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Analysis of Classes of Singular Boundary Value Problems

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By

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In this dissertation we study positive solutions to a singular p-Laplacian elliptic boundary value problem on a bounded domain with smooth boundary when a positive parameter varies. Our main focus is the analysis of a challenging class of singular p-Laplacian problems. We establish the existence of a positive solution for all positive values of the parameter and the existence of at least two positive solutions for a certain explicit range of the parameter. In the Laplacian case, we also prove the uniqueness of the positive solution for large values of the parameter. We extend our existence and multiplicity results to classes of singular systems and to the case when a domain is an exterior domain. We prove our existence and multiplicity results by the method of sub and supersolutions and our uniqueness result by establishing apriori and boundary estimates. Such results are well known in the literature for the nonsingular case. In this study, we extend these results to the more difficult singular case.

Key words:singular boundary value problems, p-Laplacian operator, positive solutions, existence, multiplicity, uniqueness, sub-supersolutions, apriori estimate

DEDICATION

To my parents.

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Δu the Laplacian of u, i.e., $\Delta u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}$

$\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ the p-Laplacian of u

$d(x, \partial\Omega)$ the distance function from x to the boundary $\partial\Omega$

$C([0,\infty),(0,\infty))$ the set of all continuous positive real-value functions on $[0,\infty)$

$C(\overline{\Omega})$ the set of all continuous real-value functions on $\overline{\Omega}$

$C^m(\Omega)$ the set of all continuously m- times differentiable functions on Ω

$C^\infty(\Omega) = \cap_{k=0}^{\infty} C^k(\Omega)$

$C_0^\infty(\Omega)$ the set of all functions in C^∞ with compact support in Ω

$W^{1,p}(\Omega)$ the set of all functions $u \in L^p(\Omega)$ such that the weak derivative $Du \in L^p(\Omega)$

$W_0^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$

CHAPTER 1

INTRODUCTION

We study quasilinear boundary value problems of the form:

$$\begin{cases} -\Delta_p u = \lambda \frac{f(u)}{u^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_p u = \text{div}\,(|\nabla u|^{p-2}\nabla u)$ is the $p-$Laplacian operator of $u, p > 1$, λ is a positive parameter, $0 < \beta < 1$, Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$ and $f : [0, \infty) \to (0, \infty)$ is a continuous function. Note that $\lim_{u\to 0} \dfrac{f(u)}{u^\beta} = \infty$, and hence (1.1) is a singular boundary value problem. In this dissertation we study positive solutions of (1.1) in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$. In particular, we establish existence and multiplicity results of (1.1) under additional assumptions on f when λ varies in $(0, \infty)$ by the method of sub and supersolutions. Further, we extend the results to classes of systems and to the case when Ω is an exterior domain in $\mathbb{R}^N, N > 2$. We also focus on establishing a uniqueness result of (1.1) when $p = 2$ for large values of the parameter λ.

In the case $p = 2$ and $\beta = 0$, there is a very rich history in the study of positive solutions of such problems (see [2, 3, 4, 9, 11, 12, 15, 17, 27, 29, 31, 36, 38, 47, 56, 60, 62]). Cohen and Laetsch in [12] show that if $\frac{u}{f(u)}$ is an increasing function for $u \geq 0$, then there exists at most one positive solution for all $\lambda > 0$. Hence, to obtain multiple positive solutions, $\frac{u}{f(u)}$ must decrease for a certain range of u. In [9], the authors discuss classes of nonlinearities

$f(u)$ when $\frac{u}{f(u)}$ has a local maximum followed by a local minimum and prove that there exist at least three positive solutions for a certain range of λ. Further, in [11] and [60], under some growth conditions on f', it was established that there exists a unique positive solution for large and small values of λ.

A typical example of such a model is the perturbed Gelfand problem:

$$\begin{cases} -\Delta u = \lambda \exp[\frac{\alpha u}{\alpha+u}] & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

This problem arises in the study of (steady state) ignition models in combustion theory, and it has been discussed in $[6,\ 53,\ 59]$. Here u is the dimensionless temperature, $\exp[\frac{\alpha u}{\alpha+u}]$ is the chemical reaction term in Arrhenius law and $\alpha > 0$ (usually large) is the activation energy. In [63], the author proves that the bifurcation curve of positive solutions of (1.2) is exactly $S-$shaped for α large when $N = 1$. This result was extended to the case when Ω is a ball in $\mathbb{R}^2$ in [24].

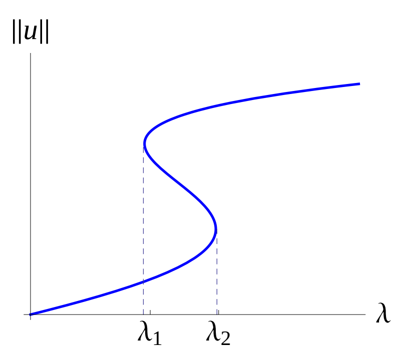

Figure 1.1

$S-$shaped bifurcation curve

The $p-$Laplacian operator Δ_p with $p > 1$ arises in the modeling of many physical and natural phenomena; non-Newtonian mechanics in [18, 33], nonlinear elasticity and glaciology in [28, 33], combustion theory in [6, 32], population biology in [52] and system of Monge-Kantorovich partial differential equations in [25]. There are many papers devoted to the study of existence and multiplicity results of $p-$Laplacian boundary value problems (see [7, 8, 10, 16, 20, 21, 22, 33, 37, 39, 45, 52, 57, 64]). In the nonsingular case $\beta = 0,$ Ramaswamy and Shivaji in [57] extend the existence and multiplicity results of the case $p = 2$ in [9] to the $p-$Laplacian case when $p > 1.$ The authors in [1] and [30] have extended these results for $p-$ Laplacian systems.

Main focus of this thesis is to establish existence and multiplicity results for singular $p-$Laplacian boundary value problems ($p > 1$ and $\beta \neq 0$) and also extend our results to singular systems. Note that in the case when $\beta = 0$ the typical multiplicity result in the literature is the existence of at least three solutions for a certain range of λ. However, our multiplicity results when $\beta \neq 0$ are restricted to two solutions for a certain range of λ. We conjecture that even for $\beta \neq 0$ there are at least three positive solutions for such a range, and we establish this fact in the study of the one dimensional $p-$Laplacian perturbed Gelfand problem:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda \frac{\exp[\frac{\alpha u}{\alpha+u}]}{u^{\beta}} \text{ in } (0,1), \\ u(0) = 0 = u(1). \end{cases} \tag{1.3}$$

Here we prove that the bifurcation curve of positive solutions of (1.3) is at least S-shaped for α large by the Quadrature method. Further, we provide computational results showing that the bifurcation curve of positive solutions of (1.3) is in fact, exactly $S-$shaped.

Additionally, for such singular models we study a uniqueness result for large values of λ when $p = 2$ (Laplacian case). We prove the uniqueness of the positive solution of (1.1) for λ large by establishing apriori and boundary estimates.

The remainder of this chapter will be mainly concerned with the statement of our results.

1.1 Singular boundary value problems on bounded domains

We first consider positive solutions of a singular boundary value problem:

$$\begin{cases} -\Delta_p u = \lambda \frac{f(u)}{u^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$, λ is a positive parameter, $0 < \beta < 1$ and $f : [0, \infty) \to (0, \infty)$ is a continuous function. To state our main result, we assume that f satisfies:

(H_1) $\lim\limits_{u\to\infty} \dfrac{f(u)}{u^{\beta+p-1}} = 0.$

We first establish:

Theorem 1.1.1 *Assume* (H_1)*. Then* (1.4) *has a positive solution for all* $\lambda > 0$.

We refer to [54] for a more general existence result for (1.4).

However, for certain classes of f we can get at least two positive solutions for a certain range of λ. To state this multiplicity result, for any $0 < a < d$ we define

$$Q(a, d) := \frac{a^{\beta+p-1}}{f(a)} / \frac{d^{\beta+p-1}}{f(d)}. \tag{1.5}$$

The motivation for the analysis of the ratio $Q(a,d) = \frac{a^{\beta+p-1}}{f(a)} / \frac{d^{\beta+p-1}}{f(d)}$ comes from the nonsingular Laplacian case. In this simpler case, $Q(a,d) = \frac{a}{f(a)} / \frac{d}{f(d)}$, and it is well known that if the function $\frac{s}{f(s)}$ is nondecreasing, then the boundary value problem has a unique solution for all $\lambda > 0$. For a multiplicity result the function $\frac{s}{f(s)}$ must decrease at least for a certain range, and our multiplicity result corresponds to a situation when you can find $a < d$ such that $\frac{a^{\beta+p-1}}{f(a)} / \frac{d^{\beta+p-1}}{f(d)}$ is large enough. Let

$$A_N := (\frac{(N+p-1)^{N+p-1}}{N^N})^{\frac{1}{p-1}}, \tag{1.6}$$

$w \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$ (see Lemma 3.1 in [26]) be the unique solution of

$$\begin{cases} -\Delta_p w = \frac{1}{w^\beta} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{1.7}$$

and assume f satisfies:

(H_2) $f(u)$ is nondecreasing for all $u \geq 0$.

(H_3) There exist a, b with $a \in (0, \frac{p}{A_N} b)$ and $\frac{f(u)}{u^\beta}$ is nondecreasing on (a, b).

We establish:

Theorem 1.1.2 *Assume* $(H_1) - (H_3)$ *and there exists* d *such that* $a < d < \frac{p}{A_N} b$ *and* $Q(a,d) > \frac{A_N^{p-1} N \|w\|_\infty^{\beta+p-1}}{(p-1)^{p-1} R^p} := C(\beta, p, N, \Omega)$*, where* R *is the radius of the largest inscribed ball* B_R *in* Ω*. Then* (1.4) *has at least two positive solutions for* $\lambda_* < \lambda < \lambda^*$*, where*

$$\lambda_* = \frac{d^{\beta+p-1}}{f(d)} \frac{A_N^{p-1} N}{(p-1)^{p-1} R^p}, \tag{1.8}$$

$$\lambda^* = \min \left\{ \frac{d^\beta}{f(d)} \frac{N}{R^p} \left(\frac{p}{p-1} \right)^{p-1} b^{p-1}, \ \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}} \right\}. \tag{1.9}$$

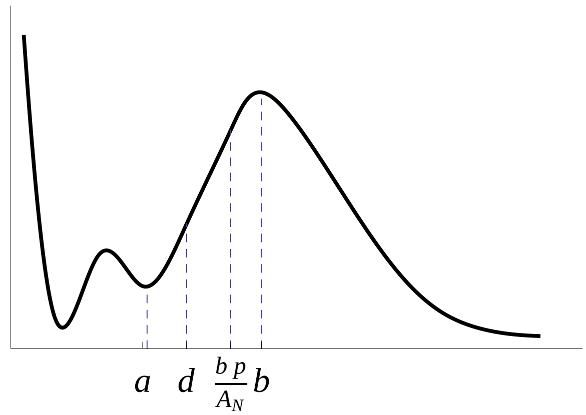

Figure 1.2

Graph of the function $\frac{f(u)}{u^\beta}$

Remark 1.1.1 *Since* $d < \frac{p}{A_N}b$, *we have* $\frac{d^{\beta+p-1}}{f(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p} < \frac{d^\beta}{f(d)}\frac{N}{R^p}(\frac{p}{p-1})^{p-1}b^{p-1}$ *and since* $Q(a,d) > \frac{A_N^{p-1}N\|w\|_\infty^{\beta+p-1}}{(p-1)^{p-1}R^p}$, *we obtain* $\frac{d^{\beta+p-1}}{f(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p} < \frac{a^{\beta+p-1}}{f(a)}\frac{1}{\|w\|_\infty^{\beta+p-1}}$. *Therefore,* (λ_*, λ^*) *is not empty.*

Remark 1.1.2 *A simple example satisfying the hypotheses of Theorem 1.1.1 and Theorem 1.1.2 is*

$$\begin{cases} -\Delta_p u = \lambda \dfrac{\exp[\frac{\alpha u}{\alpha+u}]}{u^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

Clearly, $f(u) := \exp[\frac{\alpha u}{\alpha+u}]$ *satisfies hypotheses* (H_1) *and* (H_2). *Choosing* $a = 1, d = \alpha$ *and* $b = \frac{\alpha^2}{2}$, *we can easily show that* $\frac{f(u)}{u^\beta}$ *is nondecreasing on* (a,b) *for* $\alpha \gg 1$. *Further* $Q(a,d) = \frac{a^{\beta+p-1}}{f(a)}\frac{f(d)}{d^{\beta+p-1}} = [\frac{1}{\alpha}]^{\beta+p-1}\exp[\frac{\alpha}{2} - \frac{\alpha}{\alpha+1}]$, *and hence for any given* Ω, *we have* $a < d < \frac{p}{A_N}b$ *and* $Q(1,\alpha) > C(\beta,p,N,\Omega)$, *for* α *large.*

Next we extend our results to systems of the forms:

$$\begin{cases} -\Delta_p u = \lambda \frac{f_1(v)}{u^\beta} & \text{in } \Omega, \\ -\Delta_p v = \lambda \frac{f_2(u)}{v^\beta} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

Assume that f_1 and f_2 are $C([0,\infty),(0,\infty))$ and satisfy:

(H_4) f_1 and f_2 are nondecreasing.

(H_5) $\lim\limits_{x\to\infty} \dfrac{f_1(Mf_2(x))}{x^{\beta+p-1}} = 0$ for all $M > 0$ (a combined sublinear condition at infinity).

We establish:

Theorem 1.1.3 *Assume* $(H_4)-(H_5)$. *Then* (1.11) *has a positive solution for all* $\lambda > 0$.

Next, under certain combined nonlinear effects of $\frac{x^{\beta+p-1}}{f_1(x)}$ and $\frac{x^{\beta+p-1}}{f_2(x)}$ we study the existence of multiple positive solutions to (1.11). To state the multiplicity result, for any $0 < a < d$ we define

$$Q_1(a,d) := \frac{a^{\beta+p-1}}{f_2(a)} \Big/ \frac{d^{\beta+p-1}}{f_1(d)}. \tag{1.12}$$

We also assume:

(H_6) $f_1(u) \le f_2(u)$ for all $u \ge 0$.

(H_7) There exist a, b with $a \in (0, \frac{p}{A_N} b)$ and $\frac{f_1(u)}{u^\beta}$ is nondecreasing on (a,b).

We establish:

Theorem 1.1.4 *Assume* $(H_4)-(H_7)$ *and there exists* d *such that* $a < d < \frac{p}{A_N}b$ *and* $Q_1(a,d) > C(\beta,p,N,\Omega)$*, where* $C(\beta,p,N,\Omega)$ *is as defined before. Then* (1.11) *has at least two positive solutions for* $\lambda_* < \lambda < \lambda^*$*, where*

$$\lambda_* = \frac{d^{\beta+p-1}}{f_1(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p}, \tag{1.13}$$

$$\lambda^* = \min\{\frac{d^\beta}{f_1(d)}\frac{N}{R^p}(\frac{p}{p-1})^{p-1}b^{p-1},\ \frac{a^{\beta+p-1}}{f_2(a)}\frac{1}{\|w\|_\infty^{\beta+p-1}}\}. \tag{1.14}$$

Remark 1.1.3 *Since* $d < \frac{p}{A_N}b$, *we have* $\frac{d^{\beta+p-1}}{f_1(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p} < \frac{d^\beta}{f_1(d)}\frac{N}{R^p}(\frac{p}{p-1})^{p-1}b^{p-1}$ *and since* $Q_1(a,d) > \frac{A_N^{p-1}N\|w\|_\infty^{\beta+p-1}}{(p-1)^{p-1}R^p}$, *we obtain* $\frac{d^{\beta+p-1}}{f_1(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p} < \frac{a^{\beta+p-1}}{f_2(a)}\frac{1}{\|w\|_\infty^{\beta+p-1}}$*. Therefore,* (λ_*,λ^*) *is not empty.*

Remark 1.1.4 *A simple example satisfying the hypotheses of Theorem 1.1.3 and Theorem 1.1.4 is*

$$\begin{cases} -\Delta_p u = \lambda\dfrac{\exp[\frac{\alpha v}{\alpha+v}]}{u^\beta} & \text{in } \Omega,\\ -\Delta_p v = \lambda\dfrac{u^q+M}{v^\beta} & \text{in } \Omega,\\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

where $q > 0$ *and* $M \gg 1$ *so that* (H_6) *is satisfied. Clearly,* $f_1(u) := \exp[\frac{\alpha u}{\alpha+u}]$ *and* $f_2(u) := u^q + M$ *satisfy hypotheses* (H_4) *and* (H_5)*. Choosing* $a = 1, d = \alpha$ *and* $b = \frac{\alpha^2}{2}$*, we can easily show that* $\frac{f_1(u)}{u^\beta}$ *is nondecreasing on* (a,b) *for* $\alpha \gg 1$*. Further* $Q_1(a,d) = \frac{a^{\beta+p-1}}{f_2(a)}\frac{f_1(d)}{d^{\beta+p-1}} = \left(\frac{1}{1+M}\right)\left(\frac{1}{\alpha}\right)^{\beta+p-1}\exp[\frac{\alpha}{2}]$*, and hence for any given* Ω*, we have* $a < d < \frac{p}{A_N}b$ *and* $Q_1(1,\alpha) > C(\beta,p,N,\Omega)$*, for* α *large.*

1.2 Singular boundary value problems on exterior domains

Next we consider positive radial solutions to a quasilinear boundary value problem

$$\begin{cases} -\Delta_p u = \lambda K(|x|)\frac{f(u)}{u^\beta} & \text{in } \Omega, \\ u(x) = 0 & \text{if } |x| = r_0, \\ u(x) \to 0 & \text{if } |x| \to \infty, \end{cases} \tag{1.16}$$

where $\Omega = \{x \in \mathbb{R}^N : |x| > r_0, r_0 > 0\}$, $1 < p < N, 0 \leq \beta < 1$, $f \in C([0,\infty),(0,\infty))$ and $K \in C([r_0,\infty),(0,\infty))$ such that $K(r) < \frac{1}{r^\mu}$ for $r \gg 1$ and for some $\mu > p-1$. In the nonsingular case when $\beta = 0$, such problems like (1.16) have been discussed by many authors in [5, 29, 43, 46, 49, 50, 55, 58, 65] for the case $p = 2$. Also see [19, 37, 40, 44, 48, 51] for extensions to the case $p > 1$. Here we are interested in the singular case when $0 < \beta < 1$.

Note that the change of variables $r = |x|$ and $t = (\frac{r}{r_0})^{\frac{p-N}{p-1}}$ transforms (1.16) (see Appendix) to:

$$\begin{cases} -(\varphi_p(u'(t)))' = \lambda h(t)\frac{f(u(t))}{u(t)^\beta} \ \text{in } (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.17}$$

where $\varphi_p(u) = |u|^{p-2}u$ and h is given by

$$h(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p \, t^{\frac{p(1-N)}{N-p}} K\left(r_0 \, t^{\frac{1-p}{N-p}}\right). \tag{1.18}$$

Since $K(r) < \frac{1}{r^\mu}$ for $r \gg 1$ and for some $\mu > p-1$, it turns out that

$$h \in \left\{ g \in C((0,1],(0,\infty)) : \int_0^1 s^\delta g(s)\, ds < \infty \ \text{ for some } \delta < p-1 \right\}. \tag{1.19}$$

Note that $h(t)$ is nonsingular at 0 if $\mu \geq \frac{p(N-1)}{p-1}$. In this thesis, we focus on the more challenging case when $p-1 < \mu < \frac{p(N-1)}{p-1}$ which forces h to be singular at 0. Here we establish multiplicity results of (1.17) for the case $0 \leq \beta < 1$. In the case $\beta = 0$, the authors in [40, 41] proved the existence of multiple positive solutions of (1.17) when $f(0) = 0$. However, $f(0) = 0$ helps to deal with the singularity of h. Hence, our investigation of multiple positive solutions when $f(0) > 0$ is new even for the case $\beta = 0$. To state our results precisely we first state the following hypothesis of f.

(F_1) $\displaystyle\lim_{u\to\infty} \frac{f(u)}{u^{\beta+p-1}} = 0.$

We establish:

Theorem 1.2.1 *Assume* (F_1). *Then* (1.16) *has a positive radial solution for all* $\lambda > 0$.

Further, for certain classes of f we discuss the existence of at least two positive radial solutions for a certain range of λ. To state this multiplicity result, we let

$$A_1 := p^{\frac{p}{p-1}} \quad \underline{h} := \inf_{r\in(0,1]} h(r). \tag{1.20}$$

Note that $\underline{h} > 0$. Further, let $w_1 \in C([0,1], \mathbb{R}^+) \cap C^1((0,1), \mathbb{R})$ be the unique solution (see [34]) of

$$\begin{cases} -(\varphi_p(w_1'(r)))' = \frac{h(r)}{w_1(r)^\beta} \text{ in } (0,1), \\ w_1(0) = 0 = w_1(1) \end{cases} \tag{1.21}$$

and assume f satisfies:

(F_2) $f(u)$ is nondecreasing for all $u \geq 0$.

(F_3) There exist a, b with $a \in (0, \frac{p}{A_1}b)$ and $\frac{f(u)}{u^\beta}$ is nondecreasing on (a, b).

We establish:

Theorem 1.2.2 *Assume* $(F_1) - (F_3)$ *and there exists* d *such that* $a < d < \frac{p}{A_1}b$ *and* $Q(a,d) > \tilde{C}(\beta,p,N,\Omega)$, *where* $\tilde{C}(\beta,p,N,\Omega) := \frac{(2p)^p}{(p-1)^{p-1}} \frac{\|w_1\|_\infty^{\beta+p-1}}{\underline{h}}$. *Then* (1.16) *has at least two positive radial solutions for* $\lambda \in (\lambda_*, \lambda^*)$, *where*

$$\lambda_* = \frac{d^{\beta+p-1}}{f(d)} \frac{(2p)^p}{\underline{h}(p-1)^{p-1}}, \tag{1.22}$$

$$\lambda^* = \min\left\{ \frac{d^\beta}{f(d)} \frac{2^p}{\underline{h}} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1}\,,\; \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w_1\|_\infty^{\beta+p-1}} \right\}. \tag{1.23}$$

Remark 1.2.1 *Here* $d < \frac{p}{A_1}b$ *implies that* $\frac{d^{\beta+p-1}}{f(d)} \frac{(2p)^p}{\underline{h}(p-1)^{p-1}} < \frac{d^\beta}{f(d)} \frac{2^p}{\underline{h}} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$. *Further, we obtain* $\frac{d^{\beta+p-1}}{f(d)} \frac{(2p)^p}{\underline{h}(p-1)^{p-1}} < \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w_1\|_\infty^{\beta+p-1}}$ *from* $Q(a,d) > \frac{(2p)^p}{(p-1)^{p-1}} \frac{\|w_1\|_\infty^{\beta+p-1}}{\underline{h}}$. *Therefore,* (λ_*, λ^*) *is not empty.*

Remark 1.2.2 *The function* $f(u) = \exp[\frac{\alpha u}{\alpha+u}]$ *for* $\alpha > 0$ *easily satisfies the hypothesis* (F_1) *and hence Theorem 1.2.1 holds. Also* f *satisfies the hypothesis* (F_2). *Choosing* $a = 1, d = \alpha$ *and* $b = \frac{\alpha^2}{2}$, *we can easily show that* $\frac{f(u)}{u^\beta}$ *is nondecreasing on* (a,b) *for* $\alpha \gg 1$. *Further* $Q(a,d) = \frac{a^{\beta+p-1}}{f(a)} / \frac{d^{\beta+p-1}}{f(d)} = [\frac{1}{\alpha}]^{\beta+p-1} \exp[\frac{\alpha}{2} - \frac{\alpha}{\alpha+1}]$, *and hence for any given* Ω, *we have* $a < d < \frac{p}{A_1}b$ *and* $Q(a,d) > \tilde{C}(\beta,p,N,\Omega)$, *for* α *large. Hence, the hypotheses of Theorem 1.2.2 are also satisfied for* $\alpha \gg 1$.

Next we extend our results to systems of the form:

$$\begin{cases} -\Delta_p u = \lambda K_1(|x|)\frac{f_1(v)}{u^\beta} & \text{in } \Omega, \\ -\Delta_p v = \lambda K_2(|x|)\frac{f_2(u)}{v^\beta} & \text{in } \Omega, \\ u(x) = 0 = v(x) & \text{if } |x| = r_0, \\ u(x) \to 0, v(x) \to 0 & \text{if } |x| \to \infty, \end{cases} \tag{1.24}$$

where $f_i \in C([0,\infty),(0,\infty))$ and $K_i \in C([r_0,\infty),(0,\infty))$ such that $K_i(r) < \frac{1}{r^\mu}$ for $r \gg 1$ and for some $\mu > p-1, i = 1,2$. By the same change of variables used in (1.16), (1.24) is transformed to:

$$\begin{cases} -(\varphi_p(u'(t)))' = \lambda h_1(t)\frac{f_1(v(t))}{u(t)^\beta} \text{ in } (0,1), \\ -(\varphi_p(v'(t)))' = \lambda h_2(t)\frac{f_2(u(t))}{v(t)^\beta} \text{ in } (0,1), \\ u(0) = 0 = u(1),\ v(0) = 0 = v(1), \end{cases} \tag{1.25}$$

where $h_i, i = 1,2$ is given by

$$h_i(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p\, t^{\frac{p(1-N)}{N-p}} K_i\left(r_0\, t^{\frac{1-p}{N-p}}\right), i = 1,2. \tag{1.26}$$

To state an existence result for (1.24) we assume:

(F_4) f_i are nondecreasing, $i = 1,2$.

(F_5) $\lim\limits_{u\to\infty} \dfrac{f_1(Mf_2(u))}{u^{\beta+p-1}} = 0$ for all $M > 0$.

We establish:

Theorem 1.2.3 *Assume* (F_4) *and* (F_5). *Then* (1.24) *has a positive radial solution for all* $\lambda > 0$.

Next, under certain combined nonlinear effects of $\frac{u^{\beta+p-1}}{f_1(u)}$ and $\frac{u^{\beta+p-1}}{f_2(u)}$ we study the existence of multiple positive radial solutions to (1.24). To state our multiplicity result, we let

$$h_m(r) := \min\{h_1(r), h_2(r)\}, \ \underline{h}_m := \inf_{r\in(0,1]} h_m(r), \tag{1.27}$$

$$h_M(r) := \max\{h_1(r), h_2(r)\}, \tag{1.28}$$

$w_2 \in C([0,1], \mathbb{R}^+) \cap C^1((0,1), \mathbb{R}^+)$ be the unique solution (see [34]) of

$$\begin{cases} -(\varphi_p(w_2'(r)))' = \frac{h_M(r)}{w_2(r)^\beta} \ \text{ in } (0,1), \\ w_2(0) = 0 = w_2(1) \end{cases} \tag{1.29}$$

and assume:

(F_6) $f_1(u) \le f_2(u)$ for all $u \ge 0$.

(F_7) There exist a, b with $a \in (0, \frac{p}{A_1}p)$ and $\frac{f_1(u)}{u^\beta}$ is nondecreasing on (a, b).

We establish:

Theorem 1.2.4 *Assume $(F_4) - (F_7)$ and there exists d such that $a < d < \frac{p}{A_1}p$ and $Q_1(a,d) > C_1(\beta,p,N,\Omega)$, where $C_1(\beta,p,N,\Omega) = \frac{(2p)^p}{(p-1)^{p-1}}\frac{\|w_2\|_\infty^{\beta+p-1}}{\underline{h}_m}$. Then (1.24) has at least two positive radial solutions for $\lambda \in (\lambda_*, \lambda^*)$, where*

$$\lambda_* = \frac{d^{\beta+p-1}}{f_1(d)}\frac{(2p)^p}{\underline{h}_m(p-1)^{p-1}}, \tag{1.30}$$

$$\lambda^* = \min\left\{\frac{d^\beta}{f_1(d)}\frac{2^p}{\underline{h}_m}\left(\frac{p}{p-1}\right)^{p-1} b^{p-1}\ , \ \frac{a^{\beta+p-1}}{f_2(a)}\frac{1}{\|w_2\|_\infty^{\beta+p-1}}\right\}. \tag{1.31}$$

Remark 1.2.3 *Here $d < \frac{p}{A_1}b$ implies that $\frac{d^{\beta+p-1}}{f_1(d)}\frac{(2p)^p}{\underline{h}_m(p-1)^{p-1}} < \frac{d^\beta}{f_1(d)}\frac{2^p}{\underline{h}_m}\left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$. Also since $Q_1(a,d) > \frac{(2p)^p}{(p-1)^{p-1}}\frac{\|w_2\|_\infty^{\beta+p-1}}{\underline{h}_m}$, we obtain $\frac{d^{\beta+p-1}}{f_1(d)}\frac{(2p)^p}{\underline{h}_m(p-1)^{p-1}} < \frac{a^{\beta+p-1}}{f_2(a)}\frac{1}{\|w_2\|_\infty^{\beta+p-1}}$. Therefore, (λ_*, λ^*) is not empty.*

Remark 1.2.4 *The functions* $f_1(u) := \exp[\frac{\alpha u}{\alpha+u}]$ *and* $f_2(u) := u^q + D$*, where* $\alpha > 0, q > 0$ *and* $D > 0$ *easily satisfy the hypotheses* $(F_4) - (F_5)$ *of Theorem 1.2.3. We can also choose* $D \gg 1$ *so that* (F_6) *is satisfied. Choosing* $a = 1, d = \alpha$ *and* $b = \frac{\alpha^2}{2}$*, we can easily show that* $\frac{f_1(u)}{u^\beta}$ *is nondecreasing on* (a, b) *for* $\alpha \gg 1$*. Further* $Q_1(a, d) = \frac{a^{\beta+p-1}}{f_2(a)} / \frac{d^{\beta+p-1}}{f_1(d)} = (\frac{1}{1+D})(\frac{1}{\alpha})^{\beta+p-1} \exp[\frac{\alpha}{2}]$*, and hence we have* $a < d < \frac{p}{A_1} b$ *and* $Q_1(1, \alpha) > C_1(\beta, p, N, \Omega)$*, for* α *large. Hence, the hypotheses of Theorem 1.2.4 are also satisfied for* $\alpha \gg 1$.

Remark 1.2.5 *A similar multiplicity result can be established under weaker assumptions. Namely,* (F_6) *and* (F_7) *can be replaced by the assumption:*

(F_8) *There exist* a, b *with* $a \in (0, \frac{p}{A_1} p)$ *and* $\frac{m(u)}{u^\beta}$ *is nondecreasing on* (a, b)*, where* $m(u) := \min\{f_1(u), f_2(u)\}$.

1.3 A uniqueness result for large values of λ

In this section, we study the uniqueness of the positive solution of (1.4) for large values of λ when $p = 2$. We consider singular boundary value problems of the form:

$$\begin{cases} -\Delta u = \lambda \frac{f(u)}{u^\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.32}$$

where $0 < \beta < 1, \lambda$ is a positive parameter and Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$. We assume:

(U_1) $f : [0, \infty) \to (0, \infty)$ is a C^1 nondecreasing function and there exists $\sigma > 0$ such that $\frac{f(u)}{u^\beta}$ is decreasing for $u > \sigma$.

We establish:

Theorem 1.3.1 *Assume* (U_1). *Then there exists* $\tilde{\lambda} > 0$ *such that* (1.32) *has a unique positive solution for all* $\lambda > \tilde{\lambda}$.

1.4 One dimensional perturbed Gelfand problem

Our study in this thesis is motivated by the $p-$ Laplacian perturbed Gelfand problem:

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda \frac{\exp[\frac{\alpha u}{\alpha+u}]}{u^\beta} \text{ in } (0,1), \\ u(0) = 0 = u(1), \end{cases} \tag{1.33}$$

where $p > 1, \alpha > 0$ and $0 < \beta < 1$. In the case when $p = 2$ and $\beta = 0$ the authors in [9, 62] proved that if $\alpha \geq \alpha^*$ for some α^*, the bifurcation curve of positive solutions of (1.33) is at least S-shaped, and the authors in [64] established that it is exactly $S-$shaped bifurcation when $p > 1$ and $\beta = 0$. In this section, we prove that for α large the bifurcation curve of positive solutions of (1.33) is at least S-shaped when $p > 1$ and $\beta \neq 0$.

We establish:

Theorem 1.4.1 $\forall \lambda > 0$, *the problem* (1.33) *has a positive solution. Further, there exist* $\lambda_1 > 0$ *and* $\lambda_2 > 0$ *such that* (1.33) *has at least three positive solutions for* $\lambda \in (\lambda_1, \lambda_2)$ *for* $\alpha \gg 1$.

In Chapter 2, we introduce the definition of sub and supersolutions for singular boundary problems, and describe the method of sub- supersolutions which is our main tool to prove our existence and multiplicity results. Also, we discuss the Quadrature method which we use to prove that a bifurcation curve of positive solutions is at least S-shaped in the case $N = 1$. In Chapters 3 and 4, we prove Theorems 1.1.1-1.1.4 and Theorems

1.2.1-1.2.4, respectively. In Chapter 5, the proof of Theorem 1.3.1 will be furnished. In Chapter 6, we prove that the bifurcation curve of positive solution of (1.33) is at least S-shaped for α large and provide computational results showing that the bifurcation curve is exactly $S-$shaped for α large.

CHAPTER 2

PRELIMINARIES

In this chapter, we introduce the method of sub and supersolutions for classes of singular boundary value problems, and the Quadrature method to solve two point boundary value problems.

2.1 Method of sub and supersolutions

Consider quasilinear boundary value problems of the form:

$$\begin{cases} -\Delta_p u = \lambda \frac{f(u)}{u^\beta} & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where λ is a positive parameter, $0 < \beta < 1$, Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$ and $f : [0, \infty) \to (0, \infty)$ is a continuous function.

Definition 1

A function $\psi \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$ *is a subsolution of* (2.1) *if*

$$\begin{aligned} -\Delta_p \psi \leq \lambda \frac{f(\psi)}{\psi^\beta} & \quad \textit{in } \Omega, \\ \psi > 0 & \quad \textit{in } \Omega, \\ \psi = 0 & \quad \textit{on } \partial\Omega \end{aligned} \tag{2.2}$$

and a function $\phi \in W^{1,p}(\Omega)\bigcap C(\overline{\Omega})$ *is a supersolution of* (2.1) *if*

$$
\begin{aligned}
-\Delta_p \phi &\geq \lambda \frac{f(\phi)}{\phi^{\beta}} && \text{in } \Omega, \\
\phi &> 0 && \text{in } \Omega, \\
\phi &= 0 && \text{on } \partial\Omega.
\end{aligned}
\tag{2.3}
$$

Then the following sub-supersolutions theorem to singular boundary value problems holds.

Theorem 2.1.1 *(See* [*14*], [*42*] *and* [*66*]*) If there exist a subsolution* ψ *and a supersolution* ϕ *of* (2.1) *such that* $\psi \leq \phi$ *on* $\overline{\Omega}$*, then* (2.1) *has at least one solution* $u \in W^{1,p}(\Omega)\bigcap C(\overline{\Omega})$ *satisfying* $\psi \leq u \leq \phi$ *on* $\overline{\Omega}$*.*

Proof. Let us take a sequence of subdomains of Ω with $C^{\infty}-$ boundaries, say $\{\Omega_j\}_{j=1}^{\infty}$, such that

$$
\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_j \subset\subset \Omega_{j+1} \subset\subset \cdots \tag{2.4}
$$

and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Let $g(u) := \frac{f(u)}{u^{\beta}}$. For each $j = 1, 2, \ldots,$ consider the following problem:

$$
\begin{cases}
-\Delta_p u(x) = \lambda g(u(x)) & \text{in } \Omega_j, \\
u(x) = \psi(x) & \text{on } \partial\Omega_j.
\end{cases}
\tag{2.5}
$$

Let $a_j = \min_{\bar{\Omega}_j} \psi$ and $b_j = \max_{\bar{\Omega}_j} \phi$ and define $\tilde{g} : (0,\infty) \to \mathbb{R}$ by

$$
\tilde{g}(u) =
\begin{cases}
g(a_j), & u < a_j, \\
g(u), & a_j \leq u \leq b_j, \\
g(b_j), & b_j < u.
\end{cases}
\tag{2.6}
$$

Then $\tilde{g}$ is bounded in $(0, \infty)$ and obviously the restrictions of the function ψ and ϕ on Ω_j are the subsolution and supersolution of the following problem, respectively:

$$\begin{cases} -\Delta_p u(x) = \lambda \tilde{g}(u(x)) & \text{in } \Omega_j, \\ \\ u(x) = \psi(x) & \text{on } \partial\Omega_j. \end{cases} \tag{2.7}$$

Then (2.7) has a solution $u_j \in W_0^{1,p}(\Omega_j) \times C(\overline{\Omega}_j)$ such that

$$\psi(x) \le u_j(x) \le \phi(x) \ \text{ for all } x \in \Omega_j \tag{2.8}$$

and this is also a solution of (2.5). First claim is that for fixed k, there exists $d_k > 0$ such that $\|u_j\|_{C^{1,\beta}(\overline{\Omega}_k)} \le d_k$ for all $j \ge k+1$. In fact, take Q_k such that $\Omega_k \subset\subset Q_k \subset\subset \Omega_{k+1}$. Define $g_j(x) = g(u_j(x))$. Then

$$-\Delta_p u = g_j \ \text{ on } \ Q_k. \tag{2.9}$$

Since $\{u_j\}_{j \ge k+1}$ are uniformly bounded on $\overline{\Omega}_{k+1}$, we know that there exists $c_k > 0$ such that

$$\|g_j\|_{C(\bar{Q}_k)} < c_k \ \text{ for all } \ j \ge k+1. \tag{2.10}$$

Using the proposition 3.7 in [61], we see that there exists $d_k > 0$ such that

$$\|u\|_{C^{1,\gamma}(\overline{\Omega}_k)} < d_k \ \text{ for all } \ j \ge k+1 \tag{2.11}$$

for some $\gamma \in (0, 1)$. So the claim is proven. Next, since the embedding $C^{1,\gamma}(\overline{\Omega}_k) \hookrightarrow C^1(\overline{\Omega}_k)$ is compact, for each k, sequence $\{u_j\}_{j=1}^{\infty}$ has a subsequence, renamed $\{u_j\}_{j=1}^{\infty}$, which converges to u in $C^1(\overline{\Omega}_k)$. This implies that $u \in C^1(\overline{\Omega}_k)$ for every k. Consequently, $u \in C^1(\Omega)$. Moreover, for each k, we have

$$-\int_{\Omega_k} |\nabla u_j|^{p-2} \nabla u_j \nabla \zeta = \int_{\Omega_k} g(u_j)\zeta \tag{2.12}$$

for all $\zeta \in C_0^\infty(\Omega_k)$ and $j \geq k+1$. By taking the limit of the sequence converging in $C^1(\overline{\Omega}_k)$,

$$-\int_{\Omega_k} |\nabla u|^{p-2}\nabla u \nabla\zeta = \int_{\Omega_k} g(u)\zeta \tag{2.13}$$

for all $\zeta \in C_0^\infty(\Omega_k)$ and $j \geq k+1$. Thus,

$$-\int_{\Omega} |\nabla u|^{p-2}\nabla u \nabla\zeta = \int_{\Omega} g(u)\zeta \tag{2.14}$$

for all $\zeta \in C_0^\infty(\Omega)$. Also, since $\psi(x) \leq u_j(x) \leq \phi(x)$ for all j, we have $\psi(x) \leq u(x) \leq \phi(x)$ for all $x \in \Omega$. Thus, from $\psi(x) = \phi(x) = 0$ on $x \in \partial\Omega$, we know that $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.

Remark 2.1.1 *(See Theorem B.1 in [26]) A positive weak solution $u \in W_0^{1,p}(\Omega)$ of the problem, $-\Delta_p u = g(x)$ in Ω; $u = 0$ on $\partial\Omega$, satisfies $u \in C^{1,\gamma}(\overline{\Omega})$ for some $0 < \gamma < 1$ if there exist $0 < C_1, C_2 < \infty$ and $0 < \beta < 1$ such that $0 \leq g(x) \leq C_1 d(x, \partial\Omega)^{-\beta}$ and $0 \leq u(x) \leq C_2 d(x, \partial\Omega)$, for almost all $x \in \Omega$.*

2.2 Quadrature method

In this section, we briefly describe the Quadrature technique originally due to Laetsch in [38] and extended to $p-$Laplacian problems in [8] for discussing positive solutions of

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda f(u) \ \text{ in } (0,1), \\ \\ u(0) = 0 = u(1), \end{cases} \tag{2.15}$$

where $\lambda > 0, p > 1$ and $f : [0, \infty) \to (0, \infty)$ is a continuous function. Especially, we discuss a sufficient condition for the bifurcation curve of positive solutions of (2.15) to be at least $S-$shaped. Define $G : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$G(\rho) := 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^{\rho} \frac{ds}{(F(\rho) - F(s))^{\frac{1}{p}}}, \tag{2.16}$$

where $F(u) := \int_0^u f(s)\, ds$.

Lemma 2.2.1 *(See [39])* u *is a positive solution of* (2.15) *with* $\lambda > 0$ *if and only if*

$$\lambda(\rho)^{\frac{1}{p}} = G(\rho), \tag{2.17}$$

where $\rho = \|u\|_\infty = \sup_{s\in(0,1)} u(s)$.

Lemma 2.2.2 *(See Theorem* 7 *in [8])*

A. $\lim\limits_{\rho\to 0^+} G(\rho) = 0$

B. *If* $\lim\limits_{s\to\infty} \dfrac{f(s)}{s^{p-1}} = 0$, *then* $\lim\limits_{\rho\to\infty} G(\rho) = \infty$.

Lemma 2.2.3 *(See [9])* $G(\rho)$ *is differentiable on* $\mathbb{R}_+$ *and*

$$\frac{dG(\rho)}{d\rho} = 2\left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_0^1 \frac{H(\rho) - H(\rho v)}{[F(\rho) - F(\rho v)]^{\frac{p+1}{p}}} dv, \tag{2.18}$$

where $H(s) = F(s) - \frac{1}{p} s f(s)$.

One can deduce information on the nature of the bifurcation curve by analyzing the sign of $\dfrac{dG(\rho)}{d\rho}$. It is clear that $\dfrac{dG(\rho)}{d\rho}$ has the same sign as $\dfrac{d}{d\rho}\left(\lambda(\rho)^{\frac{1}{p}}\right)$. From (2.18), a sufficient condition for $\dfrac{dG(\rho)}{d\rho}$ to be positive is:

$$H(\rho) > H(s) \;\; \forall\, s \in [0, \rho) \tag{2.19}$$

and a sufficient condition for $\frac{dG(\rho)}{d\rho}$ to be negative is:

$$H(\rho) < H(s) \;\; \forall \, s \in [0, \rho). \tag{2.20}$$

Hence, if $H'(s) > 0$ for all $s > 0$, then $G(\rho) = (\lambda(\rho))^{\frac{1}{p}}$ is a strictly increasing function, i.e. the bifurcation curve is not S-shaped. Therefore, if there exist $\rho_0 > 0$ and $\rho_1 > \rho_0$ such that $H'(s) > 0; 0 < s < \rho_0$ and $H(\rho_1) < 0$, then by (2.19)-(2.20), $\frac{dG(\rho)}{d\rho} > 0$ for $0 < \rho \leq \rho_0$ and there exists $\tilde{\rho_1} \leq \rho_1$ such that $\frac{dG}{d\rho}(\tilde{\rho_1}) < 0$ (see Figure 2.1 and Figure 2.2). Hence, from Lemma 2.2.2 this establishes that the bifurcation curve is at least S-shaped.

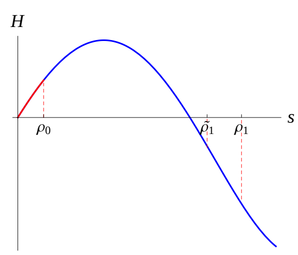

Figure 2.1

Function H

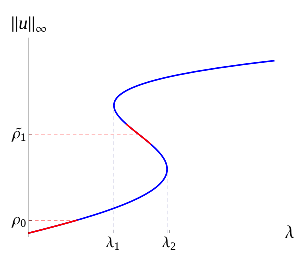

Figure 2.2

$S-$shaped bifurcaiton curve

CHAPTER 3

PROOFS OF THEOREMS 1.1.1-1.1.4

3.1 Proof of Theorem 1.1.1

We construct a positive supersolution ϕ_1 of (1.4). Let $f^*(u) = \max_{0 \le x \le u} f(x)$. Then $f^*(u)$ is nondecreasing and $\frac{f^*(u)}{u^{\beta+p-1}} \to 0$ as $u \to \infty$, since $\frac{f(u)}{u^{\beta+p-1}} \to 0$ as $u \to \infty$. So there exists $M_\lambda \gg 1$ such that

$$\frac{f^*(M_\lambda \|w\|_\infty)}{(M_\lambda \|w\|_\infty)^{\beta+p-1}} \le \frac{1}{\lambda \|w\|_\infty^{\beta+p-1}}. \tag{3.1}$$

Let $\phi_1 = M_\lambda w$. Then

$$-\Delta_p \phi_1 = \frac{M_\lambda^{p-1}}{w^\beta} \ge \lambda \frac{f^*(M_\lambda \|w\|_\infty)}{(M_\lambda w)^\beta} \ge \lambda \frac{f^*(M_\lambda w)}{(M_\lambda w)^\beta} \ge \lambda \frac{f(M_\lambda w)}{(M_\lambda w)^\beta} = \lambda \frac{f(\phi_1)}{\phi_1^\beta}, \tag{3.2}$$

showing that ϕ_1 is a positive supersolution of (1.4).

Now we construct a positive subsolution ψ_1. Let λ_1 be the first eigenvalue of $-\Delta_p$ with Dirichlet boundary condition and $e > 0$ be a corresponding eigenfunction. Hence e and λ_1 satisfy:

$$\begin{cases} -\Delta_p e = \lambda_1 e^{p-1} & \text{in } \Omega \\ \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.3}$$

Since $\frac{f(u)}{u^\beta} \to \infty$ as $u \to 0$, there exists a sufficiently small m_λ such that

$$\lambda_1 (m_\lambda e)^{p-1} \le \lambda \frac{f(m_\lambda e)}{(m_\lambda e)^\beta} \quad \text{for all} \quad \lambda > 0. \tag{3.4}$$

Let $\psi_1 = m_\lambda e$. Then

$$-\Delta_p \psi_1 = \lambda_1 (m_\lambda e)^{p-1} \le \lambda \frac{f(m_\lambda e)}{(m_\lambda e)^\beta} = \lambda \frac{f(\psi_1)}{\psi_1^\beta}. \tag{3.5}$$

Thus ψ_1 is subsolution of (1.4), and if m_λ is chosen sufficiently small, then $\psi_1 \le \phi_1$. Hence, Theorem 1.1.1 is proven.

3.2 Proof of Theorem 1.1.2

Here we construct a second positive supersolution ϕ_2 of (1.4) with $\|\phi_2\|_\infty = a$ when $\lambda < \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$. Let $\phi_2 = a \frac{w}{\|w\|_\infty}$. Since $\lambda < \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$,

$$\begin{aligned} -\Delta_p \phi_2 &= \frac{a^{p-1}}{\|w\|_\infty} \frac{1}{w^\beta} \\ &= \frac{\|w\|_\infty^\beta}{a^\beta w^\beta} \frac{a^{\beta+p-1}}{\|w\|_\infty^{\beta+p-1}} \\ &> \lambda \frac{f(a)}{\phi_2^\beta} \\ &\ge \lambda \frac{f\left(a \frac{w}{\|w\|_\infty}\right)}{\phi_2^\beta} \\ &= \lambda \frac{f(\phi_2)}{\phi_2^\beta}. \end{aligned} \tag{3.6}$$

Next we construct a second positive subsolution ψ_2 of (1.4) when $\frac{d^{\beta+p-1}}{f(d)} \frac{A_N^{p-1} N}{(p-1)^{p-1} R^p} < \lambda < \frac{d^\beta}{f(d)} \frac{N}{R^p} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$. Let $a^* \in (0, a]$ be such that $f(a^*) = \min_{0<x\le a} f(x)$ and define $\tilde{f} \in C([0,\infty))$ such that

$$\tilde{f}(u) = \begin{cases} \frac{f(a^*)}{(a^*)^\beta}, & u \le a^*, \\ \\ \frac{f(u)}{u^\beta}, & u \ge a, \end{cases} \tag{3.7}$$

so that $\tilde{f}$ is nondecreasing on $(0, a]$ and $\tilde{f}(u) \le \frac{f(u)}{u^\beta}$ for all $u \ge 0$ (see Figure 3.1).

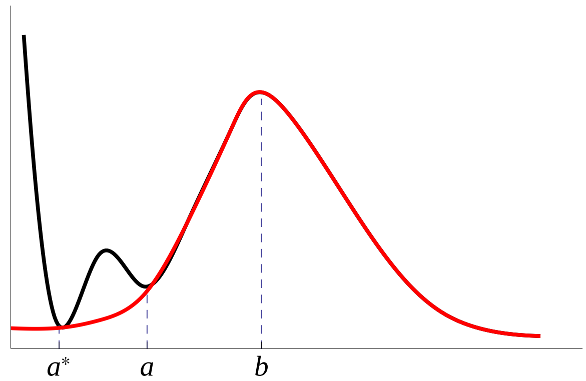

Figure 3.1

Graph of the function $\tilde{f}(u)$ below $\frac{f(u)}{u^\beta}$

Consider the following nonsingular problem:

$$\begin{cases} -\Delta_p u = \lambda \tilde{f}(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

For $0 < \epsilon < R$, and $\delta, \gamma > 1$, define $\rho(r) : [0, R] \to [0, 1]$ by

$$\rho(r) = \begin{cases} 1, & 0 \le r \le \epsilon \\ 1 - \left(1 - \left(\frac{R-r}{R-\epsilon}\right)^\gamma\right)^\delta, & \epsilon < r \le R. \end{cases} \tag{3.9}$$

Then

$$\rho'(r) = \begin{cases} 0, & 0 \le r \le \epsilon \\ -\frac{\delta\gamma}{R-\epsilon}\left(1 - \left(\frac{R-r}{R-\epsilon}\right)^\gamma\right)^{\delta-1}\left(\frac{R-r}{R-\epsilon}\right)^{\gamma-1}, & \epsilon < r \le R. \end{cases} \tag{3.10}$$

Let $v(r) = d\rho(r)$. Here note that $|v'(r)| \le \frac{d\delta\gamma}{R-\epsilon}$ since $|\ \rho'(r)\ | \le \frac{\delta\gamma}{R-\epsilon}$. Define ψ as the radially symmetric solution of

$$\begin{cases} -\Delta_p \psi(x) = \lambda \tilde{f}(v(|x|)) & \text{in } B(0, R) \\ \psi = 0 & \text{on } \partial B(0, R), \end{cases} \tag{3.11}$$

where $B(0,R) = B_R$ is the largest inscribed ball in Ω. Then ψ satisfies

$$\begin{cases} -(r^{N-1}\varphi_p(\psi'(r)))' = \lambda r^{N-1}\tilde{f}(v(r)) \text{ in } (0,R), \\ \\ \psi'(0) = 0, \quad \psi(R) = 0, \end{cases} \tag{3.12}$$

where $\varphi_p(u) = |u|^{p-2}u$. Integrating once, for $0 < r < R$, we get

$$-\varphi_p(\psi'(r)) = \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(v(s))\,ds. \tag{3.13}$$

Since φ_p is monotone, φ_p^{-1} is also continuous and monotone. Hence, we have

$$-\psi'(r) = \varphi_p^{-1}\left(\frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(v(s))\,ds\right). \tag{3.14}$$

We claim that

$$\psi(r) \geq v(r), \quad \forall\, 0 \leq r \leq R \tag{3.15}$$

and

$$\|\psi\|_\infty < b, \tag{3.16}$$

when $\frac{d^{\beta+p-1}}{f(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \frac{d^\beta}{f(d)}\frac{N}{R^p}(\frac{p}{p-1})^{p-1}b^{p-1}$. If our claim is true, ψ is a positive subsolution of the nonsingular problem (3.8) since $-\Delta_p\psi = \lambda\tilde{f}(v) \leq \lambda\tilde{f}(\psi)$. In order to show (3.15), since $\psi(R) = v(R) = 0$, it is enough to show that

$$\psi'(r) < v'(r), \quad \forall\, 0 \leq r \leq R. \tag{3.17}$$

Note that for $0 \leq r \leq \epsilon$, clearly $\psi'(r) \leq 0 = v'(r)$. Now for $r > \epsilon$, from (3.13)

$$\begin{aligned} -\varphi_p(\psi'(r)) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1}\tilde{f}(v(s))\,ds \\ &> \frac{\lambda}{R^{N-1}} \int_0^\epsilon s^{N-1}\tilde{f}(v(s))\,ds \\ &= \frac{\lambda}{R^{N-1}}\tilde{f}(d)\frac{\epsilon^N}{N} \\ &= \frac{\lambda}{R^{N-1}}\frac{f(d)}{d^\beta}\frac{\epsilon^N}{N}. \end{aligned} \tag{3.18}$$

So, we have $-\psi'(r) > \varphi_p^{-1}\left(\frac{\lambda}{R^{N-1}}\frac{f(d)}{d^\beta}\frac{\epsilon^N}{N}\right)$. Thus, (3.17) will hold for all $\epsilon \leq r \leq R$, if $\varphi_p^{-1}\left(\frac{\lambda}{R^{N-1}}\frac{f(d)}{d^\beta}\frac{\epsilon^N}{N}\right) > \frac{\delta\gamma}{R-\epsilon}d$, which is same as

$$\frac{\lambda}{R^{N-1}}\frac{f(d)}{d^\beta}\frac{\epsilon^N}{N} > \varphi_p\left(\frac{\delta\gamma}{R-\epsilon}d\right) = \left(\frac{\delta\gamma}{R-\epsilon}d\right)^{p-1}. \tag{3.19}$$

Thus, if $\lambda > \frac{d^{\beta+p-1}}{f(d)}\frac{NR^{N-1}(\delta\gamma)^{p-1}}{\epsilon^N(R-\epsilon)^{p-1}}$, inequality (3.17) will hold for all $\epsilon \leq r \leq R$. Note that

$$\inf \; \frac{d^{\beta+p-1}}{f(d)}\frac{NR^{N-1}(\delta\gamma)^{p-1}}{\epsilon^N(R-\epsilon)^{p-1}} = \frac{d^{\beta+p-1}}{f(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p}(\delta\gamma)^{p-1} \tag{3.20}$$

and is achieved at $\epsilon = \frac{NR}{N+p-1}$. Hence, if $\lambda > \frac{d^{\beta+p-1}}{f(d)}\frac{A_N^{p-1}N}{(p-1)^{p-1}R^p}$, then in the definition of the function ρ we can choose $\epsilon = \frac{NR}{N+p-1}$ and values for $\delta(>1)$ and $\gamma(>1)$ so that $\lambda > \frac{d^{\beta+p-1}}{f(d)}\frac{NR^{N-1}(\delta\gamma)^{p-1}}{\epsilon^N(R-\epsilon)^{p-1}}$ and hence (3.17) will hold for all $\epsilon \leq r \leq R$. In order to obtain (3.16), integrating (3.14) from t to R, we have

$$\int_t^R -\psi'(r)dr = \int_t^R \varphi_p^{-1}\left(\frac{\lambda}{r^{N-1}}\left(\int_0^r s^{N-1}\tilde{f}(v(s))\,ds\right)\right)dr \tag{3.21}$$

for $0 \leq t \leq R$. Hence

$$\begin{aligned}
\psi(t) &= \int_t^R \varphi_p^{-1}\left(\frac{\lambda}{r^{N-1}}\left(\int_0^r s^{N-1}\tilde{f}(v(s))\,ds\right)\right)dr \\
&\leq \int_t^R \varphi_p^{-1}\left(\frac{\lambda}{r^{N-1}}\tilde{f}(d)\frac{r^N}{N}\right)dr \\
&= \int_t^R \left(\frac{\lambda}{N}\tilde{f}(d)\right)^{\frac{1}{p-1}} r^{\frac{1}{p-1}}\,dr \\
&\leq \left(\frac{\lambda}{N}\tilde{f}(d)\right)^{\frac{1}{p-1}} \int_0^R r^{\frac{1}{p-1}}\,dr \\
&= \frac{p-1}{p}\left(\frac{\lambda R^p}{N}\frac{f(d)}{d^\beta}\right)^{\frac{1}{p-1}},
\end{aligned} \tag{3.22}$$

from which we have $\|\psi\|_\infty \leq \frac{p-1}{p}\left(\frac{\lambda R^p}{N}\frac{f(d)}{d^\beta}\right)^{\frac{1}{p-1}}$. Since $\lambda < \frac{d^\beta}{f(d)}\frac{N}{R^p}\left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$, we obtain $\|\psi\|_\infty < b$. Thus ψ satisfies

$$\begin{cases} -\Delta_p \psi < \lambda \tilde{f}(\psi) & \text{in } B(0,R), \\ \psi = 0 & \text{on } \partial B(0,R) \end{cases} \tag{3.23}$$

and $d < \|\psi\|_\infty < b$. Now, let $z(x) = \psi(x)$, if $x \in B(0,R)$ and $z(x) = 0$, if $x \in \Omega - B(0,R)$. Then $z \in W^{1,p}(\Omega) \bigcap C(\bar{\Omega})$ and $z = 0$ on $\partial\Omega$, which is subsolution of the nonsingular problem (3.8) in Ω. However, z is not strictly positive in Ω. To obtain a strictly positive subsolution of (3.8) in Ω we iterate this subsolution z once in a suitable manner. By the properties of $\tilde{f}$, there exists $\kappa_\lambda > 0$ such that $\lambda \tilde{f}(z) + \kappa_\lambda \varphi_p(z)$ is increasing for all $z \geq 0$. Define ψ_2 to be the solution of

$$\begin{cases} -\Delta_p \psi_2 + \kappa_\lambda \varphi_p(\psi_2) = \tilde{f}^*(z) & \text{in } \Omega, \\ \psi_2 = 0 & \text{on } \partial\Omega \end{cases} \tag{3.24}$$

with $\tilde{f}^*(z) = \lambda \tilde{f}(z) + \kappa_\lambda \varphi_p(z)$. Then since the operator $-\Delta_p + \kappa_\lambda \varphi_p$ satisfies the weak comparison principle (see [23]), we can have $z \leq \psi_2$ (see [45]). Further we get $\psi_2(x) > 0$ for all $x \in \Omega$ since $\tilde{f}^*(0) > 0$. Hence by the monotonicity of $\tilde{f}^*$ we have

$$-\Delta_p \psi_2 + \kappa_\lambda \varphi_p(\psi_2) = \tilde{f}^*(z) \leq \tilde{f}^*(\psi_2) = \lambda \tilde{f}(\psi_2) + \kappa_\lambda \varphi_p(\psi_2), \tag{3.25}$$

which implies that ψ_2 is a subsolution of the nonsingular problem (3.8) such that $\psi_2 > 0$ in Ω. Since $\tilde{f}(u) \leq \frac{f(u)}{u^\beta}$ for all $u \geq 0$, we have $-\Delta_p \psi_2 \leq \lambda \tilde{f}(\psi_2) \leq \lambda \frac{f(\psi_2)}{\psi_2^\beta}$, showing that ψ_2 is a positive subsolution of our singular problem (1.4).

Therefore, we obtain a positive subsolution ψ_2 and a positive supersolution ϕ_2 such that $\psi_2 \not\leq \phi_2$ when $\frac{d^{1+\beta}}{f(d)} \frac{A_N^{p-1} N}{(p-1)^{p-1} R^p} < \lambda < \min\left\{ \frac{d^\beta}{f(d)} \frac{N}{R^p} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1},\ \frac{a^{\beta+p-1}}{f(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}} \right\}$. From the proof of Theorem1.1.1 we note that we have a sufficiently small positive subsolution ψ_1 such that $\psi_1 \leq \phi_2$ and a sufficiently large positive supersolution ϕ_1 such that

$\psi_2 \le \phi_1$. Hence, there exist a positive solution u_1 of (1.4) such that $\psi_1 \le u_1 \le \phi_2$ and a positive solution u_2 of (1.4) such that $\psi_2 \le u_2 \le \phi_1$. Since $\psi_2 \not\le \phi_2$, we have $u_1 \ne u_2$. Therefore, there exist at least two positive solutions of (1.4) for $\lambda \in (\lambda_*, \lambda^*)$ and Theorem 1.1.2 is proven.

3.3 Proof of Theorem 1.1.3

We construct a positive supersolution $(\phi_1, \bar{\phi_1})$ of (1.11). If both f_1 and f_2 are bounded, let $(\phi_1, \bar{\phi_1}) = (\lambda M_\lambda w, \lambda M_\lambda w)$ and choose M_λ so large that

$$M_\lambda^{p-1} \ge \frac{1}{\lambda^{p-2}} \max\{\|f_1\|_\infty, \|f_2\|_\infty\}. \tag{3.26}$$

Then for $M_\lambda \gg 1$ we have

$$-\Delta_p \phi_1 = \lambda^{p-1} M_\lambda^{p-1} \frac{1}{w^\beta} \ge \lambda \frac{\|f_1\|_\infty}{w^\beta} \ge \lambda \frac{f_1(\lambda M_\lambda w)}{(\lambda M_\lambda w)^\beta} = \lambda \frac{f_1(\bar{\phi_1})}{{\phi_1}^\beta} \tag{3.27}$$

and

$$-\Delta_p \bar{\phi_1} = \lambda^{p-1} M_\lambda^{p-1} \frac{1}{w^\beta} \ge \lambda \frac{\|f_2\|_\infty}{w^\beta} \ge \lambda \frac{f_2(\lambda M_\lambda w)}{(\lambda M_\lambda w)^\beta} = \lambda \frac{f_2(\phi_1)}{\bar{\phi_1}^\beta}, \tag{3.28}$$

showing that $(\phi_1, \bar{\phi_1})$ is a positive supersolution of (1.11). Suppose that $f_2(x) \to \infty$ as $x \to \infty$, let $(\phi_1, \bar{\phi_1}) = \left(M_\lambda w, \lambda^{\frac{1}{\beta+p-1}} f_2(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w\right)$. Then by (H_5), we can choose M_λ large so that

$$\frac{f_1\left(\lambda^{\frac{1}{\beta+p-1}} \|w\|_\infty f_2(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}}\right)}{(M_\lambda \|w\|_\infty)^{\beta+p-1}} \le \frac{1}{\lambda \|w\|_\infty^{\beta+p-1}}. \tag{3.29}$$

Then we have

$$
\begin{aligned}
-\Delta_p \phi_1 &= \frac{M_\lambda^{p-1}}{w^\beta} \\
&\geq \lambda \frac{f_1\left(\lambda^{\frac{1}{\beta+p-1}} \|w\|_\infty f_2(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}}\right)}{(M_\lambda w)^\beta} \\
&\geq \lambda \frac{f_1\left(\lambda^{\frac{1}{\beta+p-1}} f_2(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w\right)}{(M_\lambda w)^\beta} \\
&= \lambda \frac{f_1(\bar{\phi_1})}{\phi_1{}^\beta}.
\end{aligned}
\tag{3.30}
$$

We also have

$$
\begin{aligned}
-\Delta_p \bar{\phi_1} &= \lambda^{\frac{p-1}{\beta+p-1}} f_2(M_\lambda \|w\|_\infty)^{\frac{p-1}{\beta+p-1}} \frac{1}{w^\beta} \\
&= \lambda \frac{f_2(M_\lambda \|w\|_\infty)}{\lambda^{\frac{\beta}{\beta+p-1}} f_2(M_\lambda \|w\|_\infty)^{\frac{\beta}{\beta+p-1}} w^\beta} \\
&\geq \lambda \frac{f_2(M_\lambda w)}{\left(\lambda^{\frac{1}{\beta+p-1}} f_2(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w\right)^\beta} \\
&= \lambda \frac{f_2(\phi_1)}{\bar{\phi_1}{}^\beta},
\end{aligned}
\tag{3.31}
$$

showing that $(\phi_1, \bar{\phi_1})$ is a supersolution of (1.11). (If f_2 is bounded and $f_1(x) \to \infty$ as $x \to \infty$, then $\lim_{x\to\infty} \frac{f_2(M f_1(x))}{x^{\beta+p-1}} = 0$ for all $M > 0$ and we can prove that $(\phi_1, \bar{\phi_1}) = \left(\lambda^{\frac{1}{\beta+p-1}} f_1(M_\lambda \|w\|_\infty)^{\frac{1}{\beta+p-1}} w, M_\lambda w\right)$ is a supersolution of (1.11).

Now, we construct a positive subsolution $(\psi_1, \bar{\psi_1})$ of (1.11). Let e and λ_1 be as in the proof of Theorem 1.1.1. Since $\lim_{x\to 0} \frac{f_1(0)}{x^\beta} = \infty = \lim_{x\to 0} \frac{f_2(0)}{x^\beta}$, there exist sufficiently small m_λ and m'_λ such that

$$
\lambda_1 (m_\lambda e)^{p-1} \leq \lambda \frac{f_1(0)}{(m_\lambda e)^\beta} \quad \text{and} \quad \lambda_1 (m'_\lambda e)^{p-1} \leq \lambda \frac{f_2(0)}{(m'_\lambda e)^\beta}. \tag{3.32}
$$

Let $(\psi_1, \bar{\psi_1}) = (m_\lambda e, m'_\lambda e)$. Since f_1 and f_2 are nondecreasing, we have

$$
-\Delta_p \psi_1 = \lambda_1 (m_\lambda e)^{p-1} \leq \lambda \frac{f_1(0)}{(m_\lambda e)^\beta} \leq \lambda \frac{f_1(m'_\lambda e)}{(m_\lambda e)^\beta} = \lambda \frac{f_1(\bar{\psi_1})}{\psi_1^\beta} \tag{3.33}
$$

and

$$-\Delta_p \bar{\psi_1} = \lambda_1 (m'_\lambda e)^{p-1} \le \lambda \frac{f_2(0)}{(m'_\lambda e)^\beta} \le \lambda \frac{f_2(m_\lambda e)}{(m'_\lambda e)^\beta} = \lambda \frac{f_2(\psi_1)}{\bar{\psi_1}^\beta}. \tag{3.34}$$

Thus $(\psi_1, \bar{\psi_1})$ is a positive subsolution of (1.11), and if m_λ and m'_λ are sufficiently small, then $(\psi_1, \bar{\psi_1}) \le (\phi_1, \bar{\phi_1})$. Hence Theorem 1.1.3 is proven.

3.4 Proof of Theorem 1.1.4

We construct a second positive supersolution $(\phi_2, \bar{\phi_2})$ of (1.11) for $\lambda < \frac{a^{\beta+p-1}}{f_2(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$. Let $(\phi_2, \bar{\phi_2}) = \left(a\frac{w}{\|w\|_\infty}, a\frac{w}{\|w\|_\infty} \right)$. Since $\lambda < \frac{1}{\|w\|_\infty^{\beta+p-1}} \frac{a^{\beta+p-1}}{f_2(a)}$ and $f_2(x) \ge f_1(x)$ for all $x \ge 0$, we have

$$-\Delta_p \phi_2 = \frac{a^{p-1}}{\|w\|_\infty^{p-1}} \frac{1}{w^\beta} > \lambda \frac{f_2(a)}{\left(a\frac{w}{\|w\|_\infty} \right)^\beta} \ge \lambda \frac{f_1\left(a\frac{w}{\|w\|_\infty} \right)}{\left(a\frac{w}{\|w\|_\infty} \right)^\beta} = \lambda \frac{f_1(\bar{\phi_2})}{\phi_2^\beta} \tag{3.35}$$

and

$$-\Delta_p \bar{\phi_2} = \frac{a^{p-1}}{\|w\|_\infty^{p-1}} \frac{1}{w^\beta} > \lambda \frac{f_2(a)}{\left(a\frac{w}{\|w\|_\infty} \right)^\beta} \ge \lambda \frac{f_2\left(a\frac{w}{\|w\|_\infty} \right)}{\left(a\frac{w}{\|w\|_\infty} \right)^\beta} = \lambda \frac{f_2(\phi_2)}{\bar{\phi_2}^\beta}. \tag{3.36}$$

Hence, $(\phi_2, \bar{\phi_2})$ is a positive supersolution of (1.11) with $\|\phi_2\|_\infty = a$ and $\|\bar{\phi_2}\|_\infty = a$ when $\lambda < \frac{a^{\beta+p-1}}{f_2(a)} \frac{1}{\|w\|_\infty^{\beta+p-1}}$.

Now, we construct a second positive subsolution $(\psi_2, \bar{\psi_2})$ of (1.11) for $\frac{d^{\beta+p-1}}{f_1(d)} \frac{A_N^{p-1} N}{(p-1)^{p-1} R^p} < \lambda < \frac{d^\beta}{f_1(d)} \frac{N}{R^p} \left(\frac{p}{p-1} \right)^{p-1} b^{p-1}$. Let $\tilde{f}, \rho, v, \psi, z$ and consequently ψ_2 be as defined in the proof of Theorem 1.1.2 when f is replaced by f_1. We note that $\psi_2 > 0$ in Ω and for this range of λ satisfies

$$\begin{cases} -\Delta_p \psi_2 < \lambda \dfrac{f_1(\psi_2)}{\psi_2^\beta} & \text{in } \Omega, \\ \psi_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.37}$$

Now choosing $\bar{\psi_2} = \psi_2$, we have

$$-\Delta_p \psi_2 < \lambda \frac{f_1(\psi_2)}{\psi_2^\beta} = \lambda \frac{f_1(\bar{\psi_2})}{\psi_2^\beta} \tag{3.38}$$

and

$$-\Delta_p \bar{\psi_2} < \lambda \frac{f_1(\bar{\psi_2})}{\bar{\psi_2}^\beta} \leq \lambda \frac{f_2(\psi_2)}{\bar{\psi_2}^\beta} \tag{3.39}$$

since $f_1(u) \leq f_2(u)$ for all $u \geq 0$. Hence, $(\psi_2, \bar{\psi_2})$ is a positive subsolution of (1.11) when $\frac{d^{\beta+p-1}}{f_1(d)} \frac{A_N^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \frac{d^\beta}{f_1(d)} \frac{N}{R^p} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$. Therefore, we obtain a positive supersolution $(\phi_2, \bar{\phi_2})$ and a positive subsolution $(\psi_2, \bar{\psi_2})$ such that $(\psi_2, \bar{\psi_2}) \not\leq (\phi_2, \bar{\phi_2})$ for $\frac{d^{\beta+p-1}}{f_1(d)} \frac{A_N^{p-1}N}{(p-1)^{p-1}R^p} < \lambda < \min\left\{\frac{d^\beta}{f_1(d)} \frac{N}{R^p} (\frac{p}{p-1})^{p-1} b^{p-1},\ \frac{1}{\|w\|_\infty^{\beta+p-1}} \frac{a^{\beta+p-1}}{f_2(a)}\right\}$. From the proof of Theorem 1.1.3 we note that we have a sufficiently small positive subsolution $(\psi_1, \bar{\psi_1})$ such that $(\psi_1, \bar{\psi_1}) \leq (\phi_2, \bar{\phi_2})$ and a sufficiently large positive supersolution $(\phi_1, \bar{\phi_1})$ such that $(\psi_2, \bar{\psi_2}) \leq (\phi_1, \bar{\phi_1})$. Hence, there exist a positive solution $(u_1, \bar{u}_1)$ of (1.16) such that $(\psi_1, \bar{\psi_1}) \leq (u_1, \bar{u}_1) \leq (\phi_2, \bar{\phi_2})$ and a positive solution $(u_2, \bar{u}_2)$ of (1.16) such that $(\psi_2, \bar{\psi_2}) \leq (u_2, \bar{u}_2) \leq (\phi_1, \bar{\phi_1})$. Since $(\psi_2, \bar{\psi_2}) \not\leq (\phi_2, \bar{\phi_2})$, we have $(u_1, \bar{u}_1) \neq (u_2, \bar{u}_2)$. Therefore, there exist at least two positive solutions of (1.11) for $\lambda \in (\lambda_*, \lambda^*)$ and Theorem 1.1.4 is proven.

CHAPTER 4

PROOFS OF THEOREMS 1.2.1-1.2.4

4.1 Proof of Theorem 1.2.1

First we construct a positive supersolution ϕ_1 of (1.17). Let $f^*(u) = \max_{0 \le r \le u} f(r)$. Then $f^*(u)$ is nondecreasing and $\frac{f^*(u)}{u^{\beta+p-1}} \to 0$ as $u \to \infty$ since $\frac{f(u)}{u^{\beta+p-1}} \to 0$ as $u \to \infty$. So there exists $M_\lambda \gg 1$ such that

$$\frac{f^*(M_\lambda \|w_1\|_\infty)}{(M_\lambda \|w_1\|_\infty)^{\beta+p-1}} \le \frac{1}{\lambda \|w_1\|_\infty^{\beta+p-1}}. \tag{4.1}$$

Let $\phi_1 = M_\lambda w_1$. Then

$$\begin{aligned} -(\varphi_p(\phi_1'))' &= M_\lambda^{p-1} \frac{h(r)}{w_1(r)^\beta} \\ &\ge \lambda h(r) \frac{f^*(M_\lambda \|w_1\|_\infty)}{(M_\lambda w_1(r))^\beta} \\ &\ge \lambda h(r) \frac{f^*(M_\lambda w_1(r))}{(M_\lambda w_1(r))^\beta} \\ &\ge \lambda h(r) \frac{f(M_\lambda w_1(r))}{(M_\lambda w_1(r))^\beta} \\ &= \lambda h(r) \frac{f(\phi_1)}{\phi_1^\beta}, \end{aligned} \tag{4.2}$$

showing that ϕ_1 is a positive supersolution of (1.17).

Next we construct a positive subsolution ψ_1 of (1.17). Let λ_1 be the first eigenvalue and e be a corresponding eigenfunction (see [35]) of

$$\begin{cases} -(\varphi_p(u'(t)))' = \lambda h(t)\varphi_p(u(t)) \ \text{ in } (0,1), \\ \\ u(0) = 0 = u(1). \end{cases} \tag{4.3}$$

Since $\frac{f(u)}{u^\beta} \to \infty$ as $u \to 0$, there exists a sufficiently small m_λ such that

$$\lambda_1 m_\lambda^{p-1}\varphi_p(e) \leq \lambda \frac{f(m_\lambda e)}{(m_\lambda e)^\beta} \quad \text{for all } \ \lambda > 0. \tag{4.4}$$

Let $\psi_1 = m_\lambda e$. Then

$$-(\varphi_p(\psi_1'))' = -\lambda_1 m_\lambda^{p-1}(\varphi_p(e'))' = \lambda_1 m_\lambda^{p-1} h(r)\varphi_p(e) \leq \lambda h(r)\frac{f(m_\lambda e)}{(m_\lambda e)^\beta} = \lambda \frac{f(\psi_1)}{\psi_1^\beta}. \tag{4.5}$$

Thus ψ_1 is subsolution of (1.17) such that $\psi_1 \leq \phi_1$ for sufficiently small m_λ. Hence, (1.17) has a positive solution for all $\lambda > 0$, which is equivalent that (1.16) has a positive radial solution for all $\lambda > 0$. Therefore, Theorem 1.2.1 is proven.

4.2 Proof of Theorem 1.2.2

Here we construct a second positive supersolution ϕ_2 of (1.17). Let $\phi_2 = a\frac{w_1}{\|w_1\|_\infty}$. Then for $\lambda < \frac{a^{\beta+p-1}}{f(a)}\frac{1}{\|w_1\|_\infty^{\beta+p-1}}$,

$$\begin{aligned} -(\varphi_p(\phi_2'))' &= \frac{a^{p-1}}{\|w_1\|_\infty^{p-1}}\frac{h(r)}{w_1(r)^\beta} \\ &= h(r)\frac{a^{\beta+p-1}}{\|w_1\|_\infty^{\beta+p-1}}\frac{\|w_1\|_\infty^\beta}{a^\beta w_1(r)^\beta} \\ &> \lambda h(r)\frac{f(a)}{(a\frac{w_1}{\|w_1\|_\infty})^\beta} \\ &\geq \lambda h(r)\frac{f(a\frac{w_1}{\|w_1\|_\infty})}{(a\frac{w_1}{\|w_1\|_\infty})^\beta} \\ &= \lambda h(r)\frac{f(\phi_2)}{\phi_2^\beta}. \end{aligned} \tag{4.6}$$

Hence ϕ_2 is a supersolution with $\|\phi_2\|_\infty = a$.

Next we construct a second positive subsolution ψ_2 of (1.17) when $\frac{d^{\beta+p-1}}{f(d)}\frac{(2p)^p}{\underline{h}(p-1)^{p-1}} < \lambda < \frac{d^\beta}{f(d)}\frac{2^p}{\underline{h}}\left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$. Let $a^* \in (0, a]$ be such that $f(a^*) = \min_{0<x\leq a} \frac{f(x)}{x^\beta}$. Now define $\tilde{f} \in C([0,\infty))$ to be nondecreasing function on $[0, a]$ such that

$$\tilde{f}(u) = \begin{cases} \frac{f(a^*)}{(a^*)^\beta}, & u \leq a^*, \\ \\ \frac{f(u)}{u^\beta}, & u \geq a \end{cases} \tag{4.7}$$

and $\tilde{f}(u) \leq \frac{f(u)}{u^\beta}$ for all $u \geq 0$. Next we consider the following nonsingular problem:

$$\begin{cases} -(\varphi_p(u'(r)))' = \lambda \underline{h}\, \tilde{f}(u(r)) \text{ in } (0,1), \\ \\ u(0) = 0 = u(1). \end{cases} \tag{4.8}$$

For $0 < \epsilon < \frac{1}{2}$, and $\delta, \gamma > 1$, define $\rho(r) : [0,1] \to [0,1]$ by

$$\rho(r) = \begin{cases} 1 - (1 - (\frac{r}{\epsilon})^\gamma)^\delta, & 0 < r \leq \epsilon \\ \\ 1, & \epsilon \leq r \leq \frac{1}{2} \end{cases} \tag{4.9}$$

and $\rho(r) = \rho(1-r);\ \ r \in [\frac{1}{2}, 1]$. Then

$$\rho'(r) = \begin{cases} \frac{\delta\gamma}{\epsilon}(\frac{r}{\epsilon})^{\gamma-1}\left(1 - (\frac{r}{\epsilon})^\gamma\right)^\delta, & 0 \leq r \leq \epsilon, \\ \\ 0, & \epsilon \leq r \leq \frac{1}{2}. \end{cases} \tag{4.10}$$

Let $v(r) = d\rho(r)$. Here note that $|v'(r)| \leq d\frac{\delta\gamma}{\epsilon}$ since $\mid \rho'(r) \mid \leq \frac{\delta\gamma}{\epsilon}$. Define ψ_2 on $[0, \frac{1}{2}]$ to be the solution of

$$\begin{cases} -(\varphi_p(\psi_2'(r)))' = \lambda \underline{h}\, \tilde{f}(v(r)) \text{ in } (0, \frac{1}{2}), \\ \\ \psi_2(0) = 0 = \psi_2'(\frac{1}{2}). \end{cases} \tag{4.11}$$

and extend ψ_2 to $[\frac{1}{2}, 1]$ such that $\psi_2(r) = \psi_2(1-r)$. Now for $\frac{d^{\beta+p-1}}{f(d)} \frac{(2p)^p}{\underline{h}(p-1)^{p-1}} < \lambda < \frac{d^{\beta}}{f(d)} \frac{2^p}{\underline{h}} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$ we claim that

$$\psi_2(r) \geq v(r), \;\; 0 \leq r \leq 1 \tag{4.12}$$

and

$$\|\psi_2\|_\infty < b. \tag{4.13}$$

If our claim is true, ψ_2 satisfies

$$-(\varphi_p(\psi_2'))' = \lambda \underline{h} \tilde{f}(v(r)) \leq \lambda \underline{h} \tilde{f}(\psi_2(r)) \leq \lambda h(r) \frac{f(\psi_2)}{\psi_2^{\beta}}, \tag{4.14}$$

and hence ψ_2 is a subsolution of (1.17). In order to show (4.12), it is enough to show that $\psi_2'(r) \geq v'(r), \; 0 \leq r \leq \frac{1}{2}$. Since $\psi_2(0) = 0 = v(0)$, it suffices to show that $\psi_2'(r) \geq v'(r)$ on $[0, \frac{1}{2}]$. However, this is easily satisfied on $[\epsilon, \frac{1}{2}]$ since $\psi_2'(r) > 0$ and $v'(r) = 0$. Thus it is enough to show that

$$\psi_2'(r) \geq v'(r), \; 0 \leq r \leq \epsilon. \tag{4.15}$$

From (4.11) we obtain

$$\int_r^{\frac{1}{2}} -(\varphi_p(\phi_2'(s)))' = \int_r^{\frac{1}{2}} \lambda \underline{h} \; \tilde{f}(v(s)) ds, \; 0 \leq r \leq \frac{1}{2}. \tag{4.16}$$

Since $\psi_2'(\frac{1}{2}) = 0$, we have

$$\varphi_p(\psi_2'(r)) = \int_r^{\frac{1}{2}} \lambda \underline{h} \; \tilde{f}(v(s)) ds. \tag{4.17}$$

Hence, for $0 \leq r \leq \epsilon$,

$$\begin{aligned}\psi_2'(r) &= \varphi_p^{-1}\left(\int_r^{\frac{1}{2}} \lambda \underline{h}\ \tilde{f}(v(s))ds\right)\\ &\geq \varphi_p^{-1}\left(\int_\epsilon^{\frac{1}{2}} \lambda \underline{h}\ \tilde{f}(v(s))ds\right)\\ &= \varphi_p^{-1}\left(\int_\epsilon^{\frac{1}{2}} \lambda \underline{h}\ \tilde{f}(d)ds\right)\\ &= \varphi_p^{-1}\left(\lambda \underline{h}\ \tilde{f}(d)(\frac{1}{2}-\epsilon)\right)\\ &= \left(\lambda \underline{h}\ \frac{f(d)}{d^\beta}(\frac{1}{2}-\epsilon)\right)^{\frac{1}{p-1}}.\end{aligned} \tag{4.18}$$

Thus, if $\lambda \geq \frac{d^{\beta+p-1}}{f(d)}\frac{(\delta\gamma)^{p-1}}{\underline{h}}\frac{1}{\epsilon^{p-1}(\frac{1}{2}-\epsilon)}$, then

$$\psi_2'(r) \geq v'(r)\ ;\ 0 \leq r \leq \frac{1}{2}. \tag{4.19}$$

Note that

$$\inf_\epsilon \ \frac{d^{\beta+p-1}}{f(d)}\frac{(\delta\gamma)^{p-1}}{\underline{h}}\frac{1}{\epsilon^{p-1}(\frac{1}{2}-\epsilon)} = \frac{d^{\beta+p-1}}{f(d)}\frac{(2p)^p}{\underline{h}(p-1)^{p-1}} \tag{4.20}$$

and is achieved at $\epsilon = \frac{p-1}{2p} \in (0, \frac{1}{2})$. Hence, if $\lambda > \frac{d^{\beta+p-1}}{f(d)}\frac{(2p)^p}{\underline{h}(p-1)^{p-1}}$, then in the definition of the function ρ we can choose $\epsilon = \frac{p-1}{2p}$ and values for $\delta(>1)$ and $\gamma(>1)$ so that $\lambda \geq \frac{d^{\beta+p-1}}{f(d)}\frac{(\delta\mu)^{p-1}}{\underline{h}}\frac{1}{\epsilon^{p-1}(\frac{1}{2}-\epsilon)}$ and hence (4.12) will hold for all $0 \leq r \leq 1$. In order to establish (4.13), from (4.11) we have

$$\psi_2'(r) \ = \varphi_p^{-1}\left(\int_r^{\frac{1}{2}} \lambda \underline{h}\tilde{f}(v(s))ds\right),\ 0 \leq r \leq \tfrac{1}{2} \tag{4.21}$$

and integrating (4.21) from 0 to $\frac{1}{2}$, we have

$$\begin{aligned}
\psi_2\left(\frac{1}{2}\right) &= \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\int_r^{\frac{1}{2}} \lambda \underline{h}\tilde{f}(v(s))ds\right)dr \\
&\le \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\lambda \underline{h}\tilde{f}(d)\int_r^{\frac{1}{2}} ds\right)dr \\
&= \int_0^{\frac{1}{2}} \varphi_p^{-1}\left(\lambda \underline{h}\tilde{f}(d)(\frac{1}{2}-r)\right)dr \\
&= (\lambda \underline{h}\tilde{f}(d))^{\frac{1}{p-1}} \int_0^{\frac{1}{2}} (\frac{1}{2}-r)^{\frac{1}{p-1}} dr \\
&= (\lambda \underline{h}\tilde{f}(d))^{\frac{1}{p-1}} \frac{p-1}{p}(\frac{1}{2})^{\frac{p}{p-1}} \\
&= \frac{p-1}{p}\left(\frac{\lambda \underline{h}\tilde{f}(d)}{2^p}\right)^{\frac{1}{p-1}}.
\end{aligned} \tag{4.22}$$

Therefore , if $\lambda < \frac{d^\beta}{f(d)}\frac{2^p}{\underline{h}}\left(\frac{p}{p-1}\right)^{p-1} b^{p-1}$, then $\|\psi_2\|_\infty = \psi_2(\frac{1}{2}) < b$.

From the proof of Theorem 1.2.1 we note that we have a sufficiently small positive subsolution ψ_1 such that $\psi_1 \le \phi_2$ and a sufficiently large positive supersolution ϕ_1 such that $\psi_2 \le \phi_1$. Hence, there exist a positive solution u_1 of (1.17) such that $\psi_1 \le u_1 \le \phi_2$ and a positive solution u_2 of (1.17) such that $\psi_2 \le u_2 \le \phi_1$. Since $\|\psi_2\|_\infty \ge v(\frac{1}{2}) \ge d$ and $\|\phi_2\|_\infty = a$, we have $\psi_2 \not\le \phi_2$, and hence $u_1 \ne u_2$. Therefore, there exist at least two positive solutions of (1.17) for $\lambda \in (\lambda_*, \lambda^*)$ and Theorem 1.2.2 is proven.

4.3 Proof of Theorem 1.2.3

We construct a positive supersolution $(\phi_1, \bar{\phi_1})$ of (1.25). If both f_1 and f_2 are bounded, let $(\phi_1, \bar{\phi_1}) = (\lambda M_\lambda w_2, \lambda M_\lambda w_2)$ and choose M_λ so large that

$$M_\lambda^{p-1} \ge \frac{1}{\lambda^{\beta+p-2}} \max\{\|f_1\|_\infty, \|f_2\|_\infty\}. \tag{4.23}$$

Then for $M_\lambda \gg 1$ we have

$$\begin{aligned}
-(\varphi_p(\phi_1'))' &= \lambda^{p-1} M_\lambda^{p-1} \frac{h_M(r)}{w_2(r)^\beta} \\
&\geq \lambda^{1-\beta} \frac{h_M(r)}{w_2^\beta} \|f_1\|_\infty \\
&\geq \lambda h_1(r) \frac{f_1(\lambda M_\lambda w_2)}{(\lambda M_\lambda w_2)^\beta} \\
&= \lambda h_1(r) \frac{f_1(\bar{\phi_1})}{{\phi_1}^\beta}
\end{aligned} \tag{4.24}$$

and

$$\begin{aligned}
-(\varphi_p(\bar{\phi_1}'))' &= \lambda^{p-1} M_\lambda^{p-1} \frac{h_M(r)}{w_2(r)^\beta} \\
&\geq \lambda^{1-\beta} \frac{h_M(r)}{w_2^\beta} \|f_2\|_\infty \\
&\geq \lambda h_2(r) \frac{f_2(\lambda M_\lambda w_2)}{(\lambda M_\lambda w_2)^\beta} \\
&= \lambda h_2(r) \frac{f_2(\phi_1)}{\bar{\phi_1}^\beta},
\end{aligned} \tag{4.25}$$

showing that $(\phi_1, \bar{\phi_1})$ is a positive supersolution of (1.25). Suppose that $f_2(u) \to \infty$ as $u \to \infty$, let $(\phi_1, \bar{\phi_1}) = (M_\lambda w_2, \lambda^{\frac{1}{\beta+p-1}} f_2(M_\lambda \|w_2\|_\infty)^{\frac{1}{\beta+p-1}} w_2)$. Then by (F_5), we can choose M_λ large so that

$$\frac{f_1(\lambda^{\frac{1}{\beta+p-1}} \|w_2\|_\infty f_2(M_\lambda \|w_2\|_\infty)^{\frac{1}{\beta+p-1}})}{(M_\lambda \|w_2\|_\infty)^{\beta+p-1}} \leq \frac{1}{\lambda \|w_2\|_\infty^{\beta+p-1}}. \tag{4.26}$$

Then

$$\begin{aligned}
-(\varphi_p(\phi_1'))' &= M_\lambda^{p-1} \frac{h_M(r)}{w_2^\beta} \\
&\geq \lambda h_M(r) \frac{f_1(\lambda^{\frac{1}{\beta+p-1}} \|w_2\|_\infty f_2(M_\lambda \|w_2\|_\infty)^{\frac{1}{\beta+p-1}})}{(M_\lambda w_2)^\beta} \\
&\geq \lambda h_2(r) \frac{f_1(\lambda^{\frac{1}{\beta+p-1}} f_2(M_\lambda \|w_2\|_\infty)^{\frac{1}{\beta+p-1}} w_2)}{(M_\lambda w_2)^\beta} \\
&= \lambda h_2(r) \frac{f_1(\bar{\phi_1})}{{\phi_1}^\beta}.
\end{aligned} \tag{4.27}$$

We also have

$$
\begin{aligned}
-(\varphi_p(\bar{\phi}_1'))' &= \lambda^{\frac{p-1}{\beta+p-1}} f_2(M_\lambda\|w_2\|_\infty)^{\frac{p-1}{\beta+p-1}} \frac{h_M(r)}{w_2^\beta} \\
&= \lambda h_M(r) \frac{f_2(M_\lambda\|w_2\|_\infty)}{\lambda^{\frac{\beta}{\beta+p-1}} f_2(M_\lambda\|w_2\|_\infty)^{\frac{\beta}{\beta+p-1}} w_2^\beta} \\
&\geq \lambda h_2(r) \frac{f_2(M_\lambda w_2)}{(\lambda^{\frac{1}{\beta+p-1}} f_2(M_\lambda\|w_2\|_\infty)^{\frac{1}{\beta+p-1}} w_2)^\beta} \\
&= \lambda h_2(r) \frac{f_2(\phi_1)}{\bar{\phi_1}^\beta},
\end{aligned} \tag{4.28}
$$

showing that $(\phi_1, \bar{\phi_1})$ is a supersolution of (1.25). If f_2 is bounded and $f_1(u) \to \infty$ as $u \to \infty$, then $\lim_{u\to\infty} \frac{f_2(Mf(u))}{u^{\beta+p-1}} = 0$ for all $M > 0$ and by similar argument $(\phi_1, \bar{\phi_1}) = (\lambda^{\frac{1}{\beta+p-1}} f(M_\lambda\|w_2\|_\infty)^{\frac{1}{\beta+p-1}} w_2, M_\lambda w_2)$ is a supersolution of (1.25).

Now, we construct a positive subsolution $(\psi_1, \bar{\psi_1})$ of (1.25). Let λ_1 be the first eigenvalue and e be a corresponding eigenfunction of

$$
\begin{cases}
-(\varphi_p(u(r)'))' = \lambda h_m(r)\varphi_p(u(r)) \text{ in } (0,1), \\
u(0) = 0 = u(1).
\end{cases} \tag{4.29}
$$

Since $\lim\limits_{u\to 0} \dfrac{f_1(0)}{u^\beta} = \infty = \lim\limits_{u\to 0} \dfrac{f_2(0)}{u^\beta}$, there exist sufficiently small m_λ and m'_λ such that

$$
\lambda_1(m_\lambda)^{p-1}\varphi_p(e) \leq \lambda \frac{f_1(0)}{(m_\lambda e)^\beta} \quad \text{and} \quad \lambda_1(m'_\lambda)^{p-1}\varphi_p(e) \leq \lambda \frac{f_2(0)}{(m'_\lambda e)^\beta}. \tag{4.30}
$$

Let $(\psi_1, \bar{\psi_1}) = (m_\lambda e, m'_\lambda e)$. Since f_1 and f_2 are nondecreasing, we have

$$
\begin{aligned}
-(\varphi_p(\psi_1'))' &= \lambda_1 h_m(r)(m_\lambda)^{p-1}\varphi_p(e) \\
&\leq \lambda h_m(r) \frac{f_1(0)}{(m_\lambda e)^\beta} \leq \lambda h_1(r) \frac{f_1(m'_\lambda e)}{(m_\lambda e)^\beta} = \lambda h_1(r) \frac{f_1(\bar{\psi_1})}{\psi_1^\beta}
\end{aligned} \tag{4.31}
$$

and

$$
\begin{aligned}
-(\varphi_p(\bar{\psi_1'}))' &= \lambda_1 h_m(r)(m_\lambda')^{p-1}\varphi_p(e) \\
&\leq \lambda h_m(r)\frac{f_2(0)}{(m_\lambda' e)^\beta} \leq \lambda h_2(r)\frac{f_2(m_\lambda e)}{(m_\lambda' e)^\beta} = \lambda h_2(r)\frac{f_2(\psi_1)}{\bar{\psi_1}^\beta}. \qquad (4.32)
\end{aligned}
$$

Thus $(\psi_1, \bar{\psi_1})$ is a positive subsolution of (1.25), and if m_λ and m_λ' are sufficiently small then $(\psi_1, \bar{\psi_1}) \leq (\phi_1, \bar{\phi_1})$. Hence Theorem 1.2.3 is proven.

4.4 Proof of Theorem 1.2.4

When $\lambda < \frac{a^{\beta+p-1}}{f_2(a)}\frac{1}{\|w_2\|_\infty^{\beta+p-1}}$, we first construct a second positive supersolution $(\phi_2, \bar{\phi_2})$ of (1.25) Let $(\phi_2, \bar{\phi_2}) = (a\frac{w_2}{\|w_2\|_\infty}, a\frac{w_2}{\|w_2\|_\infty})$. Since $f_2(u) \geq f_1(u)$ for all $u \geq 0$, we have

$$
\begin{aligned}
-(\varphi_p(\phi_2'))' &= \frac{a^{p-1}}{\|w_2\|_\infty^{p-1}}\frac{h_M(r)}{w_2^\beta} \\
&= \frac{a^{\beta+p-1}}{\|w_2\|_\infty^{\beta+p-1}}\frac{h_M(r)}{\frac{(aw_2)^\beta}{\|w_2\|_\infty^\beta}} \\
&> \lambda f_2(a)\frac{h_M(r)}{(a\frac{w_2}{\|w_2\|_\infty})^\beta} \geq \lambda h_1(r)\frac{f_1(a\frac{w_2}{\|w_2\|_\infty})}{(a\frac{w_2}{\|w_2\|_\infty})^\beta} = \lambda h_1(r)\frac{f_1(\bar{\phi_2})}{\phi_2^\beta} \qquad (4.33)
\end{aligned}
$$

and

$$
\begin{aligned}
-(\varphi_p(\bar{\phi_2'}))' &= \frac{a^{p-1}}{\|w_2\|^{p-1}}\frac{h_M(r)}{w_2^\beta} \\
&= \frac{a^{\beta+p-1}}{\|w_2\|_\infty^{\beta+p-1}}\frac{h_M(r)}{\frac{(aw_2)^\beta}{\|w_2\|^\beta}} \\
&> \lambda f_2(a)\frac{h_M(r)}{(a\frac{w_2}{\|w_2\|_\infty})^\beta} \geq \lambda h_2(r)\frac{f_2(a\frac{w_2}{\|w_2\|_\infty})}{(a\frac{w_2}{\|w_2\|_\infty})^\beta} = \lambda h_2(r)\frac{f_2(\phi_2)}{\bar{\phi_2}^\beta}. \qquad (4.34)
\end{aligned}
$$

Hence, $(\phi_2, \bar{\phi_2})$ is a positive supersolution of (1.25) with $\|\phi_2\|_\infty = a = \|\bar{\phi_2}\|_\infty$.

Next when $\frac{d^{\beta+p-1}}{f_1(d)}\frac{(2p)^p}{\underline{h}_m(p-1)^{p-1}} < \lambda < \frac{d^\beta}{f_1(d)}\frac{2^p}{\underline{h}_m}(\frac{p}{p-1})^{p-1}b^{p-1}$, we construct a second positive subsolution $(\psi_2, \bar{\psi_2})$ of (1.25). Let $\tilde{f}, \rho, v$ and consequently ψ_2 be as defined in the proof of Theorem 1.2.2 when f is replaced by f_1 and $\underline{h}$ is by $\underline{h}_m$. We note that ψ_2 satisfies

$$\begin{cases} -(\varphi_p(\psi_2'(t)))' \leq \lambda \underline{h}_m \tilde{f}(\psi_2(t)) \text{ in } (0,1), \\ \psi_2(0) = 0 = \psi_2(1) \end{cases} \tag{4.35}$$

and $d < \|\psi_2\|_\infty < b$. Now choosing $\bar{\psi_2} = \psi_2$,

$$-(\varphi_p(\psi_2'))' \leq \lambda \underline{h}_m \tilde{f}(\psi_2) \leq \lambda h_1 \frac{f_1(\psi_2)}{\psi_2^\beta} = \lambda h_1 \frac{f_1(\bar{\psi_2})}{\psi_2^\beta} \tag{4.36}$$

and

$$-(\varphi_p(\bar{\psi_2}'))' \leq \lambda \underline{h}_m \tilde{f}(\psi_2) \leq \lambda h_2 \frac{f_1(\psi_2)}{\psi_2^\beta} \leq \lambda h_2 \frac{f_2(\psi_2)}{\psi_2^\beta} = \lambda h_2 \frac{f_2(\psi_2)}{\bar{\psi_2}^\beta} \tag{4.37}$$

since $f_1(u) \leq f_2(u)$ for all $u \geq 0$. Hence, $(\psi_2, \bar{\psi_2})$ is a positive subsolution of (1.25) with $d < \|\psi_2\|_\infty < b$ and $d < \|\bar{\psi_2}\|_\infty < b$. Therefore, when $\frac{d^{\beta+p-1}}{f_1(d)} \frac{(2p)^p}{\underline{h}_m(p-1)^{p-1}} < \lambda < \min\left\{ \frac{d^\beta}{f_1(d)} \frac{2^p}{\underline{h}_m} \left(\frac{p}{p-1}\right)^{p-1} b^{p-1} \, , \, \frac{a^{\beta+p-1}}{f_2(a)} \frac{1}{\|w_2\|_\infty^{\beta+p-1}} \right\}$ we obtain a positive supersolution $(\phi_2, \bar{\phi_2})$ and a positive subsolution $(\psi_2, \bar{\psi_2})$ such that $(\psi_2, \bar{\psi_2}) \not\leq (\phi_2, \bar{\phi_2})$. From the proof of Theorem 1.2.3 we note that we have a sufficiently small positive subsolution $(\psi_1, \bar{\psi_1})$ such that $(\psi_1, \bar{\psi_1}) \leq (\phi_2, \bar{\phi_2})$ and a sufficiently large positive supersolution $(\phi_1, \bar{\phi_1})$ such that $(\psi_2, \bar{\psi_2}) \leq (\phi_1, \bar{\phi_1})$. Hence, there exist a positive solution $(u_1, \bar{u}_1)$ of (1.25) such that $(\psi_1, \bar{\psi_1}) \leq (u_1, \bar{u}_1) \leq (\phi_2, \bar{\phi_2})$ and a positive solution $(u_2, \bar{u}_2)$ of (1.25) such that $(\psi_2, \bar{\psi_2}) \leq (u_2, \bar{u}_2) \leq (\phi_1, \bar{\phi_1})$. Since $(\psi_2, \bar{\psi_2}) \not\leq (\phi_2, \bar{\phi_2})$, we have $(u_1, \bar{u}_1) \neq (u_2, \bar{u}_2)$. Therefore, there exist at least two positive solutions of (1.25) for $\lambda \in (\lambda_*, \lambda^*)$ and Theorem 1.2.4 is proven.

CHAPTER 5

PROOF OF THEOREM 1.3.1

5.1 Proof of Theorem 1.3.1

We first give needed lemmas to prove Theorem 1.3.1

Lemma 5.1.1 *The problem* (1.32) *has a positive solution* u_λ *in* $C^2(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$ *for some* $\gamma \in (0,1)$.

Proof. Note that w and e in Section 3.1 are in $C^2(\Omega) \cap C^1(\overline{\Omega})$ when $p = 2$ (see [13]). Hence, by the sub-supersolutions theorem (see [14]) $u_\lambda \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $\epsilon_\lambda e(x) \leq u_\lambda(x) \leq M_\lambda w(x)$, where the small ϵ_λ and large M_λ can be chosen in a similar way as in Section 3.1. Also, due to the C^1 regularity of w and e, there exist $0 < c_i < \infty, i = 1, 2, 3$ and 4 such that $c_1 d(x, \partial\Omega) \leq w(x) \leq c_2 d(x, \partial\Omega)$ and $c_3 d(x, \partial\Omega) \leq e(x) \leq c_4 d(x, \partial\Omega)$, for all $x \in \Omega$. Hence, it is true that $0 < \frac{f(u_\lambda(x))}{u_\lambda(x)^\beta} \leq \frac{f(M_\lambda w(x))}{(\epsilon_\lambda e(x))^\beta} \leq C_1 d(x, \partial\Omega)^{-\beta}$ and $0 \leq u_\lambda(x) \leq C_2 d(x, \partial\Omega)$ for some $0 < C_1, C_2 < \infty$. Therefore, by Remark 2.1.1 u_λ belongs to $C^2(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$.

Lemma 5.1.2 **(A)** *If* u_λ *is a positive solution of* (1.32)*, then* $u_\lambda \geq \delta_\lambda \tilde{e}$ *in* Ω*, where* $\delta_\lambda^{1+\beta} = \lambda \frac{f(0)}{\|\tilde{e}\|_\infty^\beta}$ *and* $\tilde{e}$ *is the solution of* $-\Delta \tilde{e} = 1 \ \text{ in } \ \Omega$ *and* $\tilde{e} = 0 \ \text{ on } \ \partial\Omega$.

(B) (1.32) *has a minimal positive solution for each* $\lambda > 0$.

Proof. **(A)** Let u_λ be a positive solution of (1.32) for arbitrary fixed $\lambda > 0$. Since f is nondecreasing and $f(0) > 0$, it turns out that on $\tilde{\Omega}_\lambda = \{x \in \Omega | u_\lambda(x) < \delta_\lambda \tilde{e}(x)\}$ we have

$$\begin{aligned} -\Delta(u_\lambda(x) - \delta_\lambda \tilde{e}(x)) &= \lambda \frac{f(u(x))}{u(x)^\beta} - \delta_\lambda \\ &\geq \lambda \frac{f(0)}{u(x)^\beta} - \delta_\lambda \\ &> \lambda \frac{f(0)}{\delta_\lambda^\beta \tilde{e}(x)^\beta} - \delta_\lambda \\ &\geq \lambda \frac{f(0)}{\delta_\lambda^\beta \|\tilde{e}\|_\infty^\beta} - \delta_\lambda \\ &= 0, \end{aligned} \tag{5.1}$$

and $u_\lambda(x) - \delta_\lambda \tilde{e}(x) = 0$ on $\partial\tilde{\Omega}_\lambda$. Hence, by the maximum principle we have $u_\lambda(x) \geq \delta_\lambda \tilde{e}(x)$ on $\tilde{\Omega}_\lambda$, which is a contradiction. Thus $\tilde{\Omega}_\lambda = \emptyset$.

(B) Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary condition and $e > 0$ be a corresponding eigenfunction. Hence e and λ_1 satisfy:

$$\begin{cases} -\Delta e = \lambda_1 e & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.2}$$

Since $\frac{f(u)}{u^\beta} \to \infty$ as $u \to 0$, there exists a sufficiently small m_λ such that

$$\lambda_1(m_\lambda e) \leq \lambda \frac{f(m_\lambda e)}{(m_\lambda e)^\beta} \quad \text{for all } \lambda > 0. \tag{5.3}$$

Let $\psi = m_\lambda e$. Then

$$-\Delta\psi = \lambda_1(m_\lambda e) \leq \lambda \frac{f(m_\lambda e)}{(m_\lambda e)^\beta} = \lambda \frac{f(\psi)}{\psi^\beta}. \tag{5.4}$$

Thus ψ is a positive subsolution of (1.32) and we can choose $m_\lambda \approx 0$ so that $m_\lambda e \leq \delta_\lambda \tilde{e}$ in Ω. But from Theorem 1.1.1 we already know that (1.32) has a positive solution for all

$\lambda > 0$ and hence by (**A**) it must satisfy $u_\lambda \geq \delta_\lambda \tilde{e}$. Therefore, there exists a minimal positive solution of (1.32).

Proof of Theorem 1.3.1. Let u_λ be the minimal positive solutions of (1.32) and assume that there exists another solution $\bar{u}_\lambda$ of (1.32) distinct from u_λ. Clearly, $u_\lambda \leq \bar{u}_\lambda$ and $u_\lambda \not\equiv \bar{u}_\lambda$ in Ω. Let $\hat{f}(u) = \frac{f(u)}{u^\beta}$. Then we have

$$\int_\Omega -\Delta(\bar{u}_\lambda - u_\lambda)(\bar{u}_\lambda - u_\lambda)dx = \lambda \int_\Omega (\hat{f}(\bar{u}_\lambda) - \hat{f}(u_\lambda))(\bar{u}_\lambda - u_\lambda)dx. \tag{5.5}$$

By the divergence theorem we get

$$\int_\Omega -\Delta(\bar{u}_\lambda - u_\lambda)(\bar{u}_\lambda - u_\lambda)dx = \int_\Omega |\nabla(\bar{u}_\lambda - u_\lambda)|^2 \, dx. \tag{5.6}$$

By the fundamental theorem of calculus

$$\begin{aligned}
&\lambda \int_\Omega (\hat{f}(\bar{u}_\lambda) - \hat{f}(u_\lambda))(\bar{u}_\lambda - u_\lambda)dx \\
&= \lambda \int_\Omega \int_0^1 \hat{f}'(u_\lambda + s(\bar{u}_\lambda - u_\lambda))(\bar{u}_\lambda - u_\lambda)ds(\bar{u}_\lambda - u_\lambda)dx \\
&= \lambda \int_\Omega \int_0^1 \hat{f}'(u_\lambda + s(\bar{u}_\lambda - u_\lambda))ds(\bar{u}_\lambda - u_\lambda)^2 dx \\
&= \lambda \int_\Omega \gamma(x)(\bar{u}_\lambda - u_\lambda)^2 dx,
\end{aligned} \tag{5.7}$$

where $\gamma(x) = \int_0^1 \hat{f}'(u_\lambda(x) + s(\bar{u}_\lambda(x) - u_\lambda(x)))ds$. Hence from (5.5) we obtain

$$\int_\Omega |\nabla(\bar{u}_\lambda - u_\lambda)|^2 \, dx = \lambda \int_\Omega \gamma(x)(\bar{u}_\lambda - u_\lambda)^2 dx. \tag{5.8}$$

Let $l > 0$ be such that $\tilde{e}(x) \geq l d(x, \partial\Omega)$ for all $x \in \bar{\Omega}$. Now if $d(x, \partial\Omega) \geq \frac{\sigma}{\delta_\lambda l}$ then by Lemma 5.1.2 $u_\lambda(x) \geq \delta_\lambda \tilde{e}(x) \geq \delta_\lambda l d(x, \partial\Omega) \geq \sigma$. Let $\Omega_1 := \{x \in \Omega | \; d(x, \partial\Omega) < \frac{\sigma}{\delta_\lambda l} =: C_1 \lambda^{-\frac{1}{1+\beta}}\}$, where $C_1 = \frac{\sigma}{l}(\frac{\|\tilde{e}\|_\infty^\beta}{f(0)})^{\frac{1}{1+\beta}}$ and $\Omega_2 := \Omega - \Omega_1$. Now we rewrite (5.8) as:

$$\int_{\Omega} |\nabla(\bar{u}_\lambda - u_\lambda)|^2 dx = \lambda \int_{\Omega_1} \gamma(x)(\bar{u}_\lambda - u_\lambda)^2 dx + \lambda \int_{\Omega_2} \gamma(x)(\bar{u}_\lambda - u_\lambda)^2 dx. \quad (5.9)$$

Since $u_\lambda + s(\bar{u}_\lambda - u_\lambda) \geq u_\lambda \geq \sigma$ for $s > 0$ in Ω_2, using(U_1), we know that $\hat{f}'(u_\lambda + s(\bar{u}_\lambda - u_\lambda)) < 0$ which gives $\gamma(x) = \int_0^1 \hat{f}'(u_\lambda(x) + s(\bar{u}_\lambda(x) - u_\lambda(x)))ds < 0$ in Ω_2. Since $\beta \in (0,1)$ and $f(0) > 0$, $\lim_{u \to 0+} \hat{f}'(u) = -\infty$. Also by (U_1), $\hat{f}'(u) \leq 0$ for $u > \sigma$. Hence $\hat{f}'$ is bounded above; let then $M_1 > 0$ be such that $\hat{f}'(u) \leq M_1$ for all $u > 0$. Hence we obtain

$$\begin{aligned} \int_{\Omega} |\nabla(\bar{u}_\lambda - u_\lambda)|^2 dx &\leq \lambda \int_{\Omega_1} \gamma(x)(\bar{u}_\lambda - u_\lambda)^2 dx \\ &\leq \lambda M_1 \int_{\Omega_1} (\bar{u}_\lambda - u_\lambda)^2 dx, \end{aligned} \quad (5.10)$$

where M_1 is such that $\hat{f}'(s) \leq M_1$ for all $s > 0$. Next if $x \in \Omega_1$, then $d(x, \partial\Omega) \to 0$ as $\lambda \to \infty$. Hence, for $\lambda \gg 1$, if $x \in \Omega_1$, then there exists a unique $p = p(x) \in \partial\Omega$ and a unique $t = t(x) \in (0, C_1 \lambda^{-\frac{1}{1+\beta}} =: a_\lambda)$ such that $x = t\eta_p + p$, where η_p is the unit inward normal at p.

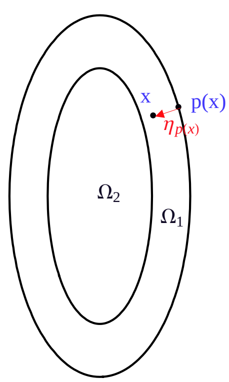

Figure 5.1

Ω_1 and inward normal vector $\eta_{p(x)}$

Now using the fact that $\bar{u}_\lambda(x) - u_\lambda(x) = 0$ on $\partial\Omega$, we have

$$
\begin{aligned}
|\bar{u}_\lambda(x) - u_\lambda(x)| &= |\int_0^t \nabla(\bar{u}_\lambda - u_\lambda)(s\eta_p + p)\eta_p ds| \\
&\leq (\int_0^{a_\lambda} |\nabla(\bar{u}_\lambda - u_\lambda)(s\eta_p + p)|^2 ds)^{\frac{1}{2}} (\int_0^{a_\lambda} |\eta_p|^2 ds)^{\frac{1}{2}} \\
&= C_1^{\frac{1}{2}} \lambda^{-\frac{1}{2(1+\beta)}} (\int_0^{a_\lambda} |\nabla(\bar{u}_\lambda - u_\lambda)(s\eta_p + p)|^2 ds)^{\frac{1}{2}} \qquad (5.11)
\end{aligned}
$$

Let C_2 be a uniform bound for the Jacobian of the transformation $x \to (y,t)$ with $y \in \partial\Omega$ and t the component in the unit inward normal direction at y. Then we get

$$
\begin{aligned}
\int_{\Omega_1} |\bar{u}_\lambda - u_\lambda|^2 dx &= \int_{\Omega_1} |\bar{u}_\lambda(p + t\eta_p) - u_\lambda(p + t\eta_p)|^2 dx \\
&\leq C_2 \int_{\partial\Omega} \int_0^{a_\lambda} |\bar{u}_\lambda(y + t\eta_y) - u_\lambda(y + t\eta_y)|^2 dt\, dy. \qquad (5.12)
\end{aligned}
$$

Now using (5.11) we have

$$
\begin{aligned}
&\int_{\Omega_1} |\bar{u}_\lambda - u_\lambda|^2 dx \\
&\leq C_2 \int_{\partial\Omega} \int_0^{a_\lambda} C_1 \lambda^{-\frac{1}{(1+\beta)}} \left(\int_0^{a_\lambda} |\nabla(\bar{u}_\lambda - u_\lambda)(s\eta_y + y)|^2 ds \right) dt\, dy. \qquad (5.13)
\end{aligned}
$$

Let $\Gamma(y) := \int_0^{a_\lambda} |\nabla(\bar{u}_\lambda - u_\lambda)(s\eta_y + y)|^2 ds$. Then

$$
\begin{aligned}
\int_{\Omega_1} |\bar{u}_\lambda - u_\lambda|^2 dx &\leq C_1 C_2 \lambda^{-\frac{1}{(1+\beta)}} \int_{\partial\Omega} \int_0^{a_\lambda} \Gamma(y)\, dt\, dy \\
&= C_1^2 C_2 \lambda^{-\frac{2}{(1+\beta)}} \int_{\partial\Omega} \Gamma(y) dy \\
&= C_1^2 C_2 \lambda^{-\frac{2}{(1+\beta)}} \int_{\partial\Omega} \int_0^{a_\lambda} |\nabla(\bar{u}_\lambda - u_\lambda)(s\eta_y + y)|^2 ds\, dy. \qquad (5.14)
\end{aligned}
$$

Hence, there exists a $C_0 > 0$ such that

$$
\int_{\Omega_1} |\bar{u}_\lambda - u_\lambda|^2 dx \leq C_0 \lambda^{-\frac{2}{(1+\beta)}} \int_{\Omega_1} |\nabla(\bar{u}_\lambda - u_\lambda)|^2 dx. \qquad (5.15)
$$

From (5.10) and (5.15), we obtain

$$\begin{aligned}\int_{\Omega} |\nabla(\bar{u}_\lambda - u_\lambda)|^2 dx &\leq M_1 C_0 \lambda^{1-\frac{2}{(1+\beta)}} \int_{\Omega_1} |\nabla(\bar{u}_\lambda - u_\lambda)|^2 dx \\ &\leq M_1 C_0 \lambda^{\frac{\beta-1}{1+\beta}} \int_{\Omega} |\nabla(\bar{u}_\lambda - u_\lambda)|^2 dx. \end{aligned} \tag{5.16}$$

This is a contradiction for λ large since $\beta < 1$. Therefore, it turns out that $u_\lambda \equiv \bar{u}_\lambda$ for $\lambda \gg 1$ and Theorem 1.3.1 is proven.

CHAPTER 6

PROOF OF THEOREM 1.4.1

6.1 Proof of Theorem 1.4.1

From our discussion in Chapter 2 it is enough to show that H has the shape in Figure 2.1 when $\alpha \gg 1$. Namely,

1. $\lim_{s\to 0^+} H'(s) > 0$.

2. there exists $\rho_1 > 0$ such that $H(\rho_1) < 0$.

Here $f(s) = \frac{\exp[\frac{\alpha s}{\alpha+s}]}{s^\beta}$. Recall that $F(u) = \int_0^u f(s)\,ds$ and $H(s) = F(s) - \frac{1}{p} s f(s)$. Clearly, $H(0) = 0$. Since $f'(s) = \exp[\frac{\alpha s}{\alpha+s}]\{\frac{\alpha^2}{s^\beta(\alpha+s)^2} - \frac{\beta}{s^{\beta+1}}\}$, we have

$$\begin{aligned} H'(s) &= \frac{1}{p}\big[(p-1)f(s) - sf'(s)\big] \\ &= \frac{1}{p}\left[(p-1)\frac{\exp[\frac{\alpha s}{\alpha+s}]}{s^\beta} - s\exp[\frac{\alpha s}{\alpha+s}]\left(\frac{\alpha^2}{s^\beta(\alpha+s)^2} - \frac{\beta}{s^{\beta+1}}\right)\right] \\ &= \frac{\exp[\frac{\alpha s}{\alpha+s}]}{ps^\beta}\left[\frac{(\beta+p-1)(\alpha+s)^2 - \alpha^2 s}{(\alpha+s)^2}\right], \end{aligned} \tag{6.1}$$

and hence $\lim_{s\to 0^+} H'(s) = +\infty$. Next, we show that there exists $\rho_1 > 0$ such that $H(\rho_1) < 0$. Take $\rho_1 = \alpha$. Then we have $H(\alpha) = \int_0^\alpha f(s)ds - \frac{\alpha}{p} f(\alpha)$. Since

$$
\begin{aligned}
\frac{dH(\alpha)}{d\alpha} &= \left(1-\frac{1}{p}\right) f(\alpha) - \frac{\alpha}{p} f'(\alpha) \\
&= \left(1-\frac{1}{p}\right) \frac{\exp[\frac{\alpha}{2}]}{\alpha^{\beta}} - \frac{\alpha}{p} \frac{\exp[\frac{\alpha}{2}]}{\alpha^{\beta}} \left(\frac{1}{4} - \frac{\beta}{\alpha}\right) \\
&= \frac{\exp[\frac{\alpha}{2}]}{\alpha^{\beta}} \left[\left(1-\frac{1}{p}\right) - \frac{\alpha}{4p} + \frac{\beta}{p}\right] \\
&= \frac{1}{p} \exp[\frac{\alpha}{2}] \alpha^{1-\beta} \left[\frac{\beta+p-1}{\alpha} - \frac{1}{4}\right], \qquad (6.2)
\end{aligned}
$$

we obtain that $\frac{dH(\alpha)}{d\alpha} \to -\infty$ as $\alpha \to \infty$. Hence, we have $H(\alpha) < 0$ for $\alpha \gg 1$. Therefore, $H(s)$ has the shape in Figure 2.1 for $\alpha \gg 1$, and hence Theorem 1.4.1 is proven.

6.2 Computational Results

Note that (2.18) describes the bifurcation curve of positive solutions of (1.33). Here by using Mathematica computations, we provide the bifurcation diagrams of positive solutions of (1.33) for the cases:

(a) $p = 1.6, \alpha = 10$ and $\beta = 0.5$

(b) $p = 10, \alpha = 50$ and $\beta = 0.5$

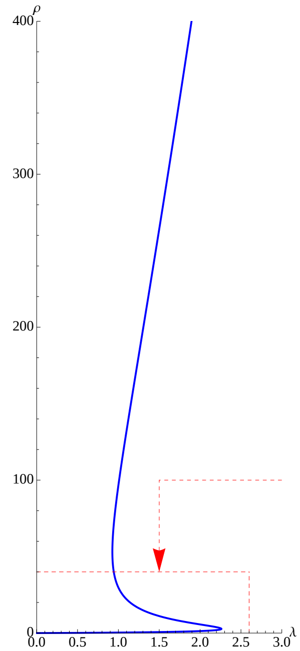

Figure 6.1

Example (a) $p = 1.6, \alpha = 10$ and $\beta = 0.5$

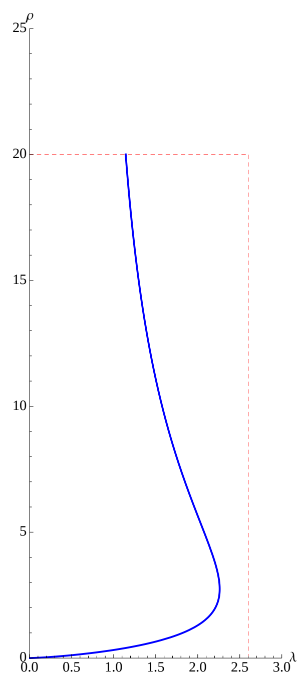

Figure 6.2

Example (a) $p = 1.6, \alpha = 10$ and $\beta = 0.5$

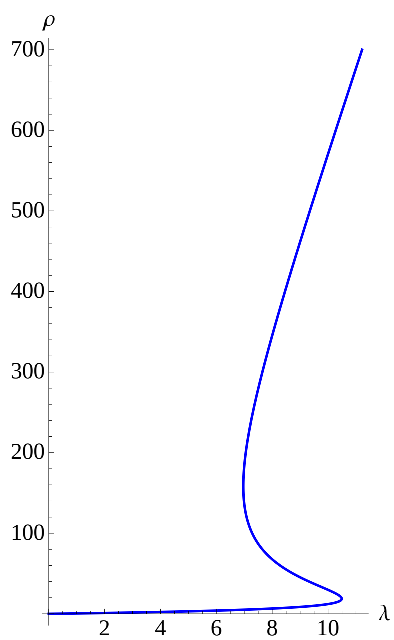

Figure 6.3

Example (b) $p = 10, \alpha = 50$ and $\beta = 0.5$

CHAPTER 7

CONCLUSIONS AND FUTURE DIRECTIONS

7.1 Conclusions

In this thesis, we established the existence and multiplicity of positive solutions to classes of singular p-Laplacian boundary value problems in bounded domains in $\mathbb{R}^N, N \geq 1$ when a parameter varies in $(0, \infty)$, and extended the results to singular p-Laplacian systems and to the case of exterior domains in $\mathbb{R}^N$. Unlike the typical multiplicity result of nonsingular cases, our multiplicity results are restricted to two positive solutions. However, we conjecture that even for the singular case there are at least three positive solutions. This was confirmed in the study of one dimensional singular p-Laplacian perturbed Gelfand problems. We also established the uniqueness of the positive solution to singular problems for large values of the parameter when $p = 2$.

7.2 Future directions

We plan to

(1) Establish a three solutions theorem via sub-supersolutions for singular boundary value problems,

(2) Extend our uniqueness result to the case of exterior domains as well as to models involving the p-Laplacian operator,

(3) Study singular boundary problems with nonlinear boundary conditions.

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APPENDIX A

TRANSFORMATION

Consider the problem

$$\begin{cases} -\Delta_p u = \lambda K(|x|)\frac{f(u)}{u^\beta} & \text{in } \Omega, \\ \\ u(x) = 0 & \text{if } |x| = r_0, \\ \\ u(x) \to 0 & \text{if } |x| \to \infty, \end{cases} \tag{A.1}$$

where $K : [r_0, \infty) \to (0, \infty)$ is continuous. Set $r = |x|$ and $v(r) = u(x)$. Then

$$\Delta_p u(x) = \frac{1}{r^{N-1}}(r^{N-1}|v'(r)|^{p-2}v'(r))', \tag{A.2}$$

which reduces (A.1) to the following:

$$\begin{cases} -(r^{N-1}|v'(r)|^{p-2}v'(r))' = \lambda r^{N-1}K(r)\frac{f(v)}{v^\beta}, & r_0 \leq r < \infty, \\ \\ v(r_0) = 0, \;\; v(r) \to 0 \;\text{ if }\; r \to \infty. \end{cases} \tag{A.3}$$

Now set $t = (\frac{r}{r_0})^{\frac{p-N}{p-1}}$ and $z(t) = v(r)$. Since

$$(r^{N-1}|v'(r)|^{p-2}v'(r))' = \left(\frac{N-p}{p-1}\right)^p \left(\frac{1}{r_0}\right)^{p-N+1} t^{\frac{1-N}{p-N}}(|z'(t)|^{p-2}z'(t))', \tag{A.4}$$

we have that

$$\begin{aligned} -(|z'(t)|^{p-2}z'(t))' &= \left(\frac{p-1}{N-p}\right)^p r_0^{p-N+1}t^{-\frac{1-N}{p-N}}\lambda(r_0 t^{\frac{p-1}{p-N}})^{N-1}K(r_0 t^{\frac{p-1}{p-N}})\frac{f(z(t))}{z(t)^\beta} \\ &= \lambda\left(\frac{p-1}{N-p}\right)^p r_0^p\, t^{\frac{p(1-N)}{N-p}}K\left(r_0\, t^{\frac{1-p}{N-p}}\right)\frac{f(z(t))}{z(t)^\beta}, \end{aligned} \tag{A.5}$$

which again reduces the problem (A.3) to the following the boundary value problem:

$$\begin{cases} -(\varphi_p(z'(t)))' = \lambda h(t)\frac{f(z(t))}{z(t)^\beta}, & 0 < t < 1, \\ \\ z(0) = 0 = z(1), \end{cases} \tag{A.6}$$

where $h(t) = \left(\frac{p-1}{N-p}\right)^p r_0^p\, t^{\frac{p(1-N)}{N-p}}K\left(r_0\, t^{\frac{1-p}{N-p}}\right)$.