 2.58.

Table 5.1 Typical values for the settling continuum, Equation 2.5 (Mehta and McAnally, 2009)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$a_w$</th>
<th>$b_w$</th>
<th>$m_w$</th>
<th>$n_w$</th>
<th>$C_1$ (kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>0.20</td>
<td>5</td>
<td>1.85</td>
<td>1.75</td>
<td>0.20</td>
</tr>
</tbody>
</table>

Table 5.2 Typical values for the consolidation equation, Equation 2.59 (Mehta and McAnally, 2009)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>$C_{s1}$ (kg/m$^3$)</th>
<th>$w_{sc1}$ (m/s)</th>
<th>$C_{s2}$ (kg/m$^3$)</th>
<th>$w_{sc2}$ (m/s)</th>
<th>$C_t$ (kg/m$^3$)</th>
<th>$m_t$</th>
<th>$n_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>20</td>
<td>5*10$^{-4}$</td>
<td>205</td>
<td>7*10$^{-6}$</td>
<td>160</td>
<td>3</td>
<td>13</td>
</tr>
</tbody>
</table>

Density of the fluid mud is a very site specific parameter, and fluid mud density and concentration are calculated from each other using Equation 5.50. Viscosity is another site specific parameter. Viscosity may be harder to test in the field, so laboratory studies may be used to determine viscosity. Based on observations and mentioned in Chapter IV, the fluid mud in this case is seen to move almost instantaneously with the water. This indicates a fluid closer to a Newtonian or shear-thinning fluid than one with a yield stress. This is supported by observations of Kusuda et al. and Mehta and Srinivas (Kusuda et al., 1993; Mehta and Srinivas, 1993). For this model, the initial viscosity is 0.2 Pa-s, a value determined in Kusuda et al. (Kusuda et al., 1993). Kusuda et al. state that “the apparent viscosity is rather independent of the concentration of the fluid mud” (Kusuda et al., 1993).
Using these assumptions, the fluid mud code is applied to the lab results seen in Chapters III and IV. Results from this application are presented in Chapter VI.

5.10 Verification and Validation

5.10.1 Flow Testing

Validation and verification tests are necessary to check for a robust and accurate model. Verification is testing for the correct solutions to the equations, while validation is testing for the correct equations for the system. In other words, verification is testing the math of the equations while validation is testing for the science when applied to the real world (Roache, 1998).

Validation and verification tests are presented in Wang et al (Wang et al., 2008). Verification tests used are Test Cases 1 and 3. The fluid mud code is modified to include the hydrodynamics of the upper layer during these tests to check the code for errors and interactions between the layers. Test Cases 1 and 3 are applied to the upper layer of the model. Additional tests are necessary to verify the lower layer equations. This will include the speed of an internal gravity wave. This is accomplished by using Test Case 1 with the perturbed surface applied at the interface between the two layers. The model was run with zero friction, no viscosity, and neglected advection terms, as specified in Wang et al. (Wang et al., 2008), for all cases. Model agreement was checked both graphically and with the error measurements from Wang et al., (Wang et al., 2008) including index of agreement $d$, $l_1$, $l_2$, $l_\infty$, and RMSE, calculated with Equations 5.91 through 5.95. $A_i$ is the analytical solution value and $N_i$ is the modeled value.
\[ d = 1 - \frac{\sum_{i=1}^{i_{\text{max}}}(A_i - N_i)^2}{\sum_{i=1}^{i_{\text{max}}}(|A_i| + |N_i|)^2} \]  
where \[ A_i' = A_i - \bar{A}_i \quad N_i' = N_i - \bar{A}_i \]  
(5.91)

\[ l_1 = \frac{\sum_{i=1}^{i_{\text{max}}}|N_i - A_i|}{\sum_{i=1}^{i_{\text{max}}}|A_i|} \]  
(5.92)

\[ l_2 = \frac{\left( \frac{\sum_{i=1}^{i_{\text{max}}}(N_i - A_i)^2}{\left( \sum_{i=1}^{i_{\text{max}}}(A_i)^2 \right)^{1/2}} \right)^{1/2}}{\left( \sum_{i=1}^{i_{\text{max}}}(A_i)^2 \right)^{1/2}} \]  
(5.93)

\[ l_\infty = \frac{\max_{i}|N_i - A_i|}{\max_{i}|A_i|} \]  
(5.94)

\[ RMSE = \sqrt{\frac{\sum_{i=1}^{i_{\text{max}}}(N_i - A_i)^2}{i_{\text{max}}}} \]  
(5.95)

Test Case 1 is “Free-Surface Seiching in a Closed Rectangular Basin with a Horizontal Bottom” (Wang et al., 2008). Mesh parameters were used with \( \Delta x = \Delta y = 4000 \text{m} \) for the low-resolution mesh and \( \Delta x = \Delta y = 2000 \text{m} \) for the high-resolution mesh. Dimensions of the mesh were specified as a length, \( L \), of 40,000 m and a width, \( B \), of 8000 m. Depth, \( H \), was specified as 12 m, with an initial water surface adjustment of 0.25 m \( \times \cos(\pi \frac{x}{L}) \), creating an initial depth at \( x=0 \) of 12.25 m and at \( x=40,000 \) of 11.75 m. Water surface elevation and x-direction velocity values were extracted at (6000, 2000) for the low-resolution case and (7000, 1000) for the high-resolution. Vertical velocities are not calculated in the two-dimensional model and were not compared.

The analytical solution for x-direction velocity and water surface elevation for Test Case 1 is given by
\[ u = 0.25 m \times \frac{\sqrt{gH}}{H} \sin\left(\frac{\pi}{L}x\right) \sin\left(\frac{\pi \sqrt{gH}}{L} t\right) \]  
and
\[ h = 0.25 m \times \cos\left(\frac{\pi}{L}x\right) \cos\left(\frac{\pi \sqrt{gH}}{L} t\right) \]
Figures 5.2 and 5.3 show water surface and x-
direction velocity for the low and high resolution cases. Table 5.3 shows the index of agreement, $l_1$, $l_2$, $l_\infty$, and RMSE for water surface and velocity for the low- and high-resolution cases.

Table 5.3  Test Case 1 Error Analysis

<table>
<thead>
<tr>
<th>Grid</th>
<th>$\Delta T$</th>
<th>SUPG $\alpha$</th>
<th>Obs. Point</th>
<th>Parameter</th>
<th>$d$</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_\infty$</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>1 s</td>
<td>0</td>
<td>(6000, 2000)</td>
<td>WSE</td>
<td>0.89</td>
<td>0.56</td>
<td>0.64</td>
<td>1.06</td>
<td>0.10 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>0.88</td>
<td>0.57</td>
<td>0.65</td>
<td>1.10</td>
<td>0.05 $\frac{m}{s}$</td>
</tr>
<tr>
<td>High</td>
<td>1 s</td>
<td>0</td>
<td>(7000, 1000)</td>
<td>WSE</td>
<td>0.99</td>
<td>0.15</td>
<td>0.18</td>
<td>0.36</td>
<td>0.03 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>0.98</td>
<td>0.20</td>
<td>0.25</td>
<td>0.50</td>
<td>0.02 $\frac{m}{s}$</td>
</tr>
</tbody>
</table>
Figure 5.4  Test Case 1 Low-Resolution Water Surface and X-direction Velocity
Figure 5.5  Test Case 1 High-Resolution Water Surface and X-direction Velocity
Test Case 3 is “Tidal Forcing in a Closed Rectangular Basin with a Horizontal Bottom” (Wang et al., 2008). Mesh parameters were used with a $\Delta x = \Delta y = 4000m$ for the low-resolution mesh and $\Delta x = \Delta y = 2000m$ for the high-resolution mesh.

Dimensions of the mesh were specified as a length, $L$, of 100,000 m and a width, $B$, of 20000 m. A water surface boundary condition was specified at the $x=0$ edge of $0.25 \, m \ast \cos(\omega_T t)$, where $\omega_T = \frac{2\pi}{12.4 \, hours}$. This was added to the depth of 12 m to specify depth at the boundary. Initial depth was specified as 12 m, with an initial water surface adjustment of $0.25 \, m \ast \left[ \frac{\cos\left(\omega_T \sqrt{gH} \left( L-x \right) \right)}{\cos\left(\omega_T \sqrt{gH} L \right)} \right]$, creating an initial depth at $x=0$ of 12.25 m and at $x=100000$ of 12.92 m. Water surface elevation and x-direction velocity values were extracted at (2500, 2500), (52500, 2500), and (92500, 2500) for the low resolution case and (1250,1250), (51250, 1250), and (96250, 1250) for the high resolution case. Vertical velocities are not calculated in the two-dimensional model and were not compared.

The analytical solution for Test Case 3 is given by

\[
u = -0.25 \, m \sqrt{\frac{gH}{L}} \sin \left( \frac{\omega_T}{\sqrt{gH}} (L-x) \right) \sin(\omega_T t) \text{ and}
\]

\[
h = 0.25 \, m \ast \left[ \frac{\cos\left(\omega_T \sqrt{gH} \left( L-x \right) \right)}{\cos\left(\omega_T \sqrt{gH} L \right)} \right] \cos(\omega_T t).
\]

Figures 5.4 through 5.9 show water surface and x-direction velocity for the low- and high-resolution cases at all three observation points. Table 5.4 shows the index of agreement, $l_1$, $l_2$, $l_\infty$, and RMSE for water surface and velocity for the low- and high-resolution cases at all three observation points.
Table 5.4  Test Case 3 Error Analysis

<table>
<thead>
<tr>
<th>Grid</th>
<th>ΔT</th>
<th>SUPG α</th>
<th>Obs. Point</th>
<th>Parameter</th>
<th>$d$</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_∞$</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>1 s</td>
<td>0.5</td>
<td>(2500, 2500)</td>
<td>WSE</td>
<td>1.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.02</td>
<td>0.003 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>1.00</td>
<td>0.11</td>
<td>0.13</td>
<td>0.19</td>
<td>0.07 $\frac{m}{s}$</td>
</tr>
<tr>
<td>Low</td>
<td>1 s</td>
<td>0.5</td>
<td>(52500, 2500)</td>
<td>WSE</td>
<td>1.00</td>
<td>0.11</td>
<td>0.12</td>
<td>0.18</td>
<td>0.06 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>0.99</td>
<td>0.13</td>
<td>0.14</td>
<td>0.20</td>
<td>0.05 $\frac{m}{s}$</td>
</tr>
<tr>
<td>Low</td>
<td>1 s</td>
<td>0.5</td>
<td>(92500, 2500)</td>
<td>WSE</td>
<td>1.00</td>
<td>0.12</td>
<td>0.13</td>
<td>0.20</td>
<td>0.09 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>1.00</td>
<td>0.08</td>
<td>0.10</td>
<td>0.15</td>
<td>0.006 $\frac{m}{s}$</td>
</tr>
<tr>
<td>High</td>
<td>1 s</td>
<td>0.5</td>
<td>(1250, 1250)</td>
<td>WSE</td>
<td>1.00</td>
<td>0.05</td>
<td>0.05</td>
<td>0.08</td>
<td>0.001 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>1.00</td>
<td>0.06</td>
<td>0.07</td>
<td>0.11</td>
<td>0.04 $\frac{m}{s}$</td>
</tr>
<tr>
<td>High</td>
<td>1 s</td>
<td>0.5</td>
<td>(51250, 1250)</td>
<td>WSE</td>
<td>1.00</td>
<td>0.06</td>
<td>0.07</td>
<td>0.11</td>
<td>0.04 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>1.00</td>
<td>0.06</td>
<td>0.07</td>
<td>0.10</td>
<td>0.03 $\frac{m}{s}$</td>
</tr>
<tr>
<td>High</td>
<td>1 s</td>
<td>0.5</td>
<td>(96250, 1250)</td>
<td>WSE</td>
<td>1.00</td>
<td>0.07</td>
<td>0.08</td>
<td>0.12</td>
<td>0.05 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$V_x$</td>
<td>1.00</td>
<td>0.15</td>
<td>0.15</td>
<td>0.17</td>
<td>0.004 $\frac{m}{s}$</td>
</tr>
</tbody>
</table>
Figure 5.6 Test Case 3 Low-Resolution Water Surface and X-direction Velocity, Point 1
Figure 5.7  Test Case 3 Low-Resolution Water Surface and X-direction Velocity, Point 2
Figure 5.8  Test Case 3 Low-Resolution Water Surface and X-direction Velocity, Point 3
Figure 5.9  Test Case 3 High-Resolution Water Surface and X-direction Velocity, Point 1
Figure 5.10  Test Case 3 High-Resolution Water Surface and X-direction Velocity, Point 2
Figure 5.11  Test Case 3 High-Resolution Water Surface and X-direction Velocity, Point 3
The two-layer test case was set up similar to Test Case 1. A simple two-layer model was developed to calculate the equations of motion for both the top and bottom layers. The bottom layer was specified with an initial slope and the upper layer was set with a constant water surface, as shown in Figure 5.10. Densities were specified as 1000 \( \frac{kg}{m^3} \) and 1050 \( \frac{kg}{m^3} \) for the upper and lower layers, respectively. The analytical solution of Test Case 1 was modified for the two-layer case by updating the wave celerity in the equations. The celerity was changed as given by Keulegan (Keulegan, 1953) to 

\[
\sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} \frac{h_1 h_2}{h_1 + h_2}}
\]

This changes the analytical solution to

\[
u = 0.25 m \ast \sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} \frac{h_1 h_2}{h_1 + h_2}} \sin \left( \frac{\pi}{L} x \right) \sin \left( \frac{\pi}{L} \sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} \frac{h_1 h_2}{h_1 + h_2}} t \right)
\]

and

\[
h = 0.25 m \ast \cos \left( \frac{\pi}{L} x \right) \cos \left( \frac{\pi}{L} \sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} \frac{h_1 h_2}{h_1 + h_2}} t \right)
\]

Two different test cases were run on both the high- and low-resolution meshes, one with an \( H_1/H_2 \) ratio of 0.5 and another with a ratio of 5. Figures 5.11 through 5.14 show water surface and x-direction velocity for the low- and high-resolution cases. Table 5.5 shows the index of agreement, \( l_1, l_2, l_\infty \), and RMSE for water surface and velocity for the low- and high-resolution cases, respectively. The different \( H_1/H_2 \) ratios impact the speed of the interfacial wave, which should be 1.37 m/s for the \( H_1/H_2=0.5 \) case and 1.53 m/s for the \( H_1/H_2=5 \) case. A visual check of Figures 5.13 and 5.15 demonstrates that the interfacial wave is moving faster for \( H_1/H_2=5 \) case, as expected. A wave should take 58394 seconds to reflect off the far wall and return for the \( H_1/H_2=0.5 \) case and 52287 seconds for the \( H_1/H_2=5 \) case. The model
shows time periods of 60600 and 51900 seconds for the $H_1/H_2=0.5$ case and the $H_1/H_2=5$ case, respectively, indicating good agreement between the model and analytical solutions.

Figure 5.12  Two-Layer Test 1 Schematic (not to scale)
<table>
<thead>
<tr>
<th>Grid</th>
<th>ΔT</th>
<th>SUPG</th>
<th>Obs. Point</th>
<th>H₁</th>
<th>H₂</th>
<th>Parameter</th>
<th>d</th>
<th>l₁</th>
<th>l₂</th>
<th>l₃</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>1 s</td>
<td>0</td>
<td>(6000, 2000)</td>
<td>6m</td>
<td>12m</td>
<td>WSE</td>
<td>1.00</td>
<td>0.07</td>
<td>0.07</td>
<td>0.11</td>
<td>0.01 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Vₓ</td>
<td>0.98</td>
<td>0.26</td>
<td>0.28</td>
<td>0.44</td>
<td>0.003 m/s</td>
</tr>
<tr>
<td>High</td>
<td>1 s</td>
<td>0</td>
<td>(7000, 1000)</td>
<td>6m</td>
<td>12m</td>
<td>WSE</td>
<td>1.00</td>
<td>0.03</td>
<td>0.04</td>
<td>0.05</td>
<td>0.005 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Vₓ</td>
<td>0.98</td>
<td>0.23</td>
<td>0.25</td>
<td>0.39</td>
<td>0.003 m/s</td>
</tr>
<tr>
<td>Low</td>
<td>1 s</td>
<td>0</td>
<td>(6000, 2000)</td>
<td>30m</td>
<td>6m</td>
<td>WSE</td>
<td>1.00</td>
<td>0.06</td>
<td>0.07</td>
<td>0.11</td>
<td>0.01 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Vₓ</td>
<td>0.98</td>
<td>0.23</td>
<td>0.26</td>
<td>0.043</td>
<td>0.005 m/s</td>
</tr>
<tr>
<td>High</td>
<td>0.5 s</td>
<td>0</td>
<td>(7000, 1000)</td>
<td>30m</td>
<td>6m</td>
<td>WSE</td>
<td>1.00</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.003 m</td>
</tr>
<tr>
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<td></td>
<td></td>
<td>Vₓ</td>
<td>0.99</td>
<td>0.16</td>
<td>0.18</td>
<td>0.32</td>
<td>0.004 m/s</td>
</tr>
</tbody>
</table>
Figure 5.13  Two Layer Test Case 1 Low-Resolution, $H_1/H_2=0.5$, Lower Layer Surface and X-direction Velocity
Figure 5.14  Two Layer Test Case 1 High-Resolution, H$_1$/H$_2$=0.5, Lower Layer Surface and X-direction Velocity
Figure 5.15  Two Layer Test Case 1 Low-Resolution, $H_1/H_2=5$, Lower Layer Surface and X-direction Velocity
Figure 5.16  Two Layer Test Case 1 High-Resolution, $H_1/H_2=5$, Lower Layer Surface and X-direction Velocity
As seen in Figures 5.13 and 5.14, oscillations can be seen in the X-direction velocity. These oscillations are typically damped out by the SUPG routine; however, for these test cases the SUPG factor was set to zero. Due to the missing temporal terms in the SUPG equations, the routine tends to demonstrate more damping of the solution. Figure 5.17 shows the results from the high-resolution 2-Layer Test Case 1 with α values of 0.0, 0.1, 0.25, and 0.5 compared to the analytical solution. As seen in the figure, higher values of α damp out the oscillations but also over damp the solution. Linkage to an existing code will likely necessitate the use of an implicit solution routine which will allow for the inclusion of the temporal terms in the SUPG equations. At that point, the equations will need to be updated to include the temporal terms for better solution stability. For the purposes of this dissertation, the small time and spatial scale are likely to minimize the effects of the missing temporal terms on the solution.
Figure 5.17  Comparison of SUPG on Two-Layer Test Case 1
Additional tests were run without the upper layer equations of motion. This eliminated the wiggles from the solution as well without the need for the SUPG terms. However, without the upper layer equations, there is no change on the effect on the bottom layer due the change in average depth of the upper layer. Without the upper layer momentum equations, the only effect is from the slope of the upper layer surface and the density difference. This is demonstrated by Equations 5.38 and 5.39, where the impact from the upper water layer is only seen in the \( \frac{\rho_w}{\rho_{fm}} \frac{\partial h_w}{\partial x} \) and \( \frac{\rho_w}{\rho_{fm}} \frac{\partial h_w}{\partial y} \) terms. This changes the celerity of an interfacial wave to \( c = \sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} h_2} \), and changes the analytical solution to \( u = 0.25 \, m \ast \frac{\sqrt{g \frac{\rho_2 - \rho_1}{\rho_2} h_2}}{H} \sin \left( \frac{\pi}{L} x \right) \sin \left( \frac{\pi}{g \frac{\rho_2 - \rho_1}{\rho_2} h_2} \frac{L}{L} t \right) \) and \( h = 0.25 \, m \ast \cos \left( \frac{\pi}{L} x \right) \cos \left( \frac{\pi}{g \frac{\rho_2 - \rho_1}{\rho_2} h_2} \frac{L}{L} t \right) \). The results from this test are presented in Figure 5.18 and Table 5.6. The interfacial wave should propagate at a velocity of 2.37 m/s in this case, and that is replicated well by the model. It should take 33,755 seconds for the wave to reflect off the far end of the domain and return to the initial position, which is well matched as can be seen in the upper portion of Figure 5.18.

Table 5.6  Two-Layer Test Case 1 Error Analysis, No Upper Layer Calculations

<table>
<thead>
<tr>
<th>Grid</th>
<th>ΔT</th>
<th>SUPG α</th>
<th>Obs. Point</th>
<th>H₁</th>
<th>H₂</th>
<th>Parameter</th>
<th>d</th>
<th>l₁</th>
<th>l₂</th>
<th>l∞</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td>1 s</td>
<td>0</td>
<td>(7000, 1000)</td>
<td>6m</td>
<td>12m</td>
<td>WSE</td>
<td>1.00</td>
<td>0.03</td>
<td>0.04</td>
<td>0.07</td>
<td>0.006 m</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Vₓ</td>
<td>0.98</td>
<td>0.25</td>
<td>0.28</td>
<td>0.50</td>
<td>0.005 m/s</td>
</tr>
</tbody>
</table>
Figure 5.18  Two Layer Test Case 1, High Resolution, $H_1/H_2=5$, Lower Layer Surface and X-direction Velocity without Upper Layer Equations
5.10.2 Bottom Layer Equation Testing

Tests were run to verify the bottom layer terms, specifically the $\frac{\partial h_w}{\partial x}$ and $\frac{\partial h_w}{\partial y}$ terms as well as the interfacial shear and bottom roughness terms. These tests were also run to verify the SUPG development, with any oscillations or errors in the solution potentially indicating problems with the SUPG terms.

The first test case was set up to check the balance between the $\frac{\partial h_w}{\partial x}$ and $\frac{\partial h^2}{\partial x}$ terms. The analytic solution was developed by starting with the x-direction momentum equation and assuming steady flow conditions and cancelling terms. Bottom and interfacial shear as well as viscosity terms were assumed to be zero for this case. Bottom elevation was set to zero as well. The development is shown below.

\[
\frac{\partial U h}{\partial t} + \frac{\partial U U h}{\partial x} + \frac{\partial U V h}{\partial y} = -\frac{1}{2} g \frac{\partial h^2}{\partial x} - g h \frac{\partial z_0}{\partial x} - \frac{\rho_{\text{water}}}{\rho_{fm}} g h \frac{\partial h_w}{\partial x} + \frac{1}{\rho_{fm}} \frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho_{fm}} \frac{\partial \tau_{xy}}{\partial y} (\tau_{\text{interface}}) + \frac{1}{\rho_{fm}} (\tau_{\text{bed}})
\]

Neglecting the unsteady, convective, friction, viscous, and bed slope terms,

\[
0 = -\frac{1}{2} g \frac{\partial h^2}{\partial x} - \frac{\rho_{\text{water}}}{\rho_{fm}} g h \frac{\partial h_w}{\partial x}
\]

\[
\frac{1}{2} g \frac{\partial h^2}{\partial x} = -\frac{\rho_{\text{water}}}{\rho_{fm}} g h \frac{\partial h_w}{\partial x}
\]

This equation is integrated over the length of the domain, from 0 to $x_{\text{max}}$, which reduces to
\[
\frac{1}{2} g (h_{x_{\text{max}}}^2 - h_0^2) = -\frac{\rho_{\text{water}}}{\rho_{\text{fm}}} gh (h_{w_{x_{\text{max}}}} - h_{w_0}) \tag{5.96}
\]

This solution is iterated to determine the analytical solution. As the bottom layer surface responds to the pressure from the upper layer, the \( h_w \) slope also changes, requiring an iterative solution for the analytic equation. Over the domain, an initial water surface slope of 19.67 at \( x=0 \) to 19.62 at \( x=40000 \text{m} \) was set. Initial mud depth of 6.835m was set constant. This initial small water surface slope should stabilize over time to a water depth slope of 1m and create a mud slope of 0.95m, which is also the ratio of the densities times the upper layer slope, \( \frac{1000 \text{kg/m}^3}{1050 \text{kg/m}^3} \times 1m = 0.952m \). The model was run for a period of 10 days with a time step of 15 seconds.

Model results produce a slope of 0.95 m, matching the analytical solution. Figure 5.19 shows the time series results for points at \( x=0 \) and \( x=40000 \) showing the initial sloshing and then stabilization of the bottom layer slope, sloping up from 0 to 40000m. Figure 5.20 shows the initial and final upper and bottom layer surface elevations. The SUPG coefficient was set to 0.5 for this test. These results show no issues from the SUPG terms.
Figure 5.19  Bottom Surface Elevation Over Time at X = 0 and X = 40000m
**Figure 5.20**  Upper and Bottom Layer Surface Elevations at Initial and Final Times
Another test was run to test the balance of the interfacial shear term with the bottom layer surface slope. By assuming a constant velocity field over the bottom layer, the analytic solution for the slope can be developed. This solution is developed as above, without the unsteady, convective, viscous, or bottom friction terms. Additionally, a constant water depth above the bottom layer was set to remove the $\frac{\partial h_w}{\partial x}$ terms from the equation. The final equation is

$$0 = -\frac{1}{2} g \frac{\partial h^2}{\partial x} - \frac{1}{\rho_{fm}} (\tau_{interface}) = -\frac{1}{2} g \frac{\partial h^2}{\partial x} - \frac{\rho_w}{\rho_{fm}} f_i u^2$$

Integrating over the domain reduces to Equation 5.97, which can be solved for the bottom layer slope.

$$\frac{1}{2} g (h_{x_{max}}^2 - h_0^2) = -\frac{\rho_w}{\rho_{fm}} f_i u^2 x_{max}$$  \hspace{1cm} (5.97)

By using an $f_i$ value of 0.03, constant uniform upper layer velocity of 0.2 m/s, and an initial bottom layer depth of 6.835m, the calculated slope should be 0.085m over the 40000m domain. The model was run for one day with a time step of 15 seconds.

Model results produce a slope of 0.085m matching the analytical solution. Figure 5.21 shows the time series results for points at x=0 and x=40000 showing the initial set up and then stabilization of the bottom layer slope. Figure 5.22 shows the initial and final surface slopes of the bottom layer. The SUPG coefficient was set to 0.5 for this test. As in the previous test, these results again show no issues from the SUPG terms.
Figure 5.21  Bottom Surface Elevation Over Time at X = 0 and X = 40000m

Figure 5.22  Bottom Layer Surface Elevations at Initial and Final Times
A third test was run to verify the implementation of the bottom friction term. The bottom friction is specified with a Manning roughness value and should match the results from the Manning equation. Constant depth boundary conditions were specified both upstream and downstream to maintain a constant 0.1m slope over the 40000 m long by 8000 wide domain. Upper layer depth was assumed constant to remove the influence of the $\frac{\partial h}{\partial x}$ term from the results. The interfacial friction factor was set to zero, so the only influences on the bottom layer were friction and the mud slope. The mud depth was set to 6.5m at the upstream end and 6.4 m at the downstream end. A roughness value of 0.02 was set throughout the domain. The model was run for 12 hours with a time step of 15 seconds. Using the Manning equation, the analytical solution should be

$$V = \frac{k}{n} R^\frac{2}{3} S^{\frac{1}{2}} = \frac{1}{0.02} \left( \frac{6.45 m + 8000 m}{8000 m + 2 \times 6.45 m} \right)^{\frac{2}{3}} \left( \frac{0.1 m}{40000 m} \right)^{\frac{1}{2}} = 0.274 \text{ m/s}$$ \hspace{1cm} (5.98)

Model results produced a velocity at the midpoint of the domain of 0.27 m/s, demonstrating agreement with the analytical solution. Figure 5.23 shows the time series of the bottom layer velocity at the midpoint of the domain. The SUPG coefficient was set to 0.5 for this test. As in the previous test, these results again show no issues from the SUPG terms.
The preceding sections have verified the speed of an interfacial wave with varying upper layer depths as well as constant water surface with no upper layer momentum equations. They have also verified the pressure effects of a sloping upper layer surface on the bottom layer, the interfacial shear on the bottom layer, and the bed friction shear. The main terms that remain to be tested are the sources and sinks in the continuity equation.

### 5.10.3 Sources and Sinks Testing

Additional testing was done on the sources and sinks terms to confirm that the terms and coding was behaving as expected. Settling was tested by assuming a quiescent fluid with a constant sediment concentration throughout. The free settling velocity was increased to 0.001 m/s to be able to observe changes rapidly. A fluid mud concentration
of 9.64 g/L and a water concentration of 5 g/L were used during the test. Based on these parameters, over a 1000 second simulation the depth increase was expected to be 0.519 meters, based on \((0.001 \text{ m/s}) \times (5 \text{ g/L} / 9.64 \text{ g/L}) \times 1000 \text{ seconds}\). The model accurately represented this change, simulating a change of 0.519 m over the simulation time.

Consolidation was tested similarly to settling, with a quiescent system. The water concentration was set at zero, so only the removal of material was calculated. A fluid mud concentration of 10 g/L was used in this test. The typical consolidation parameters were used as given in Table 5.2 and applied in Equation 2.58. Based on these values, over a 1000 second time frame a depth change of 0.301 m was expected. The model accurately represented this, simulating a change of 0.301 m over the 1000 second run.

An additional test was run with a higher concentration fluid mud, of 80 g/L. This shifted the concentration past the \(C_{s1}\) value and changed the overall settling velocity. The expected total consolidation over the 1000 second run was 0.009 m from Equation 2.58, and the model accurately predicted a change of 0.009 m.

Entrainment calculations are more complicated, as the shear force that entrains the fluid mud layer also adds to the momentum of the fluid mud layer. A constant water velocity of 0.25 m/s and a constant water depth of 1 m were used over a mud layer of 5 m thickness and concentration of 10 g/L. The interfacial friction coefficient was set to 0 to prevent momentum from being imparted on the mud. Using the entrainment formulation given in Equation 2.41, with a coefficient of 0.0052 multiplied by the inverse of the Richardson number, a depth change of 1.315 m was expected over 1000 seconds assuming the fluid mud layer was stable and using just the water velocity instead of the
difference between the water and mud velocities. The model simulated an average depth change of 1.315 m, matching the predicted depth change.
CHAPTER VI
FLUID MUD MODEL RESULTS

The following sections describe the results of the model presented in Chapter V applied to the experiments presented in Chapters III and IV.

6.1 General Model Setup

An initial model mesh is created of the experimental flume described in Chapter III. The elements are created so that spacing across the flume was 0.061 m (5 elements across) and spacing in the flow direction was 0.122 m. The elevation datum is chosen so that 0 is at the bottom of the depression in the flume. This placed the raised portions of the bed to an elevation around 0.15 m. The mesh begins at the flow straightener and ends at the end of the elevated bed. Figure 6.1 shows the mesh setup, and Figure 6.2 shows the bathymetry of the mesh.

![Flume model mesh, plan view, flow from left to right](image-url)
Model parameters including viscosity and roughness are set as constants for all the experiments. A mud viscosity value of 0.2 Pa-s and a Manning’s roughness coefficient for the mud of 0.01 are used. Initial depths in the fluid mud layer are set to the observed values, and initial velocities are set to zero. The initial water velocities are also set to zero, with a flat water surface over the mud. The Adaptive Hydraulics Model (ADH) (Berger et al., 2013) is used to create a water surface time series for the upper layer. This water surface is then used a known value in the fluid mud code. Flow is specified as uniform throughout the domain, and velocity is calculated from the flow divided by the width of the flume and water depth. This limits the model as there is no feedback to the hydrodynamic code; however, based on the small scale and laminar flow
during the flume experiments this assumption does not seem overly limiting. Figure 6.3 demonstrates the vertical layout of the two-layers.

Boundary conditions for depth of the mud layer are set at the downstream end of the mesh as an open boundary. This is calculated by using the minimum depth value as a constant value at the downstream end. This prevents the mud from piling up at the downstream end or reflecting back into the domain.

6.2 Sensitivity Tests
6.2.1 Grid and Time Step Convergence

The flume mesh is run within the fluid mud code with different grids and CFL values to determine the effects of grid size and time step on the model results. Meshes with stream-wise resolutions of 0.06 m, 0.12 m, and 0.24 m and cross-stream resolutions of 0.06 m. 0.06 m, and 0.10 m are used for the very fine, fine, and coarse grids, respectively. CFL values of 0.25 and 0.10 are used for the long and short time steps,
respectively. Results from the grid convergence testing show significant differences between the coarse and fine mesh and small differences between the fine and very fine meshes. These differences are on the order of 2-3 mm in mud depths from the coarse to the fine mesh and 0.2-0.3 mm from the fine to the very fine meshes. Based on these results, the models are run with the fine mesh described in Section 6.1, with a streamwise spacing of 0.122 m.

The fine mesh was used with CFL values of 0.25 and 0.10 to test for the effects of time step size on the solution. No difference was found between the results, and the CFL value of 0.25 is used for the remaining experiments.

6.2.2 Parameter Sensitivity

Sensitivity tests were conducted on viscosity and bed roughness parameters. The mud viscosity in the model was set to 0.2 Pa·s and a test was run with the viscosity set to 0.001 Pa·s, the same as water. A lower viscosity should increase velocity and decrease the depth of the mud due to the lessened resistance. Results from this test show very minor changes between the two runs. This indicates that viscosity has a much smaller impact on the movement of mud than the interfacial friction, bed roughness, or pressure terms.

Effects from increased bed roughness were also evaluated. The bed roughness in the model was set to a value of 0.01, and a test was conducted to see how an order of magnitude increase to 0.1 would impact the solution. Results from this test show an increase in mud surface elevation at the end of the experiment from 0.063 to 0.077 m and a decrease in velocity from $1.3 \times 10^{-3}$ to $1.0 \times 10^{-3}$ m/s, which is expected due to the increased energy losses from the higher roughness value. The lower velocity values were
seen consistently throughout the model run. This shows the roughness parameter is
behaving as expected.

6.3 Initial Fluid Mud Flow Model Results

The initial fluid mud flow experimental results are presented in Section 4.2. The
ccentration of 6.7 g/L equates to a density of 1004 kg/m^3 which is set in the model.
Changes in flow rate, water surface, and fluid mud depth are observed over the
experiment. During the 12 minute run time, water flow increases from 0 to
1.2×10^{-3} m^3/s, water surface increases from 0.235 to 0.25 m, and fluid mud depth at the
deepest part of the flume decreases from an initial depth of 0.13 m to a final depth of 0.07
m. Fluid mud is observed to flow downstream and out of the depression. The water
surface elevations are set to at the upstream and downstream end to drive the flow across
the mud and create the actual water slope.

6.3.1 Model Results

The fluid mud code is run using the above parameters. Results from this run
indicate general agreement between the model and the observed behavior. The interfacial
friction factor is initially set using values from Dermissis and Partheniades (Dermissis
and Partheniades, 1982) at 0.03. This may be incorrect, as the interfacial values are
based on fresh and salt water interactions and not the water and fluid mud interface.
Calculating an \( f \) value from the Blasius equation for smooth beds for roughness, with the
hydraulic radius used in the Reynolds formulation, a value of 0.04 is obtained.

\[
f = \frac{0.233}{Re^{0.25}} = \frac{0.233}{\left( \frac{1000 \text{ kg/m}^3}{0.03 \text{ m}^3} \left( \frac{0.30 \text{ m} \times 0.1 \text{ m}}{0.30 \text{ m} + 2 \times 0.1 \text{ m}} \right) \left( \frac{0.001 \text{ N s/m}^2}{m^2} \right) \right)^{0.25}} = 0.04
\]  

(6.1)
Dermissis and Partheniades also found that the bed friction value is “clearly smaller than the corresponding $f_i$” (Dermissis and Partheniades, 1982), which would indicate an interfacial friction factor larger than 0.04. Based on these findings, as well as the unknown relationship of the interfacial friction factor to a highly viscous fluid such as fluid mud, an appropriate value for the interfacial friction factor is not known.

Initial model results did not replicate the observed flow of mud downstream with this initial value of 0.03, and the value is tested until the observed behavior is replicated. The interfacial friction value is set to 0.7 to replicate the observed behavior. This creates the necessary drag to move the fluid mud downstream as observed. The fluid mud flow downstream is replicated by the model, and the depth of the mud layer decreases from 0.13 m to 0.065 m. Observation profiles from the beginning and final times taken longitudinally along the mesh are presented in Figure 6.4. Velocities are modeled to be on the order of 0.008 m/s. No velocity measurements are taken during this experiment. Figure 6.5 shows the velocity distribution over the model domain. Velocity is lowest at the deepest part of the flume, and increases as the mud flows out of the depression and out of the domain. Figure 6.6 shows the observed data points from the video as shown in Chapter IV compared to the model results from the middle of the mesh.
Figure 6.4  Experiment 1 Observation Profiles

(a) Beginning of experiment and (b) at final time. Flow from left to right. Vertical axis is exaggerated compared to the horizontal axis.
Increased Concentration Model Results

Experiment 2 is conducted with an increased concentration compared to Experiment 1. As previously mentioned in Section 4.3, the fluid mud concentration is 35 g/L. This equates to a density of 1022 kg/m³, which is input to the model. As in Experiment 1, changes in flow rate, water surface elevation, and fluid mud depth are observed. Additionally, velocity measurements are conducted with an ADV during this
experiment. During the 14 minute run time, flow increases from 0 to $1.0 \times 10^{-3}$ m$^3$/s, water surface increases from 0.24 to 0.25 m, and fluid mud depth at the deepest part of the flume decreases from an initial depth of 0.15 m to a final depth of 0.13 m. Velocity measurements are taken after 10 minutes, and are observed to be 0.2 to 0.25 cm/s, or 0.002 to 0.0025 m/s. Fluid mud is observed to flow downstream and out of the depression.

Water surface elevation is set as mentioned previously and updated with the new final values and a final time of 840 seconds.

### 6.4.1 Model Results

Results from this run indicate good agreement between the model and the observed behavior. The fluid mud flow downstream is replicated by the model, and the depth of the mud layer decreases from 0.15 m to 0.135 m. Observation profiles from the beginning and final times are presented in Figure 6.7. Velocities are modeled to be about 0.001 m/s. Figure 6.8 shows the velocity distribution over the model domain.
Figure 6.7  Experiment 2 Observation Profiles

(a) Beginning of experiment and (b) at final time. Flow from left to right. Vertical axis is exaggerated compared to the horizontal axis.
Figure 6.8  Velocity contours and vectors from Experiment 2 at T=840 s

6.5  Summary

Model results simulated fluid mud flow under shear from the water column, which are in good agreement with the flume experiments. The fluid mud is predicted to flow over the lip of the depression and out of the model domain for both runs. The runs are completed stably and without oscillations. Constant density and viscosity values for the fluid mud are used for both experiments. Velocities for the second experiment are lower than from the first experiment as the water flow was 20% higher during the first experiment. These results and observations are examined further in Chapter VII.
CHAPTER VII
DISCUSSION OF EXPERIMENTS

The following sections discuss the results shown in Chapter IV Fluid Mud Flow Experiments and Results, followed by a discussion of the model results shown in Chapter VI.

7.1 Initial Fluid Mud Flow Test

The results from the initial fluid mud flow test, as shown in Section 4.2, demonstrated that fluid mud flow does occur under shear stress from an overlying fluid flow. The flow is observed to occur at below measurable flow rates of the overlying water. Since no specific yield strength is observed, the fluid mud is not behaving as a Bingham plastic. This result is not entirely surprising, as some including Le Hir have postulated that fluid mud would not behave as a true Bingham plastic but would have some flow rate at low shear (Le Hir, 1997).

Flow appeared to be smooth in the fluid mud layer with minimal entrainment. A velocity profile is evident in the fluid mud with mud near the interface moving faster than the mud at the bottom. The mud velocity is lower than the overlying water velocity. This behavior is consistent with a two-fluid system with different densities as described by (Lock, 1951).
A diagram of the observed velocity profile is presented below in Figure 7.1. Since only three velocity measurements are taken of the mud in Experiment 2, a linear profile in the mud layer is assumed. This is consistent with the theoretical velocity profile known as Couette flow (Panton, 2005), which describes flow under a moving plate. In this experiment, the moving plate is analogous to the overlying water column. The velocity profile assumes some transition zone between the water and mud layers, and a logarithmic profile in the water column.

Interfacial waves are observed that are not breaking. However, some entrainment does occur in the form of small, wispy streaks. This demonstrates simultaneous flow and entrainment, with flow dominating for this case.

Figure 7.1 Observed Velocity Profile of Fluid Mud and Water Column
Based on results presented in Table 4.1
Image analysis as seen in Figures 4.5 and 4.6 show the change in elevation of the mud and water at the deepest point of the flume. Results from this indicate the greatest change in mud depth during the initial increase of water flow. This is more consistent with mud flow due to the increased slope of the water surface as opposed to the flow-induced shear. The mud continued to flow throughout the experiment, even after the water surface had stabilized, although the magnitude was not as large. This indicates that while pressure has a larger effect on the mud flow, shear does contribute to the movement of fluid mud.

7.2 Increased Concentration Flow Test

Results from the flow test with a higher concentration of sediment confirms flow of fluid mud under shear. At very small flow rates, no fluid mud movement is observed. This result could indicate one of two possibilities - a yield strength as in a Bingham plastic, or the flow rate may have been too small to observe in the given time frame. The flow rate would be smaller than the previous experiment as the increased density led to an increase in viscosity, which would slow down the mud flow. This result would follow the prediction by Le Hir and support the results from the previous experiment (Le Hir, 1997). The water flow is increased before either possibility could be definitively observed. As the flow increases to a measurable rate, the mud begins to noticeably flow. Again, the mud flowed smoothly with a visible velocity profile. The ADV tests also confirm this profile in the mud.

Entrainment is small, indicating again that entrainment and movement happened concurrently, with movement dominant at these low shear stresses. As the mud passes over the end of the slope downstream of the depression in the bed, entrainment increases
due to the increased water velocity from the decreased cross-sectional area. These observations seem to indicate that as shear over the mud increases, entrainment also increases. At some higher velocity, the fluid mud is entrained with minimal movement due to the increased shear. At these higher velocities, the shear stress at the top of the fluid mud is much greater than the internal cohesive forces holding the mud together. This causes the fluid mud to form small eddies which are entrained into the water column through individual turbulent parcels. Increased turbulence at the mud surface due to the interaction of the water and any interfacial waves present also contribute to the increased entrainment.

The bulk Richardson number, as seen in Equation 2.43, relates gravitational forces to inertial forces and can be used to estimate entrainment. Based on the ADV results, the Richardson number can be calculated as

\[ R_i = \frac{g(\rho_{fm} - \rho_w) h_{fm}}{\rho_{fm} \Delta U^2} = \frac{9.81 \frac{m}{s^2} \left(1022 \frac{kg}{m^3} - 1000 \frac{kg}{m^3}\right)}{1022 \frac{kg}{m^3} \left(0.0249 - 0.0023 \frac{m}{s}\right)^2} \frac{0.13m}{(0.216 \frac{m}{s^2})(254.5 \frac{s^2}{m})} = 54.88 \]

This value indicates that gravitational forces dominate and entrainment is small. This is supported by observations by Mehta and Srinivas (Mehta and Srinivas, 1993) which found that entrainment at Richardson numbers larger than ~20 was very low. Using Equation 7.1 from (Mehta, 2013), an entrainment rate per area and time can be calculated.
\[ \varepsilon = \frac{ufm c_{fm}}{(1+63R_i^2)^\frac{3}{4}} = \frac{0.023 m/s}{(1+63(54.88)^2)^\frac{3}{4}} = 8.7 \times 10^{-5} \frac{kg}{m^2 s} \]

This value demonstrates the small magnitude of the entrainment rate, which is less than 0.1 \( \frac{gram}{m^2 s} \) over the flume. This is also supported by the visual observations of only minor, “wispy” entrainment.

The densimetric Rayleigh number is another indication of stability in the system. It is given in Equation 7.2 and relates buoyancy to viscous diffusivity.

\[ R_a = \frac{g \Delta \rho h^3}{\mu k} \]

Where

\[ k = \text{Vertical diffusivity} \]

Evaluated for the fluid mud layer with vertical diffusivity value taken from Kullenberg (Kullenberg, 1971), the number is

\[ R_a = \frac{(9.81 m/s^2)(1004 kg/m^3 - 1000 kg/m^3)(0.1 m)^3}{(0.2 Pa \cdot s)(0.00001 m^2/s)} = \frac{(9.81 m/s^2)(4 kg/m^3)(1e^{-3} m^3)}{(0.2 Pa \cdot s)(1e^{-5} m^2/s)} = 19600 \]

This large value of the Rayleigh number indicates a stable system, which is supported by the minimal entrainment and well defined interface between the two layers.

The post experiment sensor test indicates that some bed had formed in the flume as seen in Figure 4.9. This supports the velocity profile observation. Mud moving as plug flow would likely have had very small to no bed formed during the experiment.
This result also demonstrates that the fluid layer can simultaneously move, entrain, and settle under shear from an overlying fluid flow.

7.3 Settling Experiment

Results from the fluid mud settling experiment demonstrate the formation of a firm, cohesive bed from fluid mud. As mentioned in Section 4.4, the volume of mud pumped into the flume is more than the volume occupied at the end of pumping, indicating some consolidation during pumping. This result likely is attributed to bonds forming between the sediment particles in the mixing tank during the previous flow experiment. The hand drill is not powerful enough to break these bonds, and the strong aggregates that form stayed together.

The initial sensor test confirms the presence of a formed bed, as seen in Figure 4.15. After 10 minutes of settling time, a second sensor test is conducted, with the results seen in Figure 4.16. This sensor test indicates a thicker bed, which indicates further consolidation into a firm bed. The sensors are left in the mud layer overnight; however, the mud consolidates to below the sensors so no sensor results from overnight settling is obtained. A sensor test started after the consolidation is completed, as shown in Figure 4.18, seems to indicate that there is no effective stress at the bottom of the mud, but there is at the top of the mud. These results are possibly erroneous due to clogging of the sensors with sediment particles during the overnight test. Visually, the mud layer consolidated into a thin, firm bed, demonstrating consolidation does occur with fluid mud in a quiescent fluid.
7.4 Experimental Summary

Results from these experiments indicate that at low densities, a bentonite-based fluid mud in a flume does not exhibit a specific yield strength, but rather behaves as a shear thinning fluid. As density increases, this behavior is likely to continue but with a steeper initial slope, more closely resembling a Bingham plastic due to the increased viscosity from the denser mud. The mud flowed as a fluid, with a velocity profile from zero at the bed to a maximum value at the interface. Pressure forces have a larger impact on mud movement than shear from the water velocity. Interfacial waves occur between the water and mud, indicating surface instabilities and mixing. Entrainment and movement occur simultaneously in proportion to the shear stress, with increasing entrainment at increasing velocity. Settling also occurs concurrent with entrainment and movement, with increased settling at lower velocities.

7.5 Model Results Discussion

Results from the computer code applied to the test cases described in Chapters V and VI indicate that the model is adequately representing mud flow within the flume.

7.5.1 Initial fluid mud flow model results

Results from the first fluid mud flow model demonstrate good agreement with the observed data. Fluid mud is simulated flowing downstream under the influence of shear from the water column and slope of the water column. Depth of the fluid mud decreases during the model simulation, as observed during the experiment. Depth at the end of the experiment is predicted to be 0.065 meters, which matches well to the observed depth of 0.07 meters.
The difference in depth from the model compared to the observed data could be due to several factors. The lower density of the fluid mud layer during Experiment 1 may have had a lower viscosity than that specified in the model. This would indicate the need for a density-viscosity relationship within the model.

The model depth rate of change shown in Figure 6.6 also does not exactly match the observed rate of change from the video analysis. This is possibly due to the lack of vertical momentum in the depth-averaged equations. Instead of accelerating down the slope in both the vertical and horizontal directions due to the increase in water pressure, the model only predicts acceleration in the horizontal direction. The two-dimensional model also misses the formation of a vertical eddy which could impact the movement of the fluid mud layer. This limits the model, but overall agreement is good compared to the observed data. The slope difference may also be due to the rate of water flow increase in the model. Figure 7.2 shows the video water surface elevation compared to the modeled. As seen in the figure, there is some unsteadiness in the water surface increase, while the water surface increase is smooth in the model.
The necessary increase in the interfacial friction factor is possibly due to the increased viscosity of the mud layer. Experiments conducted by Dermissis and Partheniades (Dermissis and Partheniades, 1982) involved the interface of fresh and salt water. Salt water has the same viscosity as freshwater, and the interfacial friction factor is calculated based on those values. As the viscosity of the mud is several orders of magnitude larger than water, the interaction at the interface is likely to behave differently than fresh and salt water.

Depth values obtained from the video analysis were consistently 0.015 m higher than those measured during the experiments. Based on this, 0.015 m is subtracted from the video values in Chapter VI where it is compared to the model results. This change is reflected in Figure 6.6.
7.5.2 Increased concentration fluid mud flow model results

Results from the second fluid mud flow model demonstrated good agreement with the observed data. Fluid mud is simulated flowing downstream under the influence of shear from the water column. The simulated depth of 0.135 meters compares well to the observed depth of 0.13 meters at the end of the experiment. Velocities are slightly underestimated, with simulated values of 0.001 m/s compared to the observed values of 0.002 m/s.

It is likely that the decreased velocity caused the difference in the simulated depth. A slower velocity within the mud layer would have reduced the flow of mud out of the depression in the flume.

The decreased velocity could be due to a number of factors. The roughness value of 0.01 may have been too high, though that seems unlikely. A viscosity dependent roughness value or some other form of flow resistance value may provide more accurate results.

7.6 Finite Element Summary

Results from the finite element model runs demonstrate the validity of the two-layer depth-averaged Navier-Stokes equations for describing fluid mud. The model output generally agree with the observed experimental data.

The interfacial friction factor for is increased to 0.7 for the model runs. Based on the model behavior for both cases, that value appears reasonable for this situation. While much higher than Dermissis and Partheniades (Dermissis and Partheniades, 1982) found in their study, it matches well with the observed mud layer behavior. Testing conducted in Chapter V demonstrated that the code was performing correctly in regards to the
interfacial friction factor. The existing interfacial friction factors are based on fresh water and salt water interactions and not an interaction with a much higher viscosity fluid like fluid mud. The small scale of the flume study and the slow, laminar flow of water above the mud layer may have also affected the fluid mud differently than how larger scale and turbulent water affects fluid mud in a natural system. Additional validating of the model to larger scale field studies is necessary to ensure the validity of the code and to further evaluate appropriate values of the interfacial friction factor.

Water surface elevation to drive the model is taken from ADH model output. The velocity is calculated from the ADH depth and a uniform flow throughout the domain based on observed flow. These assumptions limit the model, as there is no feedback to the hydrodynamic code. However, based on the small scale and laminar flow during the flume experiments this assumption does not seem overly limiting. A complete linkage would include the effects of the moving mud bed on the upper water column and the associated effects, which include the changing bottom surface elevation as well as the changing friction and interfacial drag from the moving mud.

Development of the finite element code necessitates the inclusion of the SUPG terms, as described in Chapter V. These terms damped out instabilities seen in the model. These instabilities can present as large oscillations in the solution values. Addition of the SUPG terms prevents these oscillations from occurring and impacting the solution. This allows the model to run smoothly and stably for the duration of the modeled experiments.

The boundary conditions specified in the fluid mud code are somewhat limiting. In general, for infinite domains the lack of a true outflow boundary is not limiting. This is also true for very large domains where the majority of fluid mud will be restricted by
walls or re-entrained back into the water column. However, this boundary condition will need to be addressed to account for mud outflow to create a complete model.

Linkage of the fluid mud code to a hydrodynamic and sediment transport code would generate more complete results. The addition of wave effects, both on formation through liquefaction as well as entrainment and movement, would also increase the usefulness of the model. Inclusion of the temporal terms to the SUPG development will increase the stability and utility of the model. Linking effects from consolidation and water column settling to a hydrodynamic and sediment transport model could potentially improve the results from the fluid mud layer. Adaption of the existing Matlab code to another programming language, such as C or FORTRAN, may increase model speed and decrease model runtimes.

7.7 Effects of Natural Sediments

The experiments conducted here were completed in fresh water with pure bentonite. Natural fluid mud may occur in salt water with various amount of organic material included. Natural mud is also unlikely to be created with a mixer as was done in the lab, but typically will form through either hindered settling or wave liquefaction.

The experiments were conducted with bentonite. Other minerals, such as kaolinite, have different physical and chemical properties, such as cation exchange capacity (CEC). CEC is a measure of the reactivity of the sediment particle, and increasing CEC is related to increasing cohesiveness (Mehta, 2013). Bentonite and other smectites have a high CEC of 80-150 mEq 100g\(^{-1}\), while kaolinite and attapulgite have a low CEC of 3-15 (Forstner and Wittmann, 1981; Horowitz, 1991; Mehta, 2013). The increasing cohesion of bentonite includes more water in the interstitial spaces and a lower
floc density (Mehta and McAnally, 2009). The lower CEC of kaolinite leads to higher floc density, and high concentration behavior that is likely less of a fluid than bentonite.

Organic material is also likely to impact behavior of natural muds. In experiments performed by de Wit and Kranenburg (de Wit and Kranenburg, 1997), a natural mud with 5% organic content resisted liquefaction while two artificial muds both underwent liquefaction under similar conditions. Data from Lake Apopka (Mehta, 2013) shows a remarkable increase in yield stress for the natural mud over pure kaolinite. This difference is apparent at all sediment concentrations, with the Lake Apopka mud having a yield stress of over two orders of magnitude larger than kaolinite at the same concentration.

Based on this information, natural muds are likely to respond less like the Newtonian fluid observed in the laboratory experiments and more like a Bingham plastic type material. This data also reinforces the need for site specific fluid mud data as opposed to a general set of parameters applicable to all muds.
CHAPTER VIII
CONCLUSIONS AND RECOMMENDATIONS

The effects of fluid mud flow under shear are investigated in flume experiments and a numerical model of fluid mud is developed and tested on the flume experiments.

The laboratory experiments demonstrate the flow of fluid mud under shear from the water column. Entrainment is also observed simultaneously with the flow. Fluid mud with higher density is observed to flow more slowly. This is evident from the smaller change in depth of the mud layer during the second flow experiment. Pressure changes due to increased water surface slope have a larger impact on mud movement than shear. Consolidation of a small bed layer is observed, demonstrating simultaneous entrainment, flow, and consolidation. The flow of mud is observed at very low water velocities, which indicates that fluid mud behaves as a fluid and not a Bingham plastic at the studied densities.

The finite element method is applied to the depth averaged Navier-Stokes equations to describe the movement of fluid mud. The finite element code is developed to include effects from friction, shear from the water column, internal viscosity of the mud, settling, consolidation, and entrainment.

Application of the fluid mud code to the experimental results demonstrates good agreement with the observed data. The model simulates flow of fluid mud driven by
shear stress from an overlying fluid along with the slope of the water surface, and the modeled velocities in the mud sufficiently match the observed velocities.

Further development of the fluid mud code will result in a more robust and accurate model, specifically effects from waves and interactions with the water column and bed, as well as better boundary conditions. Validation to field studies is needed to test model robustness and variable conditions and is necessary to ensure the validity of the code and to further evaluate appropriate values of the interfacial friction factor. A better density to viscosity relationship is needed to remove a necessary tuning step. A sediment mass balance is needed to confirm conservation of mass within the model. Linkage to an existing code will likely necessitate the use of an implicit solution routine which will allow for the inclusion of the temporal terms in the SUPG equations and an increase in stability. Full linkage to an existing hydrodynamic and sediment transport model and inclusion of better boundary conditions will allow for system-wide evaluation of fluid mud and sedimentation effects within a waterway and will assist in alternative evaluation.
REFERENCES


Krone R.B. (1962) Flume studies of the transport of sediment in estuarial shoaling processes, University of California, Berkeley, CA.


McNown J.S., Lin P.N. (1952) Sediment Concentration and Fall Velocity, 2nd Midwestern Conference on Fluid Mechanics, Ohio State University, Columbus, OH. pp. 401-411.


APPENDIX A

LABORATORY EXPERIMENTS

SUPPORTING MATERIALS
A.1 Weir Calibration for Flume

May 17 2010

Weir test

5 gallons = 0.01892705 m³

Height of notch

3.346 ft

<table>
<thead>
<tr>
<th>Water Height (ft)</th>
<th>H (m)</th>
<th>T (s)</th>
<th>Q (m³/s)</th>
<th>Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.487</td>
<td>0.0429768</td>
<td>36.88</td>
<td>0.000513206</td>
<td></td>
</tr>
<tr>
<td>3.487</td>
<td>0.0429768</td>
<td>36.09</td>
<td>0.00052444</td>
<td></td>
</tr>
<tr>
<td>3.495</td>
<td>0.0454152</td>
<td>32.09</td>
<td>0.000589811</td>
<td></td>
</tr>
<tr>
<td>3.495</td>
<td>0.0454152</td>
<td>32.87</td>
<td>0.000575815</td>
<td></td>
</tr>
<tr>
<td>3.633</td>
<td>0.0874776</td>
<td>6.54</td>
<td>0.002894044</td>
<td></td>
</tr>
<tr>
<td>3.633</td>
<td>0.0874776</td>
<td>6.69</td>
<td>0.002829155</td>
<td></td>
</tr>
<tr>
<td>3.625</td>
<td>0.0850392</td>
<td>6.78</td>
<td>0.0027916</td>
<td></td>
</tr>
</tbody>
</table>

Stopwatch

Water Height (ft) H (m) T (s) Q (m³/s) Test

\[ y = 1.0907x^{2.4337} \]

\[ R^2 = 0.9995 \]

Figure A.1 Weir Calibration Curve
A.2 GE/Druck Pore Pressure Sensor Specifications

Figure A.2 Pore Pressure Sensor Specs

(a)-(d) Pages 1 through 4 of document
Introduction

Druck’s experience with silicon diaphragm micro machining technology is considerable.

For more than 15 years a commitment to continuous research and development in this field using the very latest techniques and equipment has culminated in some remarkable achievements in pressure transducer design and performance.

The miniature series is a typical example of the benefits of such work. A range of high accuracy sensors complete with thermal compensation in a miniature package providing maximum performance with minimal size and weight.

The silicon diaphragm is intricately micro machined, and semi-conductor strain gauges are diffused into the substrate to become an atomic part of the diaphragm. Each gauge is connected to form a Wheatstone bridge configuration, which is subsequently terminated to offset and thermal compensation circuitry.

High signal outputs, excellent linearity, negligible hysteresis and good repeatability performance with considerable improvements in long term stability are the benefits of using Druck miniature pressure transducers.

Your Specific Requirements

In addition to the sensors shown, Druck have the engineering capability to design pressure transducers to specific individual requirements. By careful consideration of the configuration, environments, compatibility and other important performance characteristics, our engineering team can design, build and exhaustively test instruments for your needs.

Please contact our Sales Office for further information.

Figure A.2 (Continued)
## Miniature Series Specification

<table>
<thead>
<tr>
<th>Feature</th>
<th>POCR 110 (c)</th>
<th>POCR 110A (c)</th>
<th>POCR 110B (c)</th>
<th>POCR 110C (c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Purpose</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Operating Pressure Ranges (I)</td>
<td>75 mbar, 1.5, 2, 3, 3.5, 5, 7, 10, 15, 20, and 30 bar gauge or differential (I)</td>
<td>75 mbar, 15 mbar, 25 mbar, 30 mbar, 50 mbar, 70 mbar, 3.5 mbar, 7 mbar, 10 mbar, 15 mbar, 20 mbar, and 30 bar gauge or differential (I)</td>
<td>75 mbar, 35 mbar, 50 mbar, 75 mbar, 30 mbar, 75 mbar, 5 mbar, 7.5 mbar, 10 mbar, 15 mbar, 20 mbar, and 30 bar gauge or differential (I)</td>
<td>75 mbar, 35 mbar, 50 mbar, 75 mbar, 30 mbar, 75 mbar, 5 mbar, 7.5 mbar, 10 mbar, 15 mbar, 20 mbar, and 30 bar gauge or differential (I)</td>
</tr>
<tr>
<td>Overpressure limit with negligible effect on calibration (I)</td>
<td>3 x for all ranges (10) x for range 1 and 2 bar range on reference side</td>
<td>10 x for range 1 and 2 bar range on reference side</td>
<td>10 x for range 1 and 2 bar range on reference side</td>
<td>10 x for range 1 and 2 bar range on reference side</td>
</tr>
<tr>
<td>Reference Pressure Medium</td>
<td>Fluids, compatible with silicon, titanium and epoxy adhesives (and ceramic porous plate for POCR 110C)</td>
<td>Dry, non-conducting gases</td>
<td>Dry, non-conducting gases</td>
<td>Dry, non-conducting gases</td>
</tr>
<tr>
<td>Excitation Voltage</td>
<td>10 mA nominal</td>
<td>100 mA nominal</td>
<td>100 mA nominal</td>
<td>100 mA nominal</td>
</tr>
<tr>
<td>Zero Offset (a)</td>
<td>±15 mV maximum</td>
<td>±50 mV maximum</td>
<td>±50 mV maximum</td>
<td>±50 mV maximum</td>
</tr>
<tr>
<td>Span Setting</td>
<td>±10% of nominal output</td>
<td>±10% of nominal output</td>
<td>±10% of nominal output</td>
<td>±10% of nominal output</td>
</tr>
<tr>
<td>Output Impedance (Nominal)</td>
<td>1500 Ω</td>
<td>1000 Ω</td>
<td>1000 Ω</td>
<td>1000 Ω</td>
</tr>
<tr>
<td>Load Impedance</td>
<td>5000 Ω</td>
<td>5000 Ω</td>
<td>5000 Ω</td>
<td>5000 Ω</td>
</tr>
<tr>
<td>Resolution to 1 part in 2 parts of ± B.S.L.</td>
<td>±0.1% ± B.S.L.</td>
<td>±0.1% ± B.S.L.</td>
<td>±0.1% ± B.S.L.</td>
<td>±0.1% ± B.S.L.</td>
</tr>
<tr>
<td>Operating Temperature Range</td>
<td>-20°C to +60°C (I)</td>
<td>-40°C to +80°C (I)</td>
<td>-40°C to +80°C (I)</td>
<td>-40°C to +80°C (I)</td>
</tr>
<tr>
<td>Temperature Range</td>
<td>±5% ± total error bond 10°C to 60°C (I)</td>
<td>±0.5% ± total error bond 5°C to 35°C (I)</td>
<td>±0.5% ± total error bond 5°C to 35°C (I)</td>
<td>±0.5% ± total error bond 5°C to 35°C (I)</td>
</tr>
<tr>
<td>Natural Frequency (Mechanical)</td>
<td>75 kHz for 1 bar range increasing to 850 kHz for 60 bar range</td>
<td>75 kHz for 1 bar range increasing to 850 kHz for 60 bar range</td>
<td>75 kHz for 1 bar range increasing to 850 kHz for 60 bar range</td>
<td>75 kHz for 1 bar range increasing to 850 kHz for 60 bar range</td>
</tr>
<tr>
<td>Acceleration Sensitivity (a)</td>
<td>0.002% F.S. for 1 bar decreasing to 0.0005% F.S. for 60 bar range</td>
<td>0.002% F.S. for 1 bar decreasing to 0.0005% F.S. for 60 bar range</td>
<td>0.002% F.S. for 1 bar decreasing to 0.0005% F.S. for 60 bar range</td>
<td>0.002% F.S. for 1 bar decreasing to 0.0005% F.S. for 60 bar range</td>
</tr>
<tr>
<td>Mechanical Shock</td>
<td>100g for 1 ms in each of three mutually perpendicular axes will not affect calibration</td>
<td>100g for 1 ms in each of three mutually perpendicular axes will not affect calibration</td>
<td>100g for 1 ms in each of three mutually perpendicular axes will not affect calibration</td>
<td>100g for 1 ms in each of three mutually perpendicular axes will not affect calibration</td>
</tr>
<tr>
<td>Weight (Nominal)</td>
<td>12 grams</td>
<td>12 grams</td>
<td>12 grams</td>
<td>12 grams</td>
</tr>
<tr>
<td>Electrical Connection</td>
<td>1 metre shielded integral cable (I)</td>
<td>20 cm integral cable connects transducer and compensation package; free socket supplied</td>
<td>20 cm integral cable connects transducer and compensation package; free socket supplied</td>
<td>20 cm integral cable connects transducer and compensation package; free socket supplied</td>
</tr>
</tbody>
</table>

Key to table above:
1. Other pressure units can be specified, e.g. psi, kPa, etc.
2. For absolute pressures a vacuum can be pumped on the reference side.
3. The transducers can be used in a b direction with differential made up to ± 1 bar.
5. ±0.0% ± B.S.L. available.
6. Temperature range can be extended.
7. For special applications it is possible to give improved temperature effects over a wider temperature range.
8. Compliance with the requirements set by the British Standards Institution.
Figure A.2 (Continued)
APPENDIX B

FINITE ELEMENT FLUID MUD COMPUTER CODE

FOR EXPERIMENT 1
%Exp 1
%wse init=0.23 m
%wse max=0.25 m
%fm ioh=0.13 m
%Qmax=1.2e-3 m3/s
%TP=720
%rohfm=1004 kg/m3

%Exp 2
%wse init=0.24 m
%wse max=0.25 m
%fm ioh=0.15 m
%Qmax=1e-3 m3/s
%TP=840
%rhofm=1022 kg/m3

clear output;

%TURN ON FLUID MUD
%0 for no mud, 1 for mud
FM_FLAG=1;

%need to load element and node matrices, taken from 3dm file
%elements - element number|node1|node2|node3|material number
num_elem=length(elements);
%nodes - node number|X|Y|Z
num_node=length(nodes);

rhowater=1000; %kg/m^3
rhofm=1004; %kg/m^3
rhosed=2650; %kg/m^3
grav=9.81; %m/s^2

mu_w=0.001; %eddy viscosity for water 0.001
mufm=0.2; %Pa*s (Kusuda 1993)

concfm=(rhofm-rhowater)/(rhosed-rhowater)*(rhosed);

man_w=0.01; %roughness for flow
man_mud=0.01; %manning roughness value, need for SUPG formulation

fric_typemud=1; %1 is manning, 2 is Soulsby
C_d=1; %mannings units 1 for SI, 1.486 for English but leave SI
% interfacial friction constant, depends on density diff and ReFr^2
% assume constant 0.03 for now
% f_interface=0.1;
f_interface=0.7;

filename='Exp1_adhwse_031714_12';

alpha=0.5; % Petrov Galerkin coefficient

% for test case 3, get omega T
% omegat=2*pi/44640;

% settling factors
% specific to sediment type, site specific
C1=0.20;
aw=0.20;
bw=5;
mw=1.85;
nw=1.75;

% consolidation factors
% range 15 - 31 kg/m3
Conc_s1=20;
% range 205 - 1000
Conc_s2=205;
% range 15 - 210
Conc_t=160;
% range 6e-5 - 1e-4  0.0001
wsc1=0.0005;
% range 3e-6 - 7e-6  0.000007
wsc2=0.000007;
% range 3 - 6
m_t=3;
% range 13 - 18
n_t=13;
f_t=exp(-(concfm/Conc_t)^n_t);
wscfm=wsc1*exp(-(concfm/Conc_s1))*f_t+wsc2*(1-(concfm/Conc_s2))^(m_t)*(1-f_t);

% specify time parameters
T0=0;
T=T0;

% test various flow speed increases
%for line 1700, flow interp
%will be 30, 60, 90
T_ramp=90;

if T_ramp==30
    ADH_wse=ADH_wse_30;
elseif T_ramp==60
    ADH_wse=ADH_wse_60;
elseif T_ramp==90
    ADH_wse=ADH_wse_90;
else
    fclose('all');
    errordlg('T_ramp out of bounds','ERROR')
    return;
end

%timestep size
maxdeltaT= 0.5;
initdeltaT=0.1;

TF=750;

%min depth for wetting and drying
min_depth=0.001;

%specify output time step
outstep=0.5;
outtime=T0+outstep;

%check ioh for minimum depth
%water initial
h_water=ioh_water;
for i=1:num_node
    if h_water(i)==0
        h_water(i)=min_depth;
    end
end

%specify zero initial velocity
u_water=zeros(num_node,1);
v_water=zeros(num_node,1);
%water column concentration - for settling
conc_water=zeros(num_node,1);
% initial velocity for output
vel_water(:,1)=u_water;
vel_water(:,2)=v_water;
vel_water(:,3)=0;

% mud initial
h_mud=ioh_mud;
for i=1:num_node
    if h_mud(i)==0
        h_mud(i)=min_depth;
    end
end

if FM_FLAG==1
    % specify zero initial velocity
    u_mud=zeros(num_node,1);
    v_mud=zeros(num_node,1);
    % initial velocity for output
    vel_mud(:,1)=u_mud;
    vel_mud(:,2)=v_mud;
    vel_mud(:,3)=0;
end

% count loops
count=0;

% create output files with headers
depthwaterfile='_dep_water.dat';
velwaterfile='_ovl_water.dat';
depthmudfile='_dep_mud.dat';
velmudfile='_ovl_mud.dat';
% entrainfile='_em_mud.dat';

if exist(strcat(filename,depthwaterfile),'file')==2
    errordlg('File already exists!','ERROR')
    return;
end

depthwater_output=fopen(strcat(filename,depthwaterfile),'w');
velocitywater_output=fopen(strcat(filename,velwaterfile)','w');

if FM_FLAG==1
    depthmud_output=fopen(strcat(filename,depthmudfile),'w');
end
velocitymud_output=fopen(strcat(filename,velmudfile),'w');
%       entrain_output=fopen(strcat(filename,entrainfile),'w');
end

%print header info to files
fprintf(depthwater_output,'
DATASET
OBJTYPE "mesh2d"
BEGSCL
ND %d
nNC
%d
NAME "Water Depth %s"nTIMEUNITSECONDS\n',num_node,num_elem,filename);
fprintf(velocitywater_output,'
DATASET
OBJTYPE "mesh2d"
BEGVEC
ND %d
nNC
%d
NAME "Water Velocity %s"nTIMEUNITSECONDS\n',num_node,num_elem,filename);
if FM_FLAG==1
    fprintf(depthmud_output,'
DATASET
OBJTYPE "mesh2d"
BEGSCL
ND %d
nNC
%d
NAME "Mud Depth %s"nTIMEUNITSECONDS\n',num_node,num_elem,filename);
    fprintf(velocitymud_output,'
DATASET
OBJTYPE "mesh2d"
BEGVEC
ND %d
nNC
%d
NAME "Mud Velocity %s"nTIMEUNITSECONDS\n',num_node,num_elem,filename);
    % fprintf(entrain_output,'
DATASET
OBJTYPE "mesh2d"
BEGSCL
ND %d
nNC
%d
NAME "Entrainment %s"nTIMEUNITSECONDS\n',num_node,num_elem,filename);
end

%print initial depth timestep and values
fprintf(depthwater_output,'TS 0 %.8e\n',T);
fprintf(depthwater_output,'%.8e\n',h_water);

%print initial velocity timestep and values
fprintf(velocitywater_output,'TS 0 %.8e\n',T);
fprintf(velocitywater_output,'%8e\t%8e\t%8e\n',transpose(vel_water));
if FM_FLAG==1
    %print initial depth timestep and values
    fprintf(depthmud_output,'TS 0 %.8e\n',T);
    fprintf(depthmud_output,'%8e\n',h_mud);

    %print initial velocity timestep and values
    fprintf(velocitymud_output,'TS 0 %.8e\n',T);
    fprintf(velocitymud_output,'%8e\t%8e\t%8e\n',transpose(vel_mud));

    %print initial entrainment timestep and values
    % fprintf(entrain_output,'TS 0 %.8e\n',T);
    % fprintf(entrain_output,'%8e\n',entrain);
end

%calculate minimum deltax and deltay from the grid
%only have to do this initially, since no adaption, the grid is constant
min_deltax=9999999;

for i=1:num_elem

    %get nodes for element i
    node1=elements(i,2);
    node2=elements(i,3);
    node3=elements(i,4);

    %get x, y, and z for the nodes for element i
    %node1
    x1=nodes(node1,2);
    y1=nodes(node1,3);

    %node2
    x2=nodes(node2,2);
    y2=nodes(node2,3);

    %node3
    x3=nodes(node3,2);
    y3=nodes(node3,3);

    %calculate distance
    distance(1)=sqrt((x2-x1)^2+(y2-y1)^2);
    distance(2)=sqrt((x2-x3)^2+(y2-y3)^2);
    distance(3)=sqrt((x3-x1)^2+(y3-y1)^2);

    %get minimum values
    min_deltax_inc=min(distance);

    %check for global min
    if min_deltax_inc<min_deltax
        min_deltax=min_deltax_inc;
    end

end

%loop over time
while T<=TF
    %check for time parameters
    max_CFL=0.25;

    hwater_max=max(h_water);
    %since cwater>>cmud
mindeltaT = max_CFL * min_deltax / (sqrt(grav * hwater_max));

% use initial time step for first 1 second, then only increase by the initial time step during the next 9 seconds, then use regular time step
if T < T0 + 0.1
    deltaT = initdeltaT;
elseif T < T0 + 10
    if mindeltaT < maxdeltaT
        deltaT = mindeltaT;
    else
        deltaT = min(deltaT + initdeltaT, maxdeltaT);
    end

    if deltaT < 0.0000001
        fclose('all');
        deltaT
        return;
    end
else
    if mindeltaT < maxdeltaT
        deltaT = mindeltaT;
    else
        deltaT = maxdeltaT;
    end

    % exit code if time step gets too small
end
if deltaT < 0.0000001
    fclose('all');
    deltaT
    return;
end

% set up solution matrices
continuitywater_lhs = zeros(num_node);
xmomenwater_lhs = zeros(num_node);
ymomenwater_lhs = zeros(num_node);

% set up and zero right hand side of the equation matrices
continuitywater_rhs = zeros(num_node, 1);
xmomenwater_rhs = zeros(num_node, 1);
ymomenwater_rhs=zeros(num_node,1);

if FM_FLAG==1
    %mud solution matrices
    continuitymud_lhs=zeros(num_node);
    xmomenmud_lhs=zeros(num_node);
    ymomenmud_lhs=zeros(num_node);

    %set up and zero right hand side of the equation matrices
    continuitymud_rhs=zeros(num_node,1);
    xmomenmud_rhs=zeros(num_node,1);
    ymomenmud_rhs=zeros(num_node,1);

    txxhdxmud=zeros(num_node,1);
    txyhdxmud=zeros(num_node,1);
    txyhdymud=zeros(num_node,1);
    tyyhdymud=zeros(num_node,1);

    %zero entrainment for output routine
    %    entrain=zeros(num_node,1);
    end

%check for t-interpolation from ADH WSE output
j=floor(T);
T1=j+1;
T2=j+2;

%loop over element
for i=1:num_elem

    %get node info
    node1=elements(i,2);
    node2=elements(i,3);
    node3=elements(i,4);

    x1=nodes(node1,2);
    x2=nodes(node2,2);
    x3=nodes(node3,2);

    y1=nodes(node1,3);
    y2=nodes(node2,3);
    y3=nodes(node3,3);
z1=nodes(node1,4);
z2=nodes(node2,4);
z3=nodes(node3,4);

uw1=u_water(node1);
uw2=u_water(node2);
uw3=u_water(node3);
uw_e=(1/3)*(uw1+uw2+uw3);

vw1=v_water(node1);
vw2=v_water(node2);
vw3=v_water(node3);
vw_e=(1/3)*(vw1+vw2+vw3);

hw1=h_water(node1);
hw2=h_water(node2);
hw3=h_water(node3);
hw_e=(1/3)*(hw1+hw2+hw3);

%celerity and average
cwater1=sqrt(grav*hw1);
cwater2=sqrt(grav*hw2);
cwater3=sqrt(grav*hw3);
cwater_bar=(1/3)*(cwater1+cwater2+cwater3);

%for SUPG terms
awater_bar=sqrt(uw_e*uw_e+vw_e*vw_e+cwater_bar*cwater_bar);

if FM_FLAG==1
    umud1=u_mud(node1);
    umud2=u_mud(node2);
    umud3=u_mud(node3);
    umud_e=(1/3)*(umud1+umud2+umud3);

    vmud1=v_mud(node1);
    vmud2=v_mud(node2);
    vmud3=v_mud(node3);
    vmud_e=(1/3)*(vmud1+vmud2+vmud3);

    hmud1=h_mud(node1);
    hmud2=h_mud(node2);
    hmud3=h_mud(node3);
    hmud_e=(1/3)*(hmud1+hmud2+hmud3);
%cerelity and average mud
\[ \text{cmud1} = \sqrt{\frac{((\rho_{\text{ofm}} - \rho_{\text{water}}) / \rho_{\text{ofm}}) \cdot g \cdot ((h_{\text{mud1}} \cdot h_{\text{w1}}) / (h_{\text{mud1}} + h_{\text{w1}}))}{}}; \]
\[ \text{cmud2} = \sqrt{\frac{((\rho_{\text{ofm}} - \rho_{\text{water}}) / \rho_{\text{ofm}}) \cdot g \cdot ((h_{\text{mud2}} \cdot h_{\text{w2}}) / (h_{\text{mud2}} + h_{\text{w2}}))}{}}; \]
\[ \text{cmud3} = \sqrt{\frac{((\rho_{\text{ofm}} - \rho_{\text{water}}) / \rho_{\text{ofm}}) \cdot g \cdot ((h_{\text{mud3}} \cdot h_{\text{w3}}) / (h_{\text{mud3}} + h_{\text{w3}}))}{}}; \]
\[ \text{cmud}_{\text{bar}} = (1/3) \cdot (\text{cmud1} + \text{cmud2} + \text{cmud3}); \]

%for SUPG terms
\[ \text{amud}_{\text{bar}} = \sqrt{\text{umud}_e \cdot \text{umud}_e + \text{vmud}_e \cdot \text{vmud}_e + \text{cmud}_{\text{bar}} \cdot \text{cmud}_{\text{bar}}}; \]

%individual velocity differences
\[ \delta_u = u_{\text{w}} - u_{\text{mud}}; \]
\[ \delta_v = v_{\text{w}} - v_{\text{mud}}; \]

%for entrainment and surface shear, diff between fluid mud and water velocities
%scale to element size?!?!?!
\[ \text{delta}_u_{\text{mag}} = \sqrt{\delta_u^2 + \delta_v^2}; \]

%for friction type 2, Soulsby and Clarke
\[ \text{rec}_{\text{bedmud}} = h_{\text{mud}} \cdot \sqrt{u_{\text{mud}} \cdot u_{\text{mud}} + v_{\text{mud}} \cdot v_{\text{mud}}} / (u_{\text{fm}} / \rho_{\text{ofm}}); \]
\[ \text{cds}_{\text{bedmud}} = 0.0001615 \cdot \exp(6 \cdot \text{rec}_{\text{bedmud}}^{-0.08}); \]
else
\[ h_{\text{mud}}_1 = \text{ioh}_{\text{mud}}(\text{node1}); \]
\[ h_{\text{mud}}_2 = \text{ioh}_{\text{mud}}(\text{node2}); \]
\[ h_{\text{mud}}_3 = \text{ioh}_{\text{mud}}(\text{node3}); \]
end

%water column concentrations for settling
\[ \text{conc}_{\text{water1}} = \text{conc}_{\text{water}}(\text{node1}); \]
\[ \text{conc}_{\text{water2}} = \text{conc}_{\text{water}}(\text{node2}); \]
\[ \text{conc}_{\text{water3}} = \text{conc}_{\text{water}}(\text{node3}); \]
\[ \text{conc}_{\text{water}}_e = 1/3 \cdot (\text{conc}_{\text{water1}} + \text{conc}_{\text{water2}} + \text{conc}_{\text{water3}}); \]

%assume free settling, stokes velocity, d=6 microns
\[ \text{wsf} = 0.0000196; \ %m/s \]
% not actually used for anything yet
%     if cwater_e > C1
%         ws=aw*(cwater_e^nw)/((cwater_e^2+bw^2)^mw);
%     end

% calculate integral terms
edge1(1)=x2-x1;
edge1(2)=y2-y1;

    normal_edge1x=edge1(2);
    normal_edge1y=-edge1(1);

edge2(1)=x3-x2;
edge2(2)=y3-y2;

    normal_edge2x=edge2(2);
    normal_edge2y=-edge2(1);

edge3(1)=x1-x3;
edge3(2)=y1-y3;

    normal_edge3x=edge3(2);
    normal_edge3y=-edge3(1);

area=abs((1/2)*(x1*(y2-y3)+x2*(y3-y1)+x3*(y1-y2)));

l=sqrt(area);
dxdxi=x2-x1;
dxdeta=x3-x1;
dydx=y2-y1;
dyde=y3-y1;

det_jac=dxdxi*dydx-dydx*dxdeta;
dphiidx(1)=(dydx-dydx)/det_jac;
dphiidx(2)=(dydx)/det_jac;
dphiidx(3)=(-dydx)/det_jac;

dphidy(1)=(dxdeta-dxdxi)/det_jac;
dphidy(2)=(-dxdeta)/det_jac;
dphidy(3)=(dxdxi)/det_jac;
%Turn off water calculations since using ADH output
% continuity terms
% continuitywater_lhs(node1,node1)=continuitywater_lhs(node1,node1)+(det_jac)/6;
% continuitywater_lhs(node2,node2)=continuitywater_lhs(node2,node2)+(det_jac)/6;
% continuitywater_lhs(node3,node3)=continuitywater_lhs(node3,node3)+(det_jac)/6;
% contwater_h(1)=(det_jac/6)*hw1;
% contwater_h(2)=(det_jac/6)*hw2;
% contwater_h(3)=(det_jac/6)*hw3;
%
uhwater_total=(1/12)*(uw1*hw1+uw2*hw2+uw3*hw3)+(1/24)*(uw1*hw2+uw1*hw3+uw2*hw
1+uw2*hw3+uw3*hw1+uw3*hw2);
% contwater_uh(1)=deltaT*dphidx(1)*det_jac*(uhwater_total);
% contwater_uh(2)=deltaT*dphidx(2)*det_jac*(uhwater_total);
% contwater_uh(3)=deltaT*dphidx(3)*det_jac*(uhwater_total);
%
vhwater_total=(1/12)*(vw1*hw1+vw2*hw2+vw3*hw3)+(1/24)*(vw1*hw2+vw1*hw3+vw2*hw
1+vw2*hw3+vw3*hw1+vw3*hw2);
% contwater_vh(1)=deltaT*dphidy(1)*det_jac*(vhwater_total);
% contwater_vh(2)=deltaT*dphidy(2)*det_jac*(vhwater_total);
% contwater_vh(3)=deltaT*dphidy(3)*det_jac*(vhwater_total);
%
% sum terms
%
continuitywater_rhs(node1)=continuitywater_rhs(node1)+contwater_h(1)+contwater_uh(1)+contwater_vh(1);
% continuitywater_rhs(node2)=continuitywater_rhs(node2)+contwater_h(2)+contwater_uh(2)+contwater_vh(2);
% continuitywater_rhs(node3)=continuitywater_rhs(node3)+contwater_h(3)+contwater_uh(3)+contwater_vh(3);
% % x direction momentum terms
% xmomenwater_lhs(node1,node1)=xmomenwater_lhs(node1,node1)+(det_jac)/6;
% xmomenwater_lhs(node2,node2)=xmomenwater_lhs(node2,node2)+(det_jac)/6;
% xmomenwater_lhs(node3,node3)=xmomenwater_lhs(node3,node3)+(det_jac)/6;
%   x_uhwater(1)=(det_jac/6)*uw1*hw1;
%   x_uhwater(2)=(det_jac/6)*uw2*hw2;
%   x_uhwater(3)=(det_jac/6)*uw3*hw3;
% 
% %Cij for duuh/dx
% uuhwater1=(1/20)*(uw1*uw1*hw1)+(1/60)*(uw1*uw2*hw1+uw1*uw3*hw1+uw1*uw1*hw2+uw
  1*uw2*hw2+uw1*uw1*hw3+uw1*uw3*hw3)+(1/120)*(uw1*uw3*hw2+uw1*uw2*hw3);
% uuhwater2=(1/20)*(uw2*uw2*hw2)+(1/60)*(uw2*uw1*hw1+uw2*uw2*hw1+uw2*uw1*hw2+uw
  2*uw3*hw2+uw2*uw2*hw3+uw2*uw3*hw3)+(1/120)*(uw2*uw3*hw1+uw2*uw1*hw3);
% uuhwater3=(1/20)*(uw3*uw3*hw3)+(1/60)*(uw3*uw1*hw1+uw3*uw3*hw1+uw3*uw2*hw2+uw
  3*uw3*hw2+uw3*uw1*hw3+uw3*uw2*hw3)+(1/120)*(uw3*uw2*hw1+uw3*uw1*hw2);
% 
% x_duuhdxwater(1)=(deltaT*det_jac)*dphidx(1)*(uuhwater1+uuhwater2+uuhwater3);
% x_duuhdxwater(2)=(deltaT*det_jac)*dphidx(2)*(uuhwater1+uuhwater2+uuhwater3);
% x_duuhdxwater(3)=(deltaT*det_jac)*dphidx(3)*(uuhwater1+uuhwater2+uuhwater3);
%   x_duuhdxwater(1)=0;
%   x_duuhdxwater(2)=0;
%   x_duuhdxwater(3)=0;
% 
% %Cij for duvh/dy
% uvhwater1=(1/20)*(uw1*vw1*hw1)+(1/60)*(uw1*vw2*hw1+uw1*vw3*hw1+uw1*vw1*hw2+uw
  1*vw2*hw2+uw1*vw1*hw3+uw1*vw3*hw3)+(1/120)*(uw1*vw3*hw2+uw1*vw2*hw3);
% uvhwater2=(1/20)*(uw2*vw2*hw2)+(1/60)*(uw2*vw1*hw1+uw2*vw2*hw1+uw2*vw1*hw2+uw
  2*vw3*hw2+uw2*vw2*hw3+uw2*vw3*hw3)+(1/120)*(uw2*vw3*hw1+uw2*vw1*hw3);
% uvhwater3=(1/20)*(uw3*vw3*hw3)+(1/60)*(uw3*vw1*hw1+uw3*vw3*hw1+uw3*vw2*hw2+uw
  3*vw3*hw2+uw3*vw1*hw3+uw3*vw2*hw3)+(1/120)*(uw3*vw2*hw1+uw3*vw1*hw2);
x_duvhdywater(1)=(deltaT*det_jac)*dphidy(1)*(uvhwater1+uvhwater2+uvhwater3);

x_duvhdywater(2)=(deltaT*det_jac)*dphidy(2)*(uvhwater1+uvhwater2+uvhwater3);

x_duvhdywater(3)=(deltaT*det_jac)*dphidy(3)*(uvhwater1+uvhwater2+uvhwater3);

% x_duvhdywater(1)=0;
% x_duvhdywater(2)=0;
% x_duvhdywater(3)=0;

hwatertotal=(1/12)*(hw1*hw1+hw1*hw2+hw1*hw3+hw2*hw2+hw2*hw3+hw3*hw3);

% x_dhdxwater(1)=(deltaT*grav*det_jac/2)*dphidx(1)*(hwatertotal);
% x_dhdxwater(2)=(deltaT*grav*det_jac/2)*dphidx(2)*(hwatertotal);
% x_dhdxwater(3)=(deltaT*grav*det_jac/2)*dphidx(3)*(hwatertotal);

% edge1, nodes 1 to 2

e11dhdxwater=(deltaT*grav/2)*(normal_edge1x)*((1/4)*hw1*hw1+(1/6)*hw1*hw2+(1/12)*hw2*hw2);

% edge2, nodes 2 to 3

e21dhdxwater=(deltaT*grav/2)*(normal_edge2x)*((1/4)*hw2*hw2+(1/6)*hw2*hw3+(1/12)*hw3*hw3);

% edge3, nodes 3 to 1

e31dhdxwater=(deltaT*grav/2)*(normal_edge3x)*((1/4)*hw3*hw3+(1/6)*hw3*hw1+(1/12)*hw1*hw1);

% need gh dz/dx terms

% zhtotal=(1/12)*(z1*h1+z2*h2+z3*h3)+(1/24)*(z1*h2+z1*h3+z2*h1+z2*h3+z3*h1+z3*h2);
% dzdxwater=dphidx(1)*(z1+hmud1)+dphidx(2)*(z2+hmud2)+dphidx(3)*(z3+hmud3);
% x_dzdxwater(1)=(deltaT*det_jac*grav)*dzdxwater*(1/12*hw1+1/24*hw2+1/24*hw3);
% x_dzdxwater(2)=(deltaT*det_jac*grav)*dzdxwater*(1/24*hw1+1/12*hw2+1/24*hw3);
% x_dzdxwater(3)=(deltaT*det_jac*grav)*dzdxwater*(1/24*hw1+1/24*hw2+1/12*hw3);
%   %X direction Viscosity terms, txx*h, txy*h
%   dudxwater=dphidx(1)*uw1+dphidx(2)*uw2+dphidx(3)*uw3;
%   dvdxwater=dphidx(1)*vw1+dphidx(2)*vw2+dphidx(3)*vw3;
%   dudywater=dphidy(1)*uw1+dphidy(2)*uw2+dphidy(3)*uw3;
%   dvdywater=dphidy(1)*vw1+dphidy(2)*vw2+dphidy(3)*vw3;
%
% dtxxhdxwater(1)=(1/rhowater)*2*mu_w*(deltaT*det_jac)*dphidx(1)*dudxwater*(1/6)*(hw1+hw2+hw3);
% dtxxhdxwater(2)=(1/rhowater)*2*mu_w*(deltaT*det_jac)*dphidx(2)*dudxwater*(1/6)*(hw1+hw2+hw3);
% dtxxhdxwater(3)=(1/rhowater)*2*mu_w*(deltaT*det_jac)*dphidx(3)*dudxwater*(1/6)*(hw1+hw2+hw3);
% %
% dtxyhdywater(1)=(1/rhowater)*mu_w*(deltaT*det_jac)*dphidy(1)*(dvdxwater+dudywater)*(1/6)*(hw1+hw2+hw3);
% dtxyhdywater(2)=(1/rhowater)*mu_w*(deltaT*det_jac)*dphidy(2)*(dvdxwater+dudywater)*(1/6)*(hw1+hw2+hw3);
% dtxyhdywater(3)=(1/rhowater)*mu_w*(deltaT*det_jac)*dphidy(3)*(dvdxwater+dudywater)*(1/6)*(hw1+hw2+hw3);
% %
% dtxxhdxwater(1)=0;
% dtxxhdxwater(2)=0;
% dtxxhdxwater(3)=0;
% %
% dtxyhdywater(1)=0;
% dtxyhdywater(2)=0;
% dtxyhdywater(3)=0;
% %
% need to set to check for FM and interfacial shear
if FM_FLAG==1
if hmud_e>=2*min_depth
\[ \begin{align*}
\tau_{\text{interface water}}(1) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( f_{\text{interface}} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \right) \left( \delta u \delta u_{\text{mag}} / 8 \right); \\
\tau_{\text{interface water}}(2) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( f_{\text{interface}} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \right) \left( \delta u \delta u_{\text{mag}} / 8 \right); \\
\tau_{\text{interface water}}(3) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( f_{\text{interface}} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \right) \left( \delta u \delta u_{\text{mag}} / 8 \right);
\end{align*} \]
else %bed roughness
\[ \begin{align*}
\tau_{\text{interface water}}(1) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( \text{grav} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \frac{\text{man}_w}{\text{man}_w} \frac{\text{uw}_e}{\sqrt{\text{uw}_e \text{uw}_e + \text{vw}_e \text{vw}_e}} / \left( \frac{C_d \cdot C_d}{(\text{hw}_e)^{1/3}} \right) \right); \\
\tau_{\text{interface water}}(2) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( \text{grav} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \frac{\text{man}_w}{\text{man}_w} \frac{\text{uw}_e}{\sqrt{\text{uw}_e \text{uw}_e + \text{vw}_e \text{vw}_e}} / \left( \frac{C_d \cdot C_d}{(\text{hw}_e)^{1/3}} \right) \right); \\
\tau_{\text{interface water}}(3) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( \text{grav} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \frac{\text{man}_w}{\text{man}_w} \frac{\text{uw}_e}{\sqrt{\text{uw}_e \text{uw}_e + \text{vw}_e \text{vw}_e}} / \left( \frac{C_d \cdot C_d}{(\text{hw}_e)^{1/3}} \right) \right);
\end{align*} \]
end
else %if no mud, then bed shear
\[ \begin{align*}
\tau_{\text{interface water}}(1) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( \text{grav} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \frac{\text{man}_w}{\text{man}_w} \frac{\text{uw}_e}{\sqrt{\text{uw}_e \text{uw}_e + \text{vw}_e \text{vw}_e}} / \left( \frac{C_d \cdot C_d}{(\text{hw}_e)^{1/3}} \right) \right); \\
\tau_{\text{interface water}}(2) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( \text{grav} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \frac{\text{man}_w}{\text{man}_w} \frac{\text{uw}_e}{\sqrt{\text{uw}_e \text{uw}_e + \text{vw}_e \text{vw}_e}} / \left( \frac{C_d \cdot C_d}{(\text{hw}_e)^{1/3}} \right) \right); \\
\tau_{\text{interface water}}(3) &= \frac{1}{6} \left( \delta T \det \frac{jac} \right) \left( \text{grav} \frac{\rho_{\text{water}}}{\rho_{\text{fm}}} \frac{\text{man}_w}{\text{man}_w} \frac{\text{uw}_e}{\sqrt{\text{uw}_e \text{uw}_e + \text{vw}_e \text{vw}_e}} / \left( \frac{C_d \cdot C_d}{(\text{hw}_e)^{1/3}} \right) \right);
\end{align*} \]
end
\[ \begin{align*}
\tau_{\text{surface water}}(1) &= 0; \\
\tau_{\text{surface water}}(2) &= 0; \\
\tau_{\text{surface water}}(3) &= 0;
\end{align*} \]
xmomenwater_rhs(nodel)=xmomenwater_rhs(nodel)+x_uhwater(1)+x_duuhdxwater(1)+x_duvhdywater(1)+x_dhdxwater(1)-e32dhdxwater(1)-e11dhdxwater(1)-x_dzdxwater(1)-dtxxhdxwater(1)-dtxyhdywater(1)-tauinte waterx(1)+tausurfacewaterx(1)+tausurfacewaterx(1);
% xmomenwater_rhs(node2)=xmomenwater_rhs(node2)+x_uhwater(2)+x_duuhdxwater(2)+x_duvhdywater(2)-e12dhdxwater-e21dhdxwater-x_dzdxwater(2)-dtxxhdxwater(2)-dtxyhdywater(2)-tauinterfacialwaterx(2)+tausurfaceexwater(2);
% xmomenwater_rhs(node3)=xmomenwater_rhs(node3)+x_uhwater(3)+x_duuhdxwater(3)+x_duvhdywater(3)-e22dhdxwater-e31dhdxwater-x_dzdxwater(3)-dtxxhdxwater(3)-dtxyhdywater(3)-tauinterfacialwaterx(3)+tausurfaceexwater(3);
%
% y direction momentum terms
%
yomenwater_lhs(node1,node1)=yomenwater_lhs(node1,node1)+(det_jac)/6;
% yomenwater_lhs(node2,node2)=yomenwater_lhs(node2,node2)+(det_jac)/6;
% yomenwater_lhs(node3,node3)=yomenwater_lhs(node3,node3)+(det_jac)/6;
% % y_vhwater(1)=(det_jac/6)*vw1*hw1;
% y_vhwater(2)=(det_jac/6)*vw2*hw2;
% y_vhwater(3)=(det_jac/6)*vw3*hw3;
% % %Cij for dvuh/dx
% vuhwater1=(1/20)*(vw1*uw1*hw1)+(1/60)*(vw1*uw2*hw1+vw1*uw3*hw1+vw1*uw1*hw2+vw1*uw2*hw2+vw1*uw3*hw2+vw1*uw2*hw3+vw1*uw3*hw3)+(1/120)*(vw1*uw3*hw2+vw1*uw2*hw3);
% vuhwater2=(1/20)*(vw2*uw2*hw2)+(1/60)*(vw2*uw1*hw1+vw2*uw2*hw1+vw2*uw1*hw2+vw2*uw3*hw2+vw2*uw3*hw2+vw2*uw3*hw3)+(1/120)*(vw2*uw3*hw1+vw2*uw1*hw3);
% vuhwater3=(1/20)*(vw3*uw3*hw3)+(1/60)*(vw3*uw1*hw1+vw3*uw2*hw1+vw3*uw3*hw1+vw3*uw2*hw2+vw3*uw3*hw2+vw3*uw3*hw3)+(1/120)*(vw3*uw2*hw1+vw3*uw3*hw2);
% % y_dvuhdxwater(1)=(deltaT*det_jac)*dphidx(1)*(vuhwater1+vuhwater2+vuhwater3);
% y_dvuhdxwater(2)=(deltaT*det_jac)*dphidx(2)*(vuhwater1+vuhwater2+vuhwater3);
% y_dvuhdxwater(3)=(deltaT*det_jac)*dphidx(3)*(vuhwater1+vuhwater2+vuhwater3);
% % y_dvuhdxwater(1)=0;
% y_dvuhdxwater(2)=0;
% y_dvuhdxwater(3)=0;
% %
% Cij for dvvh/dy
%
vvhwater1 = (1/20)*(vw1*vw1*hw1)+(1/60)*(vw1*vw2*hw1+vw1*vw3*hw1+vw1*vw1*hw2+vw1*vw2*hw2+vw1*vw1*hw3+vw1*vw3*hw3)+(1/120)*(vw1*vw3*hw1+vw1*vw2*hw1+vw1*vw1*hw2+vw1*vw2*hw2+vw1*vw1*hw3+vw1*vw3*hw3);
%
vvhwater2 = (1/20)*(vw2*vw2*hw2)+(1/60)*(vw2*vw1*hw1+vw2*vw2*hw1+vw2*vw1*hw2+vw2*vw2*hw2+vw2*vw1*hw3+vw2*vw3*hw3)+(1/120)*(vw2*vw3*hw1+vw2*vw1*hw1+vw2*vw2*hw1+vw2*vw3*hw2+vw2*vw1*hw2+vw2*vw2*hw2);
%
vvhwater3 = (1/20)*(vw3*vw3*hw3)+(1/60)*(vw3*vw1*hw1+vw3*vw3*hw1+vw3*vw2*hw2+vw3*vw3*hw2+vw3*vw1*hw3+vw3*vw2*hw3)+(1/120)*(vw3*vw2*hw1+vw3*vw1*hw1+vw3*vw2*hw2+vw3*vw1*hw2+vw3*vw2*hw3);
%
% y_dvvhdywater(1) = (deltaT*det_jac)*dphidy(1)*(vvhwater1+vvhwater2+vvhwater3);
% y_dvvhdywater(2) = (deltaT*det_jac)*dphidy(2)*(vvhwater1+vvhwater2+vvhwater3);
% y_dvvhdywater(3) = (deltaT*det_jac)*dphidy(3)*(vvhwater1+vvhwater2+vvhwater3);
%
% y_dvvhdywater(1) = 0;
% y_dvvhdywater(2) = 0;
% y_dvvhdywater(3) = 0;
%
% hmudtotal = (1/12)*(hmud1*hmud1+hmud1*hmud2+hmud1*hmud3+hmud2*hmud2+hmud2*hmud3+hmud3*hmud3);
% y_dhdywater(1) = (deltaT*grav*det_jac/2)*dphidy(1)*(hwatertotal);
% y_dhdywater(2) = (deltaT*grav*det_jac/2)*dphidy(2)*(hwatertotal);
% y_dhdywater(3) = (deltaT*grav*det_jac/2)*dphidy(3)*(hwatertotal);
%
% edge1, nodes 1 to 2
%
e11dhdywater = (deltaT*grav/2)*(normal_edge1y)*((1/4)*hw1*hw1+(1/6)*hw1*hw2+(1/12)*hw2*hw2);
%
e12dhdywater = (deltaT*grav/2)*(normal_edge1y)*((1/12)*hw1*hw1+(1/6)*hw1*hw2+(1/4)*hw2*hw2);
%
% edge2, nodes 2 to 3
%
e21dhdywater = (deltaT*grav/2)*(normal_edge2y)*((1/4)*hw2*hw2+(1/6)*hw2*hw3+(1/12)*hw3*hw3);
%
e22dhdywater = (deltaT*grav/2)*(normal_edge2y)*((1/12)*hw2*hw2+(1/6)*hw2*hw3+(1/4)*hw3*hw3);
% % edge3, nodes 3 to 1
% e31dhdywater=(deltaT*grav/2)*(normal_edge3y)*((1/4)*hw3*hw3+(1/6)*hw3*hw1+(1/12)*hw1*hw1);
% e32dhdywater=(deltaT*grav/2)*(normal_edge3y)*((1/12)*hw3*hw3+(1/6)*hw3*hw1+(1/4)*hw1*hw1);
% % dzdy_terms
% dzdywater=dphidy(1)*(z1+hmud1)+dphidy(2)*(z2+hmud2)+dphidy(3)*(z3+hmud3);
% y_dzdywater(1)=(deltaT*det_jac*grav)*dzdywater*(1/12*hw1+1/24*hw2+1/24*hw3);
% y_dzdywater(2)=(deltaT*det_jac*grav)*dzdywater*(1/24*hw1+1/12*hw2+1/24*hw3);
% y_dzdywater(3)=(deltaT*det_jac*grav)*dzdywater*(1/24*hw1+1/24*hw2+1/12*hw3);
% % dtyyhdywater(1)=(1/rhowater)*2*mu_w*(deltaT*det_jac)*dphidy(1)*dvdywater*(1/6)*(hw1+hw2+hw3);
% dtyyhdywater(2)=(1/rhowater)*2*mu_w*(deltaT*det_jac)*dphidy(2)*dvdywater*(1/6)*(hw1+hw2+hw3);
% dtyyhdywater(3)=(1/rhowater)*2*mu_w*(deltaT*det_jac)*dphidy(3)*dvdywater*(1/6)*(hw1+hw2+hw3);
% % dx2yhxwater(1)=(1/rhowater)*mu_w*(deltaT*det_jac)*dphidx(1)*(dvdxwater+dudywater)*(1/6)*(hw1+hw2+hw3);
% dx2yhxwater(2)=(1/rhowater)*mu_w*(deltaT*det_jac)*dphidx(2)*(dvdxwater+dudywater)*(1/6)*(hw1+hw2+hw3);
% dx2yhxwater(3)=(1/rhowater)*mu_w*(deltaT*det_jac)*dphidx(3)*(dvdxwater+dudywater)*(1/6)*(hw1+hw2+hw3);
% % % dtyyhdywater(1)=0;
% % dtyyhdywater(2)=0;
% % dtyyhdywater(3)=0;
% % % dx2yhxwater(1)=0;
% % dx2yhxwater(2)=0;
tduhdwter(3) = 0;

if FM_FLAG == 1
    if hmud_e >= 2 * min_depth
        tauinterfacialwatery(1) = (1/6) * (deltaT * det_jac) * (f_interface * (rhowater / rhofm) * delta_v * delta_u_mag / 8);
        tauinterfacialwatery(2) = (1/6) * (deltaT * det_jac) * (f_interface * (rhowater / rhofm) * delta_v * delta_u_mag / 8);
        tauinterfacialwatery(3) = (1/6) * (deltaT * det_jac) * (f_interface * (rhowater / rhofm) * delta_v * delta_u_mag / 8);
    else
        tausurfaceywater(1) = 0;
        tausurfaceywater(2) = 0;
        tausurfaceywater(3) = 0;
    end
end

ymomenwater_rhs(node1) = ymomemenwater_rhs(node1) + y_vhwater(1) + y_dvhdxwater(1) + y
\[ \text{dvvhdywater}(1)+\text{y_dhdywater}(1)-\text{e32dhdywater}-\text{elldhdywater}-\text{y_dzdywater}(1)-\text{dtvyhdywater}(1)-\text{dtxyhdxwater}(1)-\text{tauinterfacialwatery}(1)+\text{tausurfaceywater}(1); \]
\[ \text{ymomenwater_rhs}(\text{node2})=\text{ymomenwater_rhs}(\text{node2})+\text{y_vhwater}(2)+\text{y_dvuhdxwater}(2)+\text{y_dvvhdywater}(2)+\text{y_dhdywater}(2)-\text{e12dhdywater}-\text{e21dhdywater}-\text{y_dzdywater}(2)-\text{dtyyhdywater}(1)-\text{dtxyhdxwater}(1)-\text{tauinterfacialwatery}(2)+\text{tausurfaceywater}(2); \]
\[ \text{ymomenwater_rhs}(\text{node3})=\text{ymomenwater_rhs}(\text{node3})+\text{y_vhwater}(3)+\text{y_dvuhdxwater}(3)+\text{y_dvvhdywater}(3)+\text{y_dhdywater}(3)-\text{e22dhdywater}-\text{e31dhdywater}-\text{y_dzdywater}(3)-\text{dtyyhdywater}(1)-\text{dtxyhdxwater}(1)-\text{tauinterfacialwatery}(3)+\text{tausurfaceywater}(3); \]

\textbf{if} \text{FM FLAG==1}
\textbf{if} \text{FM FLAG==1}
\text{continuity terms mud}

\text{continuimud_lhs}(\text{node1},\text{node1})=\text{continuimud_lhs}(\text{node1},\text{node1})+(\text{det Jac})/6;
\text{continuimud_lhs}(\text{node2},\text{node2})=\text{continuimud_lhs}(\text{node2},\text{node2})+(\text{det Jac})/6;
\text{continuimud_lhs}(\text{node3},\text{node3})=\text{continuimud_lhs}(\text{node3},\text{node3})+(\text{det Jac})/6;

\text{contmud_h}(1)=(\text{det Jac}/6)*\text{hmud1};
\text{contmud_h}(2)=(\text{det Jac}/6)*\text{hmud2};
\text{contmud_h}(3)=(\text{det Jac}/6)*\text{hmud3};

\text{uhmud_total}=(1/12)*\text{umud1}*\text{hmud1}+\text{umud2}+\text{umud3}+\text{umud4}+\text{umud5}+\text{umud6}+\text{umud7}+\text{umud8}+\text{umud9}+(1/24)*\text{umud1}+\text{umud2}+(\text{det Jac})/6;
\text{contmud_uh}(1)=\text{deltaT}*\text{dphidy(1)}*\text{det Jac}*(\text{uhmud_total});
\text{contmud_uh}(2)=\text{deltaT}*\text{dphidy(2)}*\text{det Jac}*(\text{uhmud_total});
\text{contmud_uh}(3)=\text{deltaT}*\text{dphidy(3)}*\text{det Jac}*(\text{uhmud_total});

\text{vhmud_total}=(1/12)*\text{vmud1}+\text{vmud2}+\text{vmud3}+\text{vmud4}+\text{vmud5}+\text{vmud6}+\text{vmud7}+\text{vmud8}+\text{vmud9}+(1/24)*\text{vmud1}+\text{vmud2}+(\text{det Jac})/6;
\text{contmud_vh}(1)=\text{deltaT}*\text{dphidy(1)}*\text{det Jac}*(\text{vhmud_total});
\text{contmud_vh}(2)=\text{deltaT}*\text{dphidy(2)}*\text{det Jac}*(\text{vhmud_total});
\text{contmud_vh}(3)=\text{deltaT}*\text{dphidy(3)}*\text{det Jac}*(\text{vhmud_total});

%entrainment value, outflow
%Mehta entrainment
%\text{Ri_num}=\text{hw}^*\text{grav}*(\text{rhofm}-\text{rhoewater})/(\text{rhoewater}^*\text{delta_u});
% \text{Ri_num_inv}=(\text{rhoewater}^*(\text{delta_u}^2))/(\text{hw}^*\text{grav}*(\text{rhofm}-\text{rhoewater}));
% if hmud1<2*min_depth
%    em(1)=0;
% else
%    em(1)=(1/6)*(deltaT*det_jac)*(delta_u_mag)*0.0052*Ri_num_inv;
% end
%
% if hmud2<2*min_depth
%    em(2)=0;
% else
%    em(2)=(1/6)*(deltaT*det_jac)*(delta_u_mag)*0.0052*Ri_num_inv;
% end
%
% if hmud3<2*min_depth
%    em(3)=0;
% else
%    em(3)=(1/6)*(deltaT*det_jac)*(delta_u_mag)*0.0052*Ri_num_inv;
% end
%
entrain(node1)=entrain(node1)+em(1);
entrain(node2)=entrain(node2)+em(2);
entrain(node3)=entrain(node3)+em(3);
%
em(1)=0;
em(2)=0;
em(3)=0;

%settling value, inflow
%settling test
%    settling(1)=(1/6)*(deltaT*det_jac)*(cwater_e/concfm)*(wsf);
%    settling(2)=(1/6)*(deltaT*det_jac)*(cwater_e/concfm)*(wsf);
%    settling(3)=(1/6)*(deltaT*det_jac)*(cwater_e/concfm)*(wsf);

settling(1)=0;
settling(2)=0;
settling(3)=0;

%consolidation value, outflow
% %assume 0
%    consol(1)=(1/6)*(deltaT*det_jac)*wscfm;
%    consol(2)=(1/6)*(deltaT*det_jac)*wscfm;
%    consol(3)=(1/6)*(deltaT*det_jac)*wscfm;

consol(1)=0;
%sum terms
continuymud_rhs(node1)=continuymud_rhs(node1)+contmud_h(1)+contmud_uh(1)+
contmud_vh(1)-em(1)+settling(1)-consol(1);

continuymud_rhs(node2)=continuymud_rhs(node2)+contmud_h(2)+contmud_uh(2)+
contmud_vh(2)-em(2)+settling(2)-consol(2);

continuymud_rhs(node3)=continuymud_rhs(node3)+contmud_h(3)+contmud_uh(3)+
contmud_vh(3)-em(3)+settling(3)-consol(3);

%error tracking
tfcontmudh=isreal(contmud_h);
tfcontmuduh=isreal(contmud_uh);
tfcontmudvh=isreal(contmud_vh);
tfcontsum=tfcontmudh+tfcontmuduh+tfcontmudvh;
if tfcontsum<3
    fclose('all');
    errordlg('Complex Number - Continuity','ERROR')
    return;
end

%x direction momentum terms
xmomenmud_lhs(node1,node1)=xmomenmud_lhs(node1,node1)+(det_jac)/6;
xmomenmud_lhs(node2,node2)=xmomenmud_lhs(node2,node2)+(det_jac)/6;
xmomenmud_lhs(node3,node3)=xmomenmud_lhs(node3,node3)+(det_jac)/6;

x_uhmud(1)=(det_jac/6)*umud1*hmd1;
x_uhmud(2)=(det_jac/6)*umud2*hmd2;
x_uhmud(3)=(det_jac/6)*umud3*hmd3;

% Cij for duuh/dx
uuhmud1=(1/20)*(umud1*umud1*hmd1)+(1/60)*(umud1*umud2*hmd1+umud1*umud3*hmd1+
    umud1*umud1*hmd2+umud1*umud2*hmd2+umud1*umud1*hmd3+umud1*umud3*hmd3)+(1/
    120)*(umud1*umud3*hmd2+umud1*umud2*hmd3);

uuhmud2=(1/20)*(umud2*umud2*hmd2)+(1/60)*(umud2*umud1*hmd1+umud2*umud2*hmd1+
    umud2*umud1*hmd2+umud2*umud3*hmd2+umud2*umud2*hmd3+umud2*umud3*hmd3)+(1/
    120)*(umud2*umud3*hmd1+umud2*umud1*hmd3);
uuhmud3 = (1/20) * (umud3*umud3*hmud3) + (1/60) * (umud3*umud1*hmud1+umud3*umud3*hmud1+umud3*umud2*hmud2+umud3*umud3*hmud3+umud3*umud1*hmud1+umud3*umud3*hmud2+umud3*umud2*hmud3) + (1/120) * (umud3*umud2*hmud1+umud3*umud3*hmud1+umud3*umud2*hmud3);

x_duuhdxmud(1) = (deltaT*det_jac)*dphidx(1) * (uuhmud1+uuhmud2+uuhmud3);
x_duuhdxmud(2) = (deltaT*det_jac)*dphidx(2) * (uuhmud1+uuhmud2+uuhmud3);
x_duuhdxmud(3) = (deltaT*det_jac)*dphidx(3) * (uuhmud1+uuhmud2+uuhmud3);

x_duuhdxmud(1) = 0;
x_duuhdxmud(2) = 0;
x_duuhdxmud(3) = 0;

% Cij for duvh/dy

uvhmud1 = (1/20) * (umud1*vmud1*hmud1) + (1/60) * (umud1*vmud2*hmud1+umud1*vmud3*hmud1+umud1*vmud3*hmud2+umud1*vmud3*hmud3) + (1/120) * (umud1*vmud3*hmud2+umud1*vmud2*hmud3);

uvhmud2 = (1/20) * (umud2*vmud2*hmud2) + (1/60) * (umud2*vmud1*hmud1+umud2*vmud2*hmud2+umud2*vmud2*hmud3+umud2*vmud3*hmud3) + (1/120) * (umud2*vmud3*hmud1+umud2*vmud1*hmud2);

uvhmud3 = (1/20) * (umud3*vmud3*hmud3) + (1/60) * (umud3*vmud1*hmud1+umud3*vmud3*hmud1+umud3*vmud3*hmud2+umud3*vmud3*hmud3) + (1/120) * (umud3*vmud2*hmud1+umud3*vmud2*hmud3);

x_duvhdymud(1) = (deltaT*det_jac)*dphidy(1) * (uvhmud1+uvhmud2+uvhmud3);
x_duvhdymud(2) = (deltaT*det_jac)*dphidy(2) * (uvhmud1+uvhmud2+uvhmud3);
x_duvhdymud(3) = (deltaT*det_jac)*dphidy(3) * (uvhmud1+uvhmud2+uvhmud3);

x_duvhdymud(1) = 0;
x_duvhdymud(2) = 0;
x_duvhdymud(3) = 0;

hmudtotal = (1/12) * (hmud1*hmud1+hmud1*hmud2+hmud1*hmud3+hmud2*hmud2+hmud2*hmud3+hmud3*hmud3);

x_dhdxmud(1) = (deltaT*grav*det_jac/2)*dphidx(1) * (hmudtotal);
x_dhdxmud(2) = (deltaT*grav*det_jac/2)*dphidx(2) * (hmudtotal);
x_dhdxmud(3) = (deltaT*grav*det_jac/2)*dphidx(3) * (hmudtotal);

% edge1, nodes 1 to 2
\[ e_{11d}d_{x_mud} = \left( \delta T \cdot \frac{grav}{2} \right) \left( normal_{edge1x} \right) \left( \frac{1}{12} \cdot h_{mud1} \cdot h_{mud1} + \frac{1}{6} \cdot h_{mud1} \cdot h_{mud2} + \frac{1}{12} \cdot h_{mud2} \cdot h_{mud2} \right); \]

\[ e_{12d}d_{x_mud} = \left( \delta T \cdot \frac{grav}{2} \right) \left( normal_{edge1x} \right) \left( \frac{1}{12} \cdot h_{mud1} \cdot h_{mud1} + \frac{1}{6} \cdot h_{mud1} \cdot h_{mud2} + \frac{1}{4} \cdot h_{mud2} \cdot h_{mud2} \right); \]

%edge2, nodes 2 to 3

\[ e_{21d}d_{x_mud} = \left( \delta T \cdot \frac{grav}{2} \right) \left( normal_{edge2x} \right) \left( \frac{1}{4} \cdot h_{mud2} \cdot h_{mud2} + \frac{1}{6} \cdot h_{mud2} \cdot h_{mud3} + \frac{1}{12} \cdot h_{mud3} \cdot h_{mud3} \right); \]

\[ e_{22d}d_{x_mud} = \left( \delta T \cdot \frac{grav}{2} \right) \left( normal_{edge2x} \right) \left( \frac{1}{12} \cdot h_{mud2} \cdot h_{mud2} + \frac{1}{6} \cdot h_{mud2} \cdot h_{mud3} + \frac{1}{4} \cdot h_{mud3} \cdot h_{mud3} \right); \]

%edge3, nodes 3 to 1

\[ e_{31d}d_{x_mud} = \left( \delta T \cdot \frac{grav}{2} \right) \left( normal_{edge3x} \right) \left( \frac{1}{4} \cdot h_{mud3} \cdot h_{mud3} + \frac{1}{6} \cdot h_{mud3} \cdot h_{mud1} + \frac{1}{12} \cdot h_{mud1} \cdot h_{mud1} \right); \]

\[ e_{32d}d_{x_mud} = \left( \delta T \cdot \frac{grav}{2} \right) \left( normal_{edge3x} \right) \left( \frac{1}{12} \cdot h_{mud3} \cdot h_{mud3} + \frac{1}{6} \cdot h_{mud3} \cdot h_{mud1} + \frac{1}{12} \cdot h_{mud1} \cdot h_{mud1} \right); \]

%need gh dz/dx terms

\[ z_{htotal} = \left( \frac{1}{12} \right) \left( z_1 \cdot h_1 + z_2 \cdot h_2 + z_3 \cdot h_3 \right) + \left( \frac{1}{24} \right) \left( z_1 \cdot h_2 + z_1 \cdot h_3 + z_2 \cdot h_1 + z_2 \cdot h_3 + z_3 \cdot h_1 + z_3 \cdot h_2 \right); \]

\[ dz_{dx_mud} = dphi_{idx}(1) \cdot z_1 + dphi_{idx}(2) \cdot z_2 + dphi_{idx}(3) \cdot z_3; \]

\[ x_{_dz_{dx_mud}(1)} = (\delta T \cdot det_jac \cdot grav) \cdot dz_{dx_mud} \cdot \left( \frac{1}{12} \cdot h_{mud1} + \frac{1}{24} \cdot h_{mud2} + \frac{1}{24} \cdot h_{mud3} \right); \]

\[ x_{_dz_{dx_mud}(2)} = (\delta T \cdot det_jac \cdot grav) \cdot dz_{dx_mud} \cdot \left( \frac{1}{24} \cdot h_{mud1} + \frac{1}{12} \cdot h_{mud2} + \frac{1}{24} \cdot h_{mud3} \right); \]

\[ x_{_dz_{dx_mud}(3)} = (\delta T \cdot det_jac \cdot grav) \cdot dz_{dx_mud} \cdot \left( \frac{1}{24} \cdot h_{mud1} + \frac{1}{24} \cdot h_{mud2} + \frac{1}{12} \cdot h_{mud3} \right); \]

%reduced gravity term

\[ dh_{wdx_mud} = dphi_{idx}(1) \cdot hw_1 + dphi_{idx}(2) \cdot hw_2 + dphi_{idx}(3) \cdot hw_3; \]

\[ x_{_dh_{w_mud}(1)} = (\delta T \cdot det_jac \cdot grav) \cdot dh_{wdx_mud} \cdot \left( \frac{1}{12} \cdot h_{mud1} + \frac{1}{24} \cdot h_{mud2} + \frac{1}{24} \cdot h_{mud3} \right) \times (rhowater/rhofm); \]
\[ x_{\text{dhwmud}}(2) = (\delta T \cdot \text{det}_\text{jac} \cdot \text{grav}) \cdot \text{dhwdxmud} \cdot (1/24 \cdot \text{hmud}_1 + 1/12 \cdot \text{hmud}_2 + 1/24 \cdot \text{hmud}_3) \cdot (\text{rho}_{\text{water}} / \text{rho}_{\text{fm}}); \]

\[ x_{\text{dhwmud}}(3) = (\delta T \cdot \text{det}_\text{jac} \cdot \text{grav}) \cdot \text{dhwdxmud} \cdot (1/24 \cdot \text{hmud}_1 + 1/24 \cdot \text{hmud}_2 + 1/12 \cdot \text{hmud}_3) \cdot (\text{rho}_{\text{water}} / \text{rho}_{\text{fm}}); \]

% X direction Viscosity terms, txx*h, txy*h

dudxmud = \text{dphidx}(1) \cdot \text{umud}_1 + \text{dphidx}(2) \cdot \text{umud}_2 + \text{dphidx}(3) \cdot \text{umud}_3;

dvdxmud = \text{dphidy}(1) \cdot \text{vmud}_1 + \text{dphidy}(2) \cdot \text{vmud}_2 + \text{dphidy}(3) \cdot \text{vmud}_3;

dudymud = \text{dphidy}(1) \cdot \text{umud}_1 + \text{dphidy}(2) \cdot \text{umud}_2 + \text{dphidy}(3) \cdot \text{umud}_3;

dvdymud = \text{dphidy}(1) \cdot \text{vmud}_1 + \text{dphidy}(2) \cdot \text{vmud}_2 + \text{dphidy}(3) \cdot \text{vmud}_3;

\textbf{if} \ \text{hmud}_e \geq 2 \cdot \text{min}_\text{depth} \\
\text{dtxxhdxmud}(1) = 2 \cdot (\mu_{\text{fm}} / \text{rho}_{\text{fm}}) \cdot (\delta T \cdot \text{det}_\text{jac}) \cdot \text{dphidx}(1) \cdot (\text{dudxmud} \cdot (1/6) \cdot (\text{hmud}_1 + \text{hmud}_2 + \text{hmud}_3));

\text{dtxxhdxmud}(2) = 2 \cdot (\mu_{\text{fm}} / \text{rho}_{\text{fm}}) \cdot (\delta T \cdot \text{det}_\text{jac}) \cdot \text{dphidx}(2) \cdot (\text{dudxmud} \cdot (1/6) \cdot (\text{hmud}_1 + \text{hmud}_2 + \text{hmud}_3));

\text{dtxxhdxmud}(3) = 2 \cdot (\mu_{\text{fm}} / \text{rho}_{\text{fm}}) \cdot (\delta T \cdot \text{det}_\text{jac}) \cdot \text{dphidx}(3) \cdot (\text{dudxmud} \cdot (1/6) \cdot (\text{hmud}_1 + \text{hmud}_2 + \text{hmud}_3));

\textbf{else}
\text{dtxxhdxmud}(1) = 0;
\text{dtxxhdxmud}(2) = 0;
\text{dtxxhdxmud}(3) = 0;

\text{dtxyhdymud}(1) = 0;
\text{dtxyhdymud}(2) = 0;
\text{dtxyhdymud}(3) = 0;
\textbf{end}

% keep for error tracking
\text{txxhdxmud} (\text{node}_1) = \text{txxhdxmud} (\text{node}_1) + \text{dtxxhdxmud}(1);
\text{txxhdxmud} (\text{node}_2) = \text{txxhdxmud} (\text{node}_2) + \text{dtxxhdxmud}(2);
\text{txxhdxmud} (\text{node}_3) = \text{txxhdxmud} (\text{node}_3) + \text{dtxxhdxmud}(3);
txyhdymud(node1)=txyhdymud(node1)+dtxyhdymud(1);

% calculate bed shear stress, 1 is Manning's, 2 for Soulsby and Clarke
if fric_typemud==1

    if hmu_e<2*min_depth
        taubedxmud(1)=0;
        taubedxmud(2)=0;
        taubedxmud(3)=0;
    else
        taubedxmud(1)=(1/6)*(deltaT*det_jac)*(grav*man_mud*man_mud*umud_e*sqrt((umud_e*umud_e)+(vmud_e*vmud_e)))/((C_d*C_d)*(hmu_e)^(1/3));
        taubedxmud(2)=(1/6)*(deltaT*det_jac)*(grav*man_mud*man_mud*umud_e*sqrt((umud_e*umud_e)+(vmud_e*vmud_e)))/((C_d*C_d)*(hmu_e)^(1/3));
        taubedxmud(3)=(1/6)*(deltaT*det_jac)*(grav*man_mud*man_mud*umud_e*sqrt((umud_e*umud_e)+(vmud_e*vmud_e)))/((C_d*C_d)*(hmu_e)^(1/3));
    end

elseif fric_typemud==2
    if rec_bedmud<=0
        taubedxmud(1)=0;
        taubedxmud(2)=0;
        taubedxmud(3)=0;
    else
        taubedxmud(1)=(1/6)*(deltaT*det_jac)*cds_bedmud*(umud_e*sqrt(umud_e*umud_e+vmud_e*vmud_e))/((C_d*C_d)*(hmu_e)^(1/3));
        taubedxmud(2)=(1/6)*(deltaT*det_jac)*cds_bedmud*(umud_e*sqrt(umud_e*umud_e+vmud_e*vmud_e))/((C_d*C_d)*(hmu_e)^(1/3));
        taubedxmud(3)=(1/6)*(deltaT*det_jac)*cds_bedmud*(umud_e*sqrt(umud_e*umud_e+vmud_e*vmud_e))/((C_d*C_d)*(hmu_e)^(1/3));
    end

end code if no friction
else
    fclose('all');
errordlg('No friction','ERROR')
    return;
end

%interfacial shear
if hmu_d_e>=2*min_depth
    tauinterfacialmudx(1)=(1/6)*(deltaT*det_jac)*(f_interface*(rhowater/rhofm)*
delta_u*delta_u_mag/8);
    tauinterfacialmudx(2)=(1/6)*(deltaT*det_jac)*(f_interface*(rhowater/rhofm)*
delta_u*delta_u_mag/8);
    tauinterfacialmudx(3)=(1/6)*(deltaT*det_jac)*(f_interface*(rhowater/rhofm)*
delta_u*delta_u_mag/8);
else %zero for below min depth of mud
    tauinterfacialmudx(1)=0;
    tauinterfacialmudx(2)=0;
    tauinterfacialmudx(3)=0;
end

xmomenmud_rhs(node1)=xmomenmud_rhs(node1)+x_uhmud(1)+x_duuhdxmud(1)+x_duvhdymud(1)-e32dhdxmud-e11dhdxmud-x_dzdxmud(1)-dttxhdxmud(1)-
dtxyhdymud(1)-taubedxmud(1)+tauinterfacialmudx(1)-x_dhwmud(1);

xmomenmud_rhs(node2)=xmomenmud_rhs(node2)+x_uhmud(2)+x_duuhdxmud(2)+x_duvhdymud(2)-e12dhdxmud-e21dhdxmud-x_dzdxmud(2)-dttxhdxmud(2)-
dtxyhdymud(2)-taubedxmud(2)+tauinterfacialmudx(2)-x_dhwmud(2);

xmomenmud_rhs(node3)=xmomenmud_rhs(node3)+x_uhmud(3)+x_duuhdxmud(3)+x_duvhdymud(3)-e22dhdxmud-e31dhdxmud-x_dzdxmud(3)-dttxhdxmud(3)-
dtxyhdymud(3)-taubedxmud(3)+tauinterfacialmudx(3)-x_dhwmud(3);

%error tracking
  tfxuh=isreal(x_uhmud);
  tfxuhu=isreal(x_duuhdxmud);
  tfxuuh=isreal(x_duvhdymud);
  tfxuhd=isreal(x_dhdxmud);
  tfxuzd=isreal(x_dzdxmud);
  tfxutxx=isreal(dttxhdxmud);
  tfxutxy=isreal(dtxyhdymud);
  tfxutbed=isreal(taubedxmud);
tfxti=isreal(tauinterfacialmudx);
tfxdhw=isreal(x_dhwmud);
tfxel1=isreal(e11dhxmdud);
tfxel2=isreal(e12dhxmdud);
tfxe21=isreal(e21dhxmdud);
tfxe22=isreal(e22dhxmdud);
tfxe31=isreal(e31dhxmdud);
tfxe32=isreal(e32dhxmdud);

tfs=sumx=tfxti+tfxtj+tfxtk+tfxtl+tfxtm+tfxtn+tfxti+tfxdhw+tfxe11+tfxe12+tfxe21+tfxe22+tfxe31+tfxe32;

if tfs<16
  fclose('all');
  errordlg('Complex Number - X momentum','ERROR')
  return;
end

%y direction momentum terms
ymomenmud_lhs(node1,node1)=ymomenmud_lhs(node1,node1)+(det_jac)/6;
ymomenmud_lhs(node2,node2)=ymomenmud_lhs(node2,node2)+(det_jac)/6;
ymomenmud_lhs(node3,node3)=ymomenmud_lhs(node3,node3)+(det_jac)/6;

y_vhmud(1)=(det_jac/6)*vmud*
hmud1;

y_vhmud(2)=(det_jac/6)*vmud2*hmud2;

y_vhmud(3)=(det_jac/6)*vmud3*hmud3;

%Cij for dvuh/dx

vuhmud1=(1/20)*(vmud1*umud1*hmud1)+(1/60)*(vmud1*umud2*hmud1+vmud1*umud3*hmud1+vmud1*umud1*hmud2+vmud1*umud2*hmud2+vmud1*umud3*hmud2+vmud1*umud1*hmud3+vmud1*umud2*hmud3+vmud1*umud3*hmud3)

vuhmud2=(1/20)*(vmud2*umud2*hmud2)+(1/60)*(vmud2*umud1*hmud1+vmud2*umud2*hmud2+vmud2*umud1*hmud2+vmud2*umud2*hmud2+vmud2*umud3*hmud2+vmud2*umud2*hmud3+vmud2*umud3*hmud3)

vuhmud3=(1/20)*(vmud3*umud3*hmud3)+(1/60)*(vmud3*umud1*hmud1+vmud3*umud2*hmud1+vmud3*umud3*hmud1+vmud3*umud2*hmud2+vmud3*umud3*hmud2+vmud3*umud1*hmud3+vmud3*umud2*hmud3+vmud3*umud3*hmud3)
%         y_dvhdxmud(1)=0;
%         y_dvhdxmud(2)=0;
%         y_dvhdxmud(3)=0;

%Cij for dvvh/dy

vvhmud1=(1/20)*(vmud1*vmud1*hmud1)+(1/60)*(vmud1*vmud2*hmud1+vmud1*vmud3*hmud1+vmud1*vmud2*hmud2+vmud1*vmud3*hmud2+vmud1*vmud2*hmud3+vmud1*vmud3*hmud3)+(1/120)*(vmud1*vmud3*hmud1+vmud2*vmud1*hmud2+vmud1*vmud2*hmud1+vmud2*vmud2*hmud1+vmud2*vmud3*hmud3+vmud2*vmud1*hmud3);

vvhmud2=(1/20)*(vmud2*vmud2*hmud2)+(1/60)*(vmud2*vmud1*hmud1+vmud2*vmud2*hmud1+vmud2*vmud3*hmud2+vmud2*vmud2*hmud3+vmud2*vmud3*hmud3)+(1/120)*(vmud2*vmud3*hmud1+vmud2*vmud1*hmud3+vmud2*vmud2*hmud2+vmud2*vmud3*hmud2+vmud3*vmud1*hmud3+vmud3*vmud3*hmud3);

vvhmud3=(1/20)*(vmud3*vmud3*hmud3)+(1/60)*(vmud3*vmud1*hmud1+vmud3*vmud3*hmud1+vmud3*vmud2*hmud2+vmud3*vmud3*hmud2+vmud3*vmud1*hmud3+vmud3*vmud3*hmud3)+(1/120)*(vmud3*vmud2*hmud1+vmud3*vmud3*hmud2+vmud3*vmud1*hmud2+vmud3*vmud2*hmud3+vmud3*vmud2*hmud3);

%         y_dvvhdymud(1)=0;
%         y_dvvhdymud(2)=0;
%         y_dvvhdymud(3)=0;

%hmudtotal=(1/12)*(hmud1*hmud1+hmud1*hmud2+hmud1*hmud3+hmud2*hmud2+hmud2*hmud3+hmud3*hmud3);

%edge1, nodes 1 to 2

e1ldhymud=(deltaT*grav/2)*(normal_edgely)*((1/4)*hmud1*hmud1+(1/6)*hmud1*hmud2+(1/12)*hmud2*hmud2);
e12dhdmud = (deltaT*grav/2)*(normal_edge1y)*((1/12)*hmud1*hmud1+(1/6)*hmud1*hmud2+(1/4)*hmud2*hmud2);
% edge2, nodes 2 to 3

e21dhdmud = (deltaT*grav/2)*(normal_edge2y)*((1/4)*hmud2*hmud2+(1/6)*hmud2*hmud3+(1/12)*hmud3*hmud3);

e22dhdmud = (deltaT*grav/2)*(normal_edge2y)*((1/12)*hmud2*hmud2+(1/6)*hmud2*hmud3+(1/4)*hmud3*hmud3);
% edge3, nodes 3 to 1

e31dhdmud = (deltaT*grav/2)*(normal_edge3y)*((1/4)*hmud3*hmud3+(1/6)*hmud3*hmud1+(1/12)*hmud1*hmud1);

e32dhdmud = (deltaT*grav/2)*(normal_edge3y)*((1/12)*hmud3*hmud3+(1/6)*hmud3*hmud1+(1/4)*hmud1*hmud1);

% bed slope terms

dzdymud = dphidy(1)*z1 + dphidy(2)*z2 + dphidy(3)*z3;

y_dzdymud(1) = (deltaT*det_jac*grav)*dzdymud*(1/12*hmud1+1/24*hmud2+1/24*hmud3);

y_dzdymud(2) = (deltaT*det_jac*grav)*dzdymud*(1/24*hmud1+1/12*hmud2+1/24*hmud3);

y_dzdymud(3) = (deltaT*det_jac*grav)*dzdymud*(1/24*hmud1+1/24*hmud2+1/12*hmud3);

% reduced gravity term

dhwdymud = dphidy(1)*hw1 + dphidy(2)*hw2 + dphidy(3)*hw3;

y_dhwmud(1) = (deltaT*det_jac*grav)*dhwdymud*(1/12*hmud1+1/24*hmud2+1/24*hmud3)*(rhowater/rhofm);

y_dhwmud(2) = (deltaT*det_jac*grav)*dhwdymud*(1/24*hmud1+1/12*hmud2+1/24*hmud3)*(rhowater/rhofm);
y_dhwmud(3)=(deltaT*det_jac*grav)*dhwdymud*(1/24*hmu1+1/24*hmu2+1/12*hmu3)*(rhowater/rhofm);

%viscous shear terms

if hmu_e>=2*min_depth
    dtuyhdmud(1) = 2 * (mufm/rhofm) * (deltaT*det_jac) * dphidy(1) * dhvymud * (1/6) * (hmu1+hmu2+hmu3);
    dtuyhdmud(2) = 2 * (mufm/rhofm) * (deltaT*det_jac) * dphidy(2) * dhvymud * (1/6) * (hmu1+hmu2+hmu3);
    dtuyhdmud(3) = 2 * (mufm/rhofm) * (deltaT*det_jac) * dphidy(3) * dhvymud * (1/6) * (hmu1+hmu2+hmu3);

    dtxyhdxmud(1) = (mufm/rhofm) * (deltaT*det_jac) * dphidx(1) * (dvhdxmud + dudymud) * (1/6) * (hmu1+hmu2+hmu3);
    dtxyhdxmud(2) = (mufm/rhofm) * (deltaT*det_jac) * dphidx(2) * (dvhdxmud + dudymud) * (1/6) * (hmu1+hmu2+hmu3);
    dtxyhdxmud(3) = (mufm/rhofm) * (deltaT*det_jac) * dphidx(3) * (dvhdxmud + dudymud) * (1/6) * (hmu1+hmu2+hmu3);
else
    dtuyhdmud(1)=0;
    dtuyhdmud(2)=0;
    dtuyhdmud(3)=0;
    dtxyhdxmud(1)=0;
    dtxyhdxmud(2)=0;
    dtxyhdxmud(3)=0;
end

%keep for error tracking

txyhdxmud(node1)=txyhdxmud(node1)+dtxyhdxmud(1);
txyhdxmud(node2)=txyhdxmud(node2)+dtxyhdxmud(2);
txyhdxmud(node3)=txyhdxmud(node3)+dtxyhdxmud(3);

tyyhdymud(node1)=tyyhdymud(node1)+dtuyhdmud(1);
tyyhdymud(node2)=tyyhdymud(node2)+dtuyhdmud(2);
tyyhdymud(node3)=tyyhdymud(node3)+dtuyhdmud(3);

if fric_typemud==1
taubedymud(1) = (1/6)*(deltaT*det_jac)*(grav*man_mud*man_mud*vmud_e*sqrt((umud_e*umud_e) + (vmud_e*vmud_e))/((C_d*C_d)*(hmud_e)^(1/3)));

if fric_typemud==2
    elseif fric_typemud==2
        if rec_bedmud<=0
            taubedymud(1)=0;
            taubedymud(2)=0;
            taubedymud(3)=0;
        else
            taubedymud(1)=(1/6)*(deltaT*det_jac)*cds_bedmud*(vmud_e*sqrt(umud_e*umud_e+vmud_e*vmud_e));
            taubedymud(2)=(1/6)*(deltaT*det_jac)*cds_bedmud*(vmud_e*sqrt(umud_e*umud_e+vmud_e*vmud_e));
            taubedymud(3)=(1/6)*(deltaT*det_jac)*cds_bedmud*(vmud_e*sqrt(umud_e*umud_e+vmud_e*vmud_e));
        end
    else
        fclose('all');
        errordlg('No friction','ERROR')
        return;
    end
endif

if hmud_e>=2*min_depth
    %set same as from water column

tauinterfacialmudy(1)=(1/6)*(deltaT*det_jac)*(f_interface*(rhowater/rhofm)*delta_v*delta_u_mag/8);

tauinterfacialmudy(2)=(1/6)*(deltaT*det_jac)*(f_interface*(rhowater/rhofm)*delta_v*delta_u_mag/8);
\[
\text{tauinterfacialmudy}(3) = \frac{1}{6} \ast (\delta T \ast \text{det}_j \ast \text{f\_interface} \ast (\text{rhowater} \ast \text{rhofm}) \ast \text{delta}_v \ast \text{delta}_{u\_mag} / 8);
\]

\[
\text{else} \quad \%\text{zero for below min depth of mud}
\]
\[
\text{tauinterfacialmudy}(1) = 0;
\]
\[
\text{tauinterfacialmudy}(2) = 0;
\]
\[
\text{tauinterfacialmudy}(3) = 0;
\]
\[
\text{end}
\]

\[
\text{ymomenmud\_rhs(\text{node1}) = ymomemnud\_rhs(\text{node1})} + \text{y\_vhmud}(1) + \text{y\_dvuhdxmud}(1) + \text{y\_dvvhdxmud}(1) + \text{e32dhdydmud} - \text{e11dhdydmud} - \text{y\_dzdydmud}(1) - \text{dtyyhdymud}(1) - \text{dtxyhdxmud}(1) - \text{taubedymud}(1) - \text{tauinterfacialmudy}(1) - \text{y\_dhwmdud}(1);
\]

\[
\text{ymomenmud\_rhs(\text{node2}) = ymomemnud\_rhs(\text{node2})} + \text{y\_vhmud}(2) + \text{y\_dvuhdxmud}(2) + \text{y\_dvvhdxmud}(2) + \text{e12dhdydmud} - \text{e21dhdydmud} - \text{y\_dzdydmud}(2) - \text{dtyyhdymud}(1) - \text{dtxyhdxmud}(1) - \text{taubedymud}(2) - \text{tauinterfacialmudy}(2) - \text{y\_dhwmdud}(2);
\]

\[
\text{ymomenmud\_rhs(\text{node3}) = ymomemnud\_rhs(\text{node3})} + \text{y\_vhmud}(3) + \text{y\_dvuhdxmud}(3) + \text{y\_dvvhdxmud}(3) + \text{e22dhdydmud} - \text{e31dhdydmud} - \text{y\_dzdydmud}(3) - \text{dtyyhdymud}(1) - \text{dtxyhdxmud}(1) - \text{taubedymud}(3) - \text{tauinterfacialmudy}(3) - \text{y\_dhwmdud}(3);
\]

\[
\%\text{error tracking}
\]
\[
\text{tfyvh} = \text{isreal}(\text{y\_vhmud});
\]
\[
\text{tfyuvh} = \text{isreal}(\text{y\_dvuhdxmud});
\]
\[
\text{tfyvvh} = \text{isreal}(\text{y\_dvvhdxmud});
\]
\[
\text{tfydh} = \text{isreal}(\text{y\_dhdydmud});
\]
\[
\text{tfydz} = \text{isreal}(\text{y\_dzdydmud});
\]
\[
\text{tfytyy} = \text{isreal}(\text{dtyyhdymud});
\]
\[
\text{tfytxy} = \text{isreal}(\text{dtxyhdxmud});
\]
\[
\text{tfybed} = \text{isreal}(\text{taubedymud});
\]
\[
\text{tfyi} = \text{isreal}(\text{tauinterfacialmudy});
\]
\[
\text{tfydhw} = \text{isreal}(\text{y\_dhwmdud});
\]
\[
\text{tfye11} = \text{isreal}(\text{e11dhdydmud});
\]
\[
\text{tfye12} = \text{isreal}(\text{e12dhdydmud});
\]
\[
\text{tfye21} = \text{isreal}(\text{e21dhdydmud});
\]
\[
\text{tfye22} = \text{isreal}(\text{e22dhdydmud});
\]
\[
\text{tfye31} = \text{isreal}(\text{e31dhdydmud});
\]
\[
\text{tfye32} = \text{isreal}(\text{e32dhdydmud});
\]
\[
\text{tfsumy} = \text{tfyvh} + \text{tfyuvh} + \text{tfyvvh} + \text{tfydh} + \text{tfydz} + \text{tfytyy} + \text{tfybed} + \text{tfyi} + \text{tfydhw} + \text{tfye11} + \text{tfye12} + \text{tfye21} + \text{tfye22} + \text{tfye31} + \text{tfye32};
\]
if tfsumy<16
    fclose('all');
    errordlg('Complex Number','ERROR')
    return;
end

end

% Petrov Galerkin Terms
% neglect time terms

% if alpha>0
%     % calculate bigC, bigX, and bigY, where those are the portions of
%     % the equations not including the time derivatives
%     %
%     % leave out all time derivatives for now
%     %
%     if awater_bar==0
%         A_hatwater(1)=0;
%         A_hatwater(2)=0;
%         A_hatwater(3)=0;
%         A_hatwater(4)=0;
%         A_hatwater(5)=0;
%         A_hatwater(6)=0;
%         A_hatwater(7)=0;
%         A_hatwater(8)=0;
%         A_hatwater(9)=0;
%     %
%         B_hatwater(1)=0;
%         B_hatwater(2)=0;
%         B_hatwater(3)=0;
%         B_hatwater(4)=0;
%         B_hatwater(5)=0;
%         B_hatwater(6)=0;
%         B_hatwater(7)=0;
%         B_hatwater(8)=0;
%         B_hatwater(9)=0;
%     else
%         A_hatwater(1)=uw_e/awater_bar;
%         A_hatwater(2)=1/awater_bar;
%         A_hatwater(3)=0;
%         A_hatwater(4)=(cwater_bar*cwater_bar)/awater_bar;
% A_hatchwater(5)=uw_e/awater_bar;
% A_hatchwater(6)=0;
% A_hatchwater(7)=0;
% A_hatchwater(8)=0;
% A_hatchwater(9)=uw_e/awater_bar;
% 
% B_hatchwater(1)=vw_e/awater_bar;
% B_hatchwater(2)=0;
% B_hatchwater(3)=1/awater_bar;
% B_hatchwater(4)=0;
% B_hatchwater(5)=vw_e/awater_bar;
% B_hatchwater(6)=0;
% B_hatchwater(7)=(cwater_bar*cwater_bar)/awater_bar;
% B_hatchwater(8)=0;
% B_hatchwater(9)=vw_e/awater_bar;
% end

%need spatial derivatives for integrals
%already know du/dx and dv/dy from viscosity calcs

dhdxwater=dphidx(1)*hw1+dphidx(2)*hw2+dphidx(3)*hw3;

dhdywater=dphidy(1)*hw1+dphidy(2)*hw2+dphidy(3)*hw3;

vel_mag=sqrt(uw_e*uw_e+vw_e*vw_e);

if hw_e<=2*min_depth
    Sxwater=0;
    Sywater=0;
else
    Sxwater=grav*man_w*man_w*uw_e*vel_mag/((C_d*C_d)*(hw_e)^(1/3));
    %Sx=cds_bed*(u_e*sqrt(u_e*u_e+v_e*v_e));
    Sywater=grav*man_w*man_w*vw_e*vel_mag/((C_d*C_d)*(hw_e)^(1/3));
    %Sy=cds_bed*(v_e*sqrt(u_e*u_e+v_e*v_e));
end

bigCwater=dhdxwater*(1/6)*(uw1+uw2+uw3)+dudxwater*(1/6)*(hw1+hw2+hw3)+dhdywater*(1/6)*(vw1+vw2+vw3)+dvdywater*(1/6)*(hw1+hw2+hw3);

bigXwater=(dudxwater*(uhwater_total))+(dudywater*(vhwater_total))+(grav*dhdxwater);
water*(1/6)*(hw1+hw2+hw3))+(grav*dzdxwater*(1/6)*(hw1+hw2+hw3))+(1/2)*(Sxwater);
%

bigYwater=(dvdxwater*(uhwater_total))+(dvdywater*(vhwater_total))+(grav*dhdywater*(1/6)*(hw1+hw2+hw3))+(grav*dzdywater*(1/6)*(hw1+hw2+hw3))+(1/2)*(Sywater);
%
%            continuitywater_rhs(node1)=continuitywater_rhs(node1) -
%                                     (det_jac*deltaT)*(alpha*l*(dphidx(1)*(A_hatwater(1)*bigCwater+A_hatwater(2)*bigXwater+A_hatwater(3)*bigYwater) +
%                                    dphidy(1)*(B_hatwater(1)*bigCwater+B_hatwater(2)*bigXwater+B_hatwater(3)*bigYwater))));
%
%            continuitywater_rhs(node2)=continuitywater_rhs(node2) -
%                                     (det_jac*deltaT)*(alpha*l*(dphidx(2)*(A_hatwater(1)*bigCwater+A_hatwater(2)*bigXwater+A_hatwater(3)*bigYwater) +
%                                    dphidy(2)*(B_hatwater(1)*bigCwater+B_hatwater(2)*bigXwater+B_hatwater(3)*bigYwater))));
%
%            continuitywater_rhs(node3)=continuitywater_rhs(node3) -
%                                     (det_jac*deltaT)*(alpha*l*(dphidx(3)*(A_hatwater(1)*bigCwater+A_hatwater(2)*bigXwater+A_hatwater(3)*bigYwater) +
%                                    dphidy(3)*(B_hatwater(1)*bigCwater+B_hatwater(2)*bigXwater+B_hatwater(3)*bigYwater))));
%
%            xmomenwater_rhs(node1)=xmomenwater_rhs(node1) -
%                                     (det_jac*deltaT)*(alpha*l*(dphidx(1)*(A_hatwater(4)*bigCwater+A_hatwater(5)*bigXwater+A_hatwater(6)*bigYwater) +
%                                    dphidy(1)*(B_hatwater(4)*bigCwater+B_hatwater(5)*bigXwater+B_hatwater(6)*bigYwater))));
%
%            xmomenwater_rhs(node2)=xmomenwater_rhs(node2) -
%                                     (det_jac*deltaT)*(alpha*l*(dphidx(2)*(A_hatwater(4)*bigCwater+A_hatwater(5)*bigXwater+A_hatwater(6)*bigYwater) +
%                                    dphidy(2)*(B_hatwater(4)*bigCwater+B_hatwater(5)*bigXwater+B_hatwater(6)*bigYwater))));
%
%            xmomenwater_rhs(node3)=xmomenwater_rhs(node3) -
%                                     (det_jac*deltaT)*(alpha*l*(dphidx(3)*(A_hatwater(4)*bigCwater+A_hatwater(5)*bigXwater+A_hatwater(6)*bigYwater) +
%                                    dphidy(3)*(B_hatwater(4)*bigCwater+B_hatwater(5)*bigXwater+B_hatwater(6)*bigYwater))));
%
%            ymomenwater_rhs(node1)=ymomenwater_rhs(node1) -
%                                     (det_jac*deltaT)*(alpha*l*(dphidx(1)*(A_hatwater(7)*bigCwater+A_hatwater(8)*bigXwater+A_hatwater(9)*bigYwater) +
%                                    dphidy(1)*(B_hatwater(7)*bigCwater+B_hatwater(8)*bigXwater+B_hatwater(9)*bigYwater))));
% ymomenwater_rhs(node2)=ymomenwater_rhs(node2)-(det_jac*deltaT)*(alpha*l*(dphidx(2)*(A_hatwater(7)*bigCwater+A_hatwater(8)*bigXwater+A_hatwater(9)*bigYwater)+dphidy(2)*(B_hatwater(7)*bigCwater+B_hatwater(8)*bigXwater+B_hatwater(9)*bigYwater)));
% ymomenwater_rhs(node3)=ymomenwater_rhs(node3)-(det_jac*deltaT)*(alpha*l*(dphidx(3)*(A_hatwater(7)*bigCwater+A_hatwater(8)*bigXwater+A_hatwater(9)*bigYwater)+dphidy(3)*(B_hatwater(7)*bigCwater+B_hatwater(8)*bigXwater+B_hatwater(9)*bigYwater)));
%
% end

if FM_FLAG==1
  if alpha>0
    %calculate bigC, bigX, and bigY, where those are the portions of
    %the equations not including the time derivatives or viscous

    if amud_bar==0
      A_hatmud(1)=0;
      A_hatmud(2)=0;
      A_hatmud(3)=0;
      A_hatmud(4)=0;
      A_hatmud(5)=0;
      A_hatmud(6)=0;
      A_hatmud(7)=0;
      A_hatmud(8)=0;
      A_hatmud(9)=0;

      B_hatmud(1)=0;
      B_hatmud(2)=0;
      B_hatmud(3)=0;
      B_hatmud(4)=0;
      B_hatmud(5)=0;
      B_hatmud(6)=0;
      B_hatmud(7)=0;
      B_hatmud(8)=0;
      B_hatmud(9)=0;

    else
      A_hatmud(1)=umud_e/amud_bar;
      A_hatmud(2)=1/amud_bar;
      A_hatmud(3)=0;
      A_hatmud(4)=(cmud_bar*cmud_bar)/amud_bar;
      A_hatmud(5)=umud_e/amud_bar;
      A_hatmud(6)=0;
      A_hatmud(7)=0;
    end
  end
end
A_hatmud(8)=0;
A_hatmud(9)=umud_e/amud_bar;

B_hatmud(1)=vmud_e/amud_bar;
B_hatmud(2)=0;
B_hatmud(3)=1/amud_bar;
B_hatmud(4)=0;
B_hatmud(5)=vmud_e/amud_bar;
B_hatmud(6)=0;
B_hatmud(7)=(cmud_bar*cmud_bar)/amud_bar;
B_hatmud(8)=0;
B_hatmud(9)=vmud_e/amud_bar;
end

%need spatial derivatives for integrals
%already know du/dx and dv/dy from viscosity calcs
dhdxmud=dphidx(1)*hmud1+dphidx(2)*hmud2+dphidx(3)*hmud3;
dhdymud=dphidy(1)*hmud1+dphidy(2)*hmud2+dphidy(3)*hmud3;

vel_mag=sqrt((umud_e*umud_e) + (vmud_e*vmud_e));

if hmud_e<=2*min_depth
   Sxmud=0;
   Symud=0;
else
   Sxmud=grav*man_mud*man_mud*umud_e*vel_mag/((C_d*C_d)*(hmud_e)^(1/3));
   %Sx=cds_bed*(u_e*sqrt(u_e*u_e+v_e*v_e));
   Symud=grav*man_mud*man_mud*vmud_e*vel_mag/((C_d*C_d)*(hmud_e)^(1/3));
   %Sy=cds_bed*(v_e*sqrt(u_e*u_e+v_e*v_e));
end

if hmud_e<=2*min_depth
   Sixmud=0;
   Siymud=0;
else
   Sixmud=f_interface*(rhowater/rhofm)*delta_u*delta_u_mag/8;
   Siymud=f_interface*(rhowater/rhofm)*delta_v*delta_u_mag/8;
end
\texttt{bigCmud=hdhdmdud*(1/6)*(umud1+umud2+umud3) +
dudxmud*(1/6)*(hmud1+hmud2+hmud3) + dhdxmud*(1/6)*(vmud1+vmud2+vmud3) +
dvdxmud*(1/6)*(hmud1+hmud2+hmud3);}

\texttt{%X= hu(du/dx) +hv(du/dy) +gh(dh/dx) +gh(dz/dx)
+Xmud = Sixmud}

\texttt{bigXmud=(dudxmud*(uhmud_total)) + (dudymud*(vhmud_total)) +
(grav*hdhmud*(1/6)*(hmud1+hmud2+hmud3)) +
(grav*zdhmud*(1/6)*(hmud1+hmud2+hmud3)) +
(grav*hdhwdmud*(1/6)*(hmud1+hmud2+hmud3)*(rhowater/rhofm)) + (1/2)*(Xmud) -
(1/2)*(Sixmud);}

\texttt{bigYmud=(dvdxmud*(uhmud_total)) + (dvdxmud*(vhmud_total)) +
(grav*dhdxmud*(1/6)*(hmud1+hmud2+hmud3)) +
(grav*dzdxmud*(1/6)*(hmud1+hmud2+hmud3)) +
(grav*dhwdymud*(1/6)*(hmud1+hmud2+hmud3)*(rhowater/rhofm)) + (1/2)*(Ymud) -
(1/2)*(Siymud);}

\texttt{%error tracking}
\texttt{tfsupgc=isreal(bigCmud);}
\texttt{tfsupgx=isreal(bigXmud);}
\texttt{tfsupgy=isreal(bigYmud);}
\texttt{tfahatmud=isreal(A_hatmud);}
\texttt{tfbhatmud=isreal(B_hatmud);}

\texttt{tfsupgsum=tfsupgc+tfsupgx+tfsupgy+tfahatmud+tfbhatmud;}

\texttt{if tfsupgsum<5
  fclose('all');
  errordlg('Complex Number - SUPG','ERROR')
  return;
end}

\texttt{continuitymud_rhs(node1)=continuitymud_rhs(node1)-
(det_jac*deltaT)*(alpha*1*(dphidx(1)*(A_hatmud(1)*bigCmud+A_hatmud(2)*bigXmud
+A_hatmud(3)*bigYmud) +
dphidy(1)*(B_hatmud(1)*bigCmud+B_hatmud(2)*bigXmud+B_hatmud(3)*bigYmud)));}

\texttt{continuitymud_rhs(node2)=continuitymud_rhs(node2)-
(det_jac*deltaT)*(alpha*1*(dphidy(1)*(A_hatmud(1)*bigCmud+A_hatmud(2)*bigXmud
+A_hatmud(3)*bigYmud) +
dphidy(2)*(B_hatmud(1)*bigCmud+B_hatmud(2)*bigXmud+B_hatmud(3)*bigYmud)));}
%Enforce water boundary conditions
%zero out rows where H is specified
%specifying depth NOT SURFACE ELEVATION

% %right hand side, outflow
% continuitywater_rhs(546,:)=0;
% continuitywater_rhs(545,:)=0;
% continuitywater_lhs(543,:) = 0;  
% continuitywater_lhs(538,:) = 0;  
% continuitywater_lhs(531,:) = 0;  
% continuitywater_lhs(523,:) = 0;

% %% Left Hand Side  
% continuitywater_lhs(1,:) = 0;  
% continuitywater_lhs(2,:) = 0;  
% continuitywater_lhs(3,:) = 0;  
% continuitywater_lhs(4,:) = 0;  
% continuitywater_lhs(5,:) = 0;  
% continuitywater_lhs(6,:) = 0;

% right hand side, outflow  
% continuitywater_lhs(546,546) = 1;  
% continuitywater_lhs(545,545) = 1;  
% continuitywater_lhs(543,543) = 1;  
% continuitywater_lhs(538,538) = 1;  
% continuitywater_lhs(531,531) = 1;  
% continuitywater_lhs(523,523) = 1;

% % left side, flow from left to right  
% continuitywater_lhs(1,1) = 1;  
% continuitywater_lhs(2,2) = 1;  
% continuitywater_lhs(3,3) = 1;  
% continuitywater_lhs(4,4) = 1;  
% continuitywater_lhs(5,5) = 1;  
% continuitywater_lhs(6,6) = 1;

% % % right hand side, outflow  
% % wse - hmud - bed  
% continuitywater_rhs(546) = min(0.249,(h_water(544)+nodes(544,4)+h_mud(544))) - nodes(546,4);  
% continuitywater_rhs(545) = min(0.249,(h_water(542)+nodes(542,4)+h_mud(542))) - nodes(545,4);  
% continuitywater_rhs(543) = min(0.249,(h_water(541)+nodes(541,4)+h_mud(541))) - nodes(543,4);  
% continuitywater_rhs(538) = min(0.249,(h_water(536)+nodes(536,4)+h_mud(536))) - nodes(538,4);  
% continuitywater_rhs(531) = min(0.249,(h_water(529)+nodes(529,4)+h_mud(529))) - nodes(531,4);  
% continuitywater_rhs(523) = min(0.249,(h_water(522)+nodes(521,4)+h_mud(521))) - nodes(523,4);
% continuitywater_rhs(546)=min(0.249, (h_water(544)+nodes(544,4)))-nodes(546,4);-%h_mud(546);
% continuitywater_rhs(545)=min(0.249, (h_water(542)+nodes(542,4)))-nodes(545,4);-%h_mud(545);
% continuitywater_rhs(543)=min(0.249, (h_water(541)+nodes(541,4)))-nodes(543,4);-%h_mud(543);
% continuitywater_rhs(538)=min(0.249, (h_water(536)+nodes(536,4)))-nodes(538,4);-%h_mud(538);
% continuitywater_rhs(531)=min(0.249, (h_water(529)+nodes(529,4)))-nodes(531,4);-%h_mud(531);
% continuitywater_rhs(523)=min(0.249, (h_water(522)+nodes(521,4)))-nodes(523,4);-%h_mud(523);

% continuitywater_rhs(546)=h_water(544);
% continuitywater_rhs(545)=h_water(542);
% continuitywater_rhs(543)=h_water(541);
% continuitywater_rhs(538)=h_water(536);
% continuitywater_rhs(531)=h_water(529);
% continuitywater_rhs(523)=h_water(522);

% continuitywater_rhs(546)=0.235-nodes(546,4);-%h_mud(546);
% continuitywater_rhs(545)=0.235-nodes(545,4);-%h_mud(545);
% continuitywater_rhs(543)=0.235-nodes(543,4);-%h_mud(543);
% continuitywater_rhs(538)=0.235-nodes(538,4);-%h_mud(538);
% continuitywater_rhs(531)=0.235-nodes(531,4);-%h_mud(531);
% continuitywater_rhs(523)=0.235-nodes(523,4);-%h_mud(523);

% left side, flow from left to right, inflow
% if T<100
%     us_wse= ( (T-40)/60 ) * 0.017 + 0.235;
%     %us_wse=0.235;
% else
%     us_wse=0.252;
%     %us_wse=0.235;
% end

% continuitywater_rhs(1)=us_wse-nodes(1,4)-h_mud(1);
% continuitywater_rhs(2)=us_wse-nodes(2,4)-h_mud(2);
% continuitywater_rhs(3)=us_wse-nodes(3,4)-h_mud(3);
% continuitywater_rhs(4)=us_wse-nodes(4,4)-h_mud(4);
% continuitywater_rhs(5)=us_wse-nodes(5,4)-h_mud(5);
% continuitywater_rhs(6)=us_wse-nodes(6,4)-h_mud(6);

% % set mud outflow as open boundary  
% % coarse
% % continuitymud_lhs(181,:) = 0;
% % continuitymud_lhs(182,:) = 0;
% % continuitymud_lhs(183,:) = 0;
% % continuitymud_lhs(184,:) = 0;
% 
% % Fine
continuitymud_lhs(546,:) = 0;
continuitymud_lhs(545,:) = 0;
continuitymud_lhs(543,:) = 0;
continuitymud_lhs(538,:) = 0;
continuitymud_lhs(531,:) = 0;
continuitymud_lhs(523,:) = 0;

% coarse
% continuitymud_lhs(181,181) = 1;
% continuitymud_lhs(182,182) = 1;
% continuitymud_lhs(183,183) = 1;
% continuitymud_lhs(184,184) = 1;

% fine
continuitymud_lhs(546,546) = 1;
continuitymud_lhs(545,545) = 1;
continuitymud_lhs(543,543) = 1;
continuitymud lhs(538,538) = 1;
continuitymud_lhs(531,531) = 1;
continuitymud_lhs(523,523) = 1;

% coarse
% continuitymud_rhs(181) = h_mud(177);
% continuitymud_rhs(182) = h_mud(178);
% continuitymud_rhs(183) = h_mud(179);
% continuitymud_rhs(184) = h_mud(180);

% fine
continuitymud_rhs(546) = min_depth;
continuitymud_rhs(545) = min_depth;
continuitymud_rhs(543) = min_depth;
continuitymud_rhs(538) = min_depth;
continuymud_rhs(531)=min_depth;
continuymud_rhs(523)=min_depth;

%Solve equations
%turn off water calcs
%    Hwater=continuitywater_lhs\continuitywater_rhs;
%    for i=1:length(H)
%       if H(i)<0
%          i
%             return; %exit code and print node number if depth goes negative
%       end
%    end
%
%truncate H to 8 decimal places
%    Hwater=round(Hwater*1e8)/1e8;
%    for i=1:num_node
%       if isnan(Hwater(i))||Hwater(i)<=min_depth
%          Hwater(i)=min_depth;
%       else
%          Hwater(i)=Hwater(i);
%    end
%
%truncate UH and VH to 8 decimal places
%    UHwater=xmomenwater lhs\xmomenwater rhs;
%    %UH=round(UH*1e8)/1e8;
%    Uwater=UHwater./Hwater;
%    for i=1:num_node
%       if Hwater(i)<=2*min_depth
%          Uwater(i)=0;
%       else
%          Uwater(i)=Uwater(i);
%    end
%
%Uwater=round(Uwater*1e8)/1e8;
%
%    %Replace NaN and Inf values with 0
%    for i=1:length(U)
%       if isnan(U(i))
\% \% U(i)=0; 
\% \% elseif isinf(U(i)) 
\% \% \hspace{1em} U(i)=0; 
\% \% end 
\% \% end 
\% VHwater=ymomenwater_{lhs}/ymomenwater_{rhs}; 
\% \% VH=round(VH*1e8)/1e8; 
\% \hspace{1em} Vwater=VHwater./Hwater; 
\% \hspace{1em} for i=1:num_node 
\% \hspace{2em} if Hwater(i)<=2*min_depth 
\% \hspace{3em} Vwater(i)=0; 
\% \hspace{2em} else 
\% \hspace{3em} Vwater(i)=Vwater(i); 
\% \hspace{2em} end 
\% \hspace{1em} end 
\% Vwater=round(Vwater*1e8)/1e8; 
\% \% \hspace{1em} %Replace NaN and Inf values with 0 
\% \% \hspace{1em} for i=1:length(V) 
\% \hspace{2em} if isnan(V(i)) 
\% \hspace{3em} V(i)=0; 
\% \hspace{2em} elseif isinf(V(i)) 
\% \hspace{3em} V(i)=0; 
\% \hspace{2em} end 
\% \hspace{1em} end 
\% \% \hspace{1em} %update old time info with current time info 
\% \% \hspace{1em} %WSE=H+nodes(:,4); 
\% \hspace{1em} %Solve equations 
\% \% \hspace{1em} for i=1:length(H) 
\% \hspace{2em} if H(i)<0 
\% \hspace{3em} i 
\% \hspace{2em} return; \%exit code and print node number if depth goes 
\% \hspace{2em} negative 
\% \hspace{2em} end 
\% \% end 
\% \hspace{1em} h_water=Hwater; 
\% \hspace{1em} u_water=Uwater; 
\% \hspace{1em} v_water=Vwater; 
\% \hspace{1em} vel_water(:,1)=u_water;
%   vel_water(:,2)=v_water;

if FM_FLAG==1
    
    %error tracking
    tfclhs=isreal(continuitymud_lhs);
    tfcrhs=isreal(continuitymud_rhs);

    tfxlhs=isreal(xmomenmud_lhs);
    tfxrhs=isreal(xmomenmud_rhs);

    tfylhs=isreal(ymomenmud_lhs);
    tfyrhs=isreal(ymomenmud_rhs);

    tfsum=tfclhs+tfcrhs+txlhs+txrhs+tylhs+tyrhs;

    if tfsum<6
        fclose('all');
        errordlg('Complex Number - Solution Matrices','ERROR')
        return;
    end

    Hmud=continuitymud_lhs\continuitymud_rhs;
    %truncate H to 8 decimal places
    Hmud=round(Hmud*1e8)/1e8;
    for i=1:num_node
        if isnan(Hmud(i))||Hmud(i)<=min_depth
            Hmud(i)=min_depth;
        else
            Hmud(i)=Hmud(i);
        end
    end

    %truncate UH and VH to 8 decimal places
    UHmud=xmomenmud_lhs\xmomenmud_rhs;
    %UH=round(UH*1e8)/1e8;
    Umud=UHmud./Hmud;
    for i=1:num_node
        if Hmud(i)<=2*min_depth
            Umud(i)=0;
        else
            Umud(i)=Umud(i);
Umud=round(Umud*1e8)/1e8;

%Replace NaN and Inf values with 0
% for i=1:length(U)
%   if isnan(U(i))
%       U(i)=0;
%   elseif isinf(U(i))
%       U(i)=0;
%   end
% end

VHmud=ymomenmud_lhs\ymomenmud_rhs;
%VH=round(VH*1e8)/1e8;
Vmud=VHmud./Hmud;
for i=1:num_node
    if Hmud(i)<=2*min_depth
        Vmud(i)=0;
    else
        Vmud(i)=Vmud(i);
    end
end
Vmud=round(Vmud*1e8)/1e8;

%Replace NaN and Inf values with 0
% for i=1:length(V)
%   if isnan(V(i))
%       V(i)=0;
%   elseif isinf(V(i))
%       V(i)=0;
%   end
% end

%update old time info with current time info

h_mud=Hmud;
u_mud=Umud;
v_mud=Vmud;
vel_mud(:,1)=u_mud;
vel_mud(:,2)=v_mud;
newbed=h_mud+nodes(:,4);

if T<750
    wse=(T-T1)*(ADH_wse(:,T2)-ADH_wse(:,T1))+ADH_wse(:,T1);
else
    wse=ADH_wse(:,751);
end

%h_water=wse-newbed;

if T<40
    flow=0;
elseif T<T_ramp+40
    flow=(0.0012)*((T-40)/T_ramp);
else
    flow=0.0012;
end

% if T<=T_ramp
%     wse=(T/T_ramp)*(0.000077).*((10.9728-nodes(:,2))/10.9728)+(0.23+(T/TF)*(0.019957138));
% else
%     wse=(0.000077).*((10.9728-nodes(:,2))/10.9728)+(0.23+(T/TF)*(0.019957138));
%     flow=(T/T_ramp)*0.0012;
% end

h_water=wse-newbed;

u_water=flow./(0.3048*h_water);

v_water=zeros(num_node,1);
vel_water(:,1)=u_water;
vel_water(:,2)=v_water;

end

%increment time and output to Matlab
T=T+deltaT
count=count+1;

%check for outtime value
if T>=outtime
    outT=round(T*1000)/1000;
%Write output files
%print depth timestep and values
fprintf(depthwater_output,'TS 0 %.8e\n',outT);
fprintf(depthwater_output,'%.8e\n',h_water);

%print velocity timestep and values
fprintf(velocitywater_output,'TS 0 %.8e\n',outT);
fprintf(velocitywater_output,'%.8e %.8e %.8e\n',transpose(vel_water));

if FM_FLAG==1
  %Write mud output files
  %print depth timestep and values
  fprintf(depthmud_output,'TS 0 %.8e\n',outT);
  fprintf(depthmud_output,'%.8e\n',h_mud);

  %print velocity timestep and values
  fprintf(velocitymud_output,'TS 0 %.8e\n',outT);
  fprintf(velocitymud_output,'%.8e %.8e %.8e\n',transpose(vel_mud));

  %print entrain timestep and values
  %fprintf(entrain_output,'TS 0 %.8e\n',T);
  fprintf(entrain_output,'%8e\n',entrain);
end
outtime=outtime+outstep;
end
end
%write ENDDS
fprintf(depthwater_output,'ENDDS');
fprintf(velocitywater_output,'ENDDS');
if FM_FLAG==1
fprintf(depthmud_output,'ENDDS');
fprintf(velocitymud_output,'ENDDS');
fprintf(entrain_output,'ENDDS');
end
%close files
fclose('all');