8-7-2020

Analysis of positive solutions for singular p-Laplacian problems via fixed point methods

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By

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A Thesis
Submitted to the Faculty of
Mississippi State University
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
in Mathematical Science
in the Department of Mathematics and Statistics

Mississippi State, Mississippi

August 2020

Analysis of positive solutions for singular p-Laplacian problems via fixed point methods

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Title of Study: Analysis of positive solutions for singular p-Laplacian problems via fixed point methods

Pages of Study: 54

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In this dissertation, we study the existence and nonexistence of positive solutions to some classes of singular p-Laplacian boundary value problems with a parameter.

In the first study, we discuss positive solutions for a class of sublinear Dirichlet p-Laplacian equations and systems with sign-changing coefficients on a bounded domain of $\mathbb{R}^n$ via Schauder Fixed Point Theorem and the method of sub- and supersolutions. Under certain conditions, we show the existence of positive solutions when the parameter is large and nonexistence when the parameter is small.

In the second study, we discuss positive radial solutions for a class of superlinear p-Laplacian problems with nonlinear boundary conditions on an exterior domain via degree theory and fixed point approach. Under certain conditions, we show the existence of positive solutions when the paprameter is small and nonexistence when the paramter is large.

Our results provide extensions of corresponding ones in the literature from the Laplacian to the p-Laplacian, and can be applied to the challenging infinite semipositone case.

Key words: p-Laplacian, sign-changing coefficients, positive solutions, positive radial solutions, Nonlinear boundary conditions

ACKNOWLEDGEMENTS

My deep gratitude and profound respect goes first to my supervisor, Dr. Hai Dang, who has convincingly guided and encouraged me to be professional even when the road got tough. I am indebted to his helpful comments and constant encouragement. Without his persistent guidance, the goal of this dissertation would not have been accomplished.

I would like to thank my committee members, Dr. Mohsen Razzaghi, Dr. Robert Smith, Dr. Jon Woody, Dr. Chuanxi Qian. Also, I would like to thank my department chair Dr. Razzaghi and associate head and graduate coordinator Dr. Mohammad Sepehrifar for supporting me as a graduate student. I wish to thank all the faculty members at Mississippi State University whose assistance was invaluable during my study.

I wish to express my deepest gratitude to my parents Sara Algothami and Haza Alotaibi, to my wife Haifa Algothami, to my daughters Shahad and Raghad, to my sons Bandar, Yasir and Omar. To my brothers and sisters, thank you from my deep heart.

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LIST OF SYMBOLS

Ω : bounded domain in $\mathbb{R}^n$

$B(0,R) := \{x \in \mathbb{R}^n : ||x|| < R\}$

$\partial\Omega$: boundary of the Ω

$\overline{\Omega}$: closure of Ω

$C[0,1]$:= { $u : [0,1] \longrightarrow \mathbb{R}$: u is continuous }

$C^k[0,1]$:= { $u : [0,1] \longrightarrow \mathbb{R}$: u is k- times continuously differentiable }

$C^{0,\alpha}(\overline{\Omega})$: = $\{u \in C(\overline{\Omega}) : \sup_{x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty\}$, $\alpha \in (0,1)$

$C^{1,\alpha}(\overline{\Omega}) := \{u \in C(\overline{\Omega}) : D^\beta u \in C^{0,\alpha}(\overline{\Omega})\ \forall\beta$ with $|\beta| = 1$

$||u||_\infty := \sup_{x\in[0,1]} |u(x)|$

$|u|_{1,\alpha} := ||u||_\infty + \sup_{x\neq y} \frac{|\nabla u(x)-\nabla u(y)|}{|x-y|^\alpha}$

$L^p(\Omega) := \{u : (\Omega) \longrightarrow \mathbb{R}$: u is measurable and $\int_\Omega |u|^p < \infty\}, 1 \leq p < \infty$

$L^p_{loc}(\Omega) := \{u : (\Omega) \longrightarrow \mathbb{R}$: u is measurable and $u \in L^p(K)$ for every compact set $K \subset \Omega\}$, $1 \leq p < \infty$

$||u||_p := (\int_\Omega |u|^p)^{\frac{1}{p}}$

$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \exists g_1, ..., g_n \in L^p(0,1)$ such that $\frac{\partial u}{\partial x_i} \in L^p(\Omega), i = 1, ..., n\}$

$W^{1,p}_0(\Omega) := \{u \in W^{1,p}(\Omega) : u = 0$ on $\partial\Omega\}$

CHAPTER 1

INTRODUCTION

1.1 History of the problems and abstract of results

The p-Laplacian operator has appeared in mathematical models in physics such as the Thomas -Fermi equation [17, 30], the study of equilibrium configuration of mass in a spherical clous of gas [29, 39, 50], chemical reactions [25], and non-Newtonian fluids [7]. In non-Newtonian fluids, the case $p \in (1, 2)$ represents pseudo-plastic while $p > 2$ corresponds to dilatant fluids. In this dissertation, we discuss the existence and nonexistence of positive solutions to the p-Laplacian problems

$$\begin{cases} -\Delta_p u = \lambda a(x) f(u) \text{ in } \Omega, \\ \quad u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega,\ f : (0, \infty) \to \mathbb{R}$ is p-sublinear at ∞ i.e. $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$ and is allowed to be singular $(\pm\infty)$ at $0,\ a : \Omega \to \mathbb{R}$ is a sign-changing coefficient, and

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} + \tilde{c}(u) u = 0 \text{ on } |x| = r_0, \\ \quad u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{1.2}$$

where $N > p > 1,\ \Omega = \{x \in \mathbb{R}^N : |x| > r_0 > 0\},\ \tilde{c} : [0, \infty) \to (0, \infty),\ f : (0, \infty) \to \mathbb{R}$ is p-superlinear at ∞ i.e. $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty,\ K : [r_0, \infty) \to (0, \infty)$ with $r^{N+\mu} K(r)$

bounded for some $\mu > 0$. Here n denotes the outer unit normal vector on $\partial\Omega$. Here $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u), p > 1$ and λ is a positive parameter. We also consider a system version of (1.1) i.e.

$$\begin{cases} -\Delta_{p_i} u_i = \lambda a_i(x) f_i(u_j) \text{ in } \Omega, \\ \qquad u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where $i, j \in \{1, 2\}, i \neq j$ and $p_i > 1, a_i : \Omega \to \mathbb{R}$ are a sign-changing coefficients and $f_i : (0, \infty) \to \mathbb{R}$ are bounded above.

In Chapter 2, we prove that if the negative part of a is sufficiently small with respect to the positive part of a, and $f(s) \sim ls^q$ for some $l > 0, q \in [0, p-1)$ at ∞ then for λ large, problem (1.1) has a positive solution $u_\lambda \in C^{1,\nu}(\bar{\Omega})$ with $u_\lambda(x) \to \infty$ as $\lambda \to \infty$ uniformly on compact subsets of Ω. An analogous existence results for the system (1.3) is obtained when $f(s) \sim l > 0$ at ∞. We also prove the nonexistence of positive solutions for λ small under semipositone assumptions of f and f_i at 0. Our approach for the existence results is based on the method of sub- and supersolutions and Schauder Fixed Point Theorem. The results in Chapter 2 was published in [43].

In Chapter 3, we establish the existence of positive radial solutions to (1.2), i.e. to the ODE problem

$$\begin{cases} -(\phi(u'))' = \lambda h(t) f(u), \ t \in (0, 1), \\ u(0) = 0, \ u'(1) + c(u(1))u(1) = 0, \end{cases}$$

for $\lambda > 0$ small, where $\phi(z) = |z|^{p-2}z, \ h(t) = \left(\frac{p-1}{N-p} r_0\right)^p t^{\frac{p(N-1)}{p-N}} K(r_0 t^{\frac{p-1}{p-N}})$, and $c(s) = \frac{p-1}{N-p} r_0 \tilde{c}(s)$. We also prove the nonexistence of positive solutions for λ large under semipositone assumption of f at 0. Our approach depends on degree theory, fixed point arguments, and comparison principle. The results in Chapter 3 have been accepted for publication in [44].

Our results extend corresponding ones in the literature from the case $p = 2$ to the general case $p > 1$. Note that the p-Laplacian operator is nonlinear when $p \neq 2$ and therefore previous approaches using the Green's function can not be used. We also allow the semipositone case where the maximum principle can not be directly applied.

We recall some important results in the literature that will be needed in the proofs of our results.

1.2 Backgrounds

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = h(x,u) \text{ in } \Omega, \\ \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $h : \Omega \times (0,\infty) \to \mathbb{R}$ is continuous and there exists $\alpha \in [0,1)$ such that

$$\lim_{z \to 0^+} z^\alpha |h(x,z)| < \infty \tag{1.5}$$

uniformly for a.e. $x \in \Omega$.

We say that $\phi \in C^1(\bar{\Omega})$ is a subsolution of (1.4) if for all $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geq 0$ in Ω,

$$\int_\Omega |\nabla \phi|^{p-2} \nabla \phi . \nabla \xi dx \leq \int_\Omega h(x,\phi)\xi dx$$

and $\phi \leq 0$ on $\partial\Omega$. Similarly, we say that $\psi \in C^1(\bar{\Omega})$ is a supersolution of (1.4) if for all $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geq 0$ in Ω,

$$\int_\Omega |\nabla \psi|^{p-2} \nabla \psi . \nabla \xi dx \geq \int_\Omega h(x,\psi)\xi dx$$

and $\psi \geq 0$ on $\partial\Omega$. Let $d(x)$ denote the distance from $x \in \Omega$ to Ω.

Theorem A ([9] - Sub-supersolutions principle)

Suppose ϕ and ψ are sub- and supersolutions of (1.4) respectively with $\inf\limits_{\Omega} \frac{\phi}{d} > 0$ and $\phi \leq \psi$ in Ω. Then there exists $\nu \in (0,1)$ such that (1.4) has a solution $u \in C^{1,\nu}(\bar{\Omega})$ with $\phi \leq u \leq \psi$ in Ω.

Remark A:

Note that $\int_\Omega h(x,\phi)\xi dx$ and $\int_\Omega h(x,\psi)\xi dx$ exist for all $\xi \in W_0^{1,p}(\Omega)$ in view of (1.5) and $\inf\limits_{\Omega} \frac{\phi}{d} > 0$. Indeed,

$$\int_\Omega |h(x,\phi)|\xi dx \leq C\int_\Omega \left|\frac{\xi}{d^\alpha}\right| dx \leq C||d||_\infty^{1-\alpha}\int_\Omega \left|\frac{\xi}{d}\right| dx$$

$$\leq C||d||_\infty^{1-\alpha}\left|\left|\frac{\xi}{d}\right|\right|_{1,p} \leq C_1||\xi||_{1,p},$$

where $C_1 = C||d||_\infty^{1-\alpha}k$ and $k > 0$ is a constant in Hardy's inequality $||\xi/d||_{1,p} \leq k||\xi||_{1,p}$ for all $\xi \in W_0^{1,p}(\Omega)$. (See [23]).

Theorem B ([42] - Weak comparison principle)

Let $u, v \in W^{1,p}(\Omega)$ satisfy

$$-\Delta_p u \leq -\Delta_p v \text{ in } \Omega$$

in the weak sense i.e.,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\xi dx \leq \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\xi dx$$

for all $\xi \in W_0^{1,p}(\Omega)$ with $\xi \geq 0$. If $u \leq v$ on $\partial\Omega$, then

$$u \leq v \text{ in } \Omega$$

Theorem C ([24] - Strong maximum principle)

Let $u \in C^1(\bar{\Omega})$ be such that $\Delta_p u \in L^2_{loc}(\Omega), u \geq 0$ in Ω, and

$$\Delta_p u \leq \beta(u) \text{ in } \Omega,$$

where $\beta : [0, \infty) \to R$ is continuous, nondecreasing, $\beta(0) = 0$ and either $\beta(s) = 0$ for some $s > 0$ or $\beta(s) > 0$ for all $s > 0$ but $\int_0^1 (\beta(s)s)^{-1/p} ds = \infty$.

Then if u does not vanish identically on Ω, it is positive everywhere in Ω. Moreover, if $u(x_0) = 0$ for some $x_0 \in \partial\Omega$ then

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where ν denotes the outer unit normal vector at x_0.

Theorem D (Schauder Fixed Point Theorem)

Let E be a Banach space and let C be a nonempty closed convex subset of E. Let

$$T : C \to C$$

be a continuous map such that $T(C) \subset K$, where K is a compact subset of C. Then T has a fixed point in C.

CHAPTER 2

ON SINGULAR QUASILINEAR PROBLEMS WITH SIGN-CHANGING COEFFICIENTS

2.1 Introduction

Consider the p-Laplacian problems

$$\begin{cases} -\Delta_p u = \lambda a(x) f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u), p > 1,\ \Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega,\ f : (0, \infty) \to \mathbb{R}$ is p-sublinear at ∞ i.e. $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$ and is allowed to be singular $(\pm\infty)$ at $0,\ a : \Omega \to \mathbb{R}$ is a sign-changing coefficient, and λ is a positive parameter.

We are interested in studying positive solutions of (2.1), which has appeared in mathematical models in physics such as the Thomas -Fermi equation [17, 30] and the study of equilibrium configuration of mass in a spherical clous of gas [29, 39, 50]. In the positone case i.e. $f(0) > 0$, the existence of a positive solution to the problem

$$\begin{cases} -\Delta u = \lambda a(x) f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.2}$$

was established in [20, 33, 34] for small λ when $a \not\equiv 0$ is such that there exists a constant $\varepsilon > 0$ for which the problem

$$\begin{cases} -\Delta z = a^+(x) - (1+\varepsilon)a^-(x) \text{ in } \Omega, \\ \\ z = 0 \text{ on } \partial\Omega \end{cases} \tag{2.3}$$

has a nonnegative solution. Here $a^+ = \max(a, 0), a^- = \max(-a, 0)$. These results were extended to the quasilinear problem (2.1) in [11] under the assumption that the problem

$$\begin{cases} -\Delta_p z = a(x) \text{ in } \Omega, \\ \\ z = 0 \text{ on } \partial\Omega \end{cases} \tag{2.4}$$

has a positive solution $z \in C^1(\bar{\Omega})$ with $\frac{\partial z}{\partial n} < 0$ on $\partial\Omega$, where n denotes the outer unit normal on $\partial\Omega$. Note that this assumption is equivalent to (2.3) when $p = 2$. In the semipositone case i.e. $f(0) < 0$, the study of positive solutions to (2.1) is rather challenging due to the absence of the maximum principles even when $a \geq 0$ (see [18] for a survey of this case). When $a \geq 0$, it was proved in [1, Theorem 8] that (2.1) has a large positive solution for λ large when $f(0) < 0$ and $f : [0, \infty) \to \mathbb{R}$ is continuous with $f(s) \sim s^q$ at ∞ for some $q \in [0, 1)$. The case when a may change sign was recently discussed in [48, Theorem 1.2] where the existence of a $C^{1.\theta}(\bar{\Omega})$ positive solution to (2.1) with $f(0) < 0$ was established for λ large when $\lim_{s\to\infty} f(s) \in (0, \infty)$ and $a \in L^\infty(\Omega)$ satisfies (2.3), or when $\lim_{s\to\infty} \frac{f(s)}{s^q} = 1$ for some $q \in (0, 1)$ and $a \in L^\infty(\Omega)$ is such that there exists $\varepsilon > 0$ for which the problem

$$\begin{cases} -\Delta z = a^+(x)z^q - (1+\varepsilon)a^-(x)z^q \text{ in } \Omega, \\ \\ z = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.5}$$

has a positive solution $z \in C^1(\bar{\Omega})$ with $\frac{\partial z}{\partial n} < 0$ on $\partial\Omega$. The condition on a in (2.3) or (2.5) is satisfied if $||a^-||_\infty$ is sufficiently small and examples can be found in [33, p 1398], [48, Remark 2.5], and [47, Lemma 2.2] for (2.3), and [45, Theorems 2.1 (ii), 3.1 and 3.3] and [46, Theorem 3.1] for (2.5). Note that the proofs in [48] depends on the linearity of the Laplacian and the fact that $f(0)$ is finite and can not be applied to singular p-Laplacian. In this paper, we shall extend the results in [48] to the p-Laplacian where f is also allowed to be singular at 0 i.e. $\lim_{s\to 0^+} f(s) = \pm\infty$, which has not been treated before to the best of our knowledge. A nonexistence result for λ small is also given in the semipositone case when f is bounded above. In particular, our results when applied to the model example

$$\begin{cases} -\Delta_p u = \lambda a(x)\left(\frac{C}{u^\alpha} + u^q e^{\frac{u}{u+1}}\right) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

where $C \in \mathbb{R}, \alpha \in [0,1), q \in [0, p-1)$ and $a(x)$ satisfies condition (A3) below, gives the existence of a positive solution for λ large, and if $\alpha = q = 0, C < 0$, the nonexistence of positive solutions for λ small. We also discuss the existence and nonexistence of positive solutions to the Laplacian system

$$\begin{cases} -\Delta_{p_i} u_i = \lambda a_i(x) f_i(u_j) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.6}$$

where $i, j \in \{1,2\}, i \neq j$ and $p_i > 1$ and f_i are bounded above.

For the single equation, we shall make the following assumptions:

(A1) $f : (0,\infty) \to \mathbb{R}$ is continuous and there exist constants $q \in [0, p-1)$ and $l > 0$ such that

$$\lim_{s\to\infty} \frac{f(s)}{s^q} = l.$$

(A2) There exists a constant $\alpha \in [0,1)$ such that

$$\limsup_{s\to 0^+} s^\alpha |f(s)| < \infty.$$

(A3) $a \in L^\infty(\Omega)$ and there exists a constant $\varepsilon > 0$ such that the problem

$$\begin{cases} -\Delta_p z = a^+(x)z^q - (1+\varepsilon)a^-(x)z^q \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega \end{cases}$$

has a positive solution $z \in C^1(\bar{\Omega})$ with $\frac{\partial z}{\partial n} < 0$ on $\partial\Omega$.

For systems, we assume

(B1) $f_i : (0,\infty) \to \mathbb{R}$ are continuous and there exist constants $l_i > 0, i = 1,2$, such that

$$\lim_{s\to\infty} f_i(s) = l_i.$$

(B2) There exists a constant $\alpha \in [0,1)$ such that

$$\limsup_{s\to 0^+} s^\alpha |f_i(s)| < \infty$$

for $i \in \{1,2\}$.

(B3) $a_i \in L^\infty(\Omega)$ and there exists a constant $\varepsilon > 0$ such that the problem

$$\begin{cases} -\Delta_p z_i = a_i^+(x)z_i - (1+\varepsilon)a_i^-(x)z_i \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega, \end{cases}$$

has a positive solution $z_i \in C^1(\bar{\Omega})$ with $\frac{\partial z_i}{\partial n} < 0$ on $\partial\Omega, i \in \{1,2\}$.

2.2 Main Results

Theorem 1 *Let (A1)-(A3) hold. Then there exists a constant* $\lambda_0 > 0$ *such that (2.1) has a positive solution* u_λ *for* $\lambda > \lambda_0$*. Furthermore,* $u_\lambda(x) \to \infty$ *as* $\lambda \to \infty$ *uniformly in compact subsets of* Ω*.*

Theorem 2 *Let* $q \in [0, p-1)$ *and* $\varepsilon > 0$*. Let* $a, a_0 \in L^\infty(\Omega)$ *with* $a_0 \geq 0$ *in* $\Omega, a_0 \not\equiv 0$*, and* $a^+ \geq a_0$ *in* Ω*. Then the problem*

$$\begin{cases} -\Delta_p z = \left(a^+(x) - (1+\varepsilon)a^-(x)\right) z^q \text{ in } \Omega, \\ z = 0 \text{ on } \partial\Omega \end{cases}$$

has a positive solution z *with* $\frac{\partial z}{\partial n} < 0$ *on* $\partial\Omega$*, provided that* $\int_{\{x:a(x)<0\}} a^-(x)dx$ *is sufficiently small (depending on* a_0*).*

Theorem 3 *Let (B1)-(B3) hold.Then there exists a constant* $\tilde{\lambda}_0 > 0$ *such that (2.6) has a positive solution* $u_\lambda = (u_{1\lambda}, u_{1\lambda})$ *for* $\lambda > \tilde{\lambda}_0$*. Furthermore,* $u_{i\lambda}(x) \to \infty$ *as* $\lambda \to \infty$ *uniformly in compact subsets of* $\Omega, i = 1, 2$*.*

Theorem 4 *(i) Let (A1)-(A2 hold. Suppose (A3) holds with* $q = 0$ *and* f *is continuous at* 0 *with* $f(0) < 0$*. Then there exists a constant* $\hat{\lambda} > 0$ *such that (2.1) has no positive solutions for* $\lambda < \hat{\lambda}$*.*

(ii) Let (B1)-(B2) hold. Suppose (B3) holds for some $i \in \{1, 2\}$*,* f_i *is continuous at* 0 *with* $f_i(0) < 0$*, and* $\lim_{s\to 0^+} f_j(s) < 0$ *(could be* $-\infty$*) for* $j \neq i$*. Then there exists a constant* $\tilde{\lambda} > 0$ *such that (2.6) has no positive solutions for* $\lambda < \tilde{\lambda}$*.*

2.3 Preliminary Results

We shall denote the norms in $L^p(\Omega)$, $C^1(\bar{\Omega})$, and $C^{1,\alpha}(\bar{\Omega})$ by $||.||_p$, $|.|_1$, and $|.|_{1,\alpha}$ respectively. Let $d(x)$ denote the distance from $x \in \Omega$ to $\partial\Omega$.

We first recall a regularity result and a comparison-like principle that will be needed for the proofs of the main results.

Lemma A. [9, Lemma 3.1]

Let $h \in L^{\infty}_{loc}(\Omega)$ *and suppose there exist numbers* $\alpha \in (0,1)$ *and* $C > 0$ *such that*

$$|h(x)| \leq \frac{C}{d^{\alpha}(x)} \tag{2.7}$$

for a.e. $x \in \Omega$. Let $u \in W_0^{1,p}(\Omega)$ be the weak solution of

$$\begin{cases} -\Delta_p u = h \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases}$$

Then there exist constants $\nu \in (0,1)$ *and* $M > 0$ *depending only on* C, α, Ω *such that* $u \in C^{1,\nu}(\bar{\Omega})$ *and* $|u|_{1,\nu} < M$.

Lemma B. [10, Lemma 2.3]

Let $h_i \in L^1(\Omega)$ *satisfy* (2.7) *for some constants* $C > 0$ *and* $\alpha \in (0,1)$, $i = 1, 2$. *Let* $w_i \in W_0^{1.p}(\Omega)$ *satisfy*

$$\begin{cases} -\Delta_p w_i = h_i \text{ in } \Omega, \\ w_i = 0 \text{ on } \partial\Omega, \end{cases}$$

$i = 1, 2$. *Then* $|w_1 - w_2|_1 \to 0$ *as* $||h_1 - h_2||_1 \to 0$.

Lemma C. [4, p 6]

$$\int_{\Omega} \frac{1}{d^{\gamma}(x)} dx < \infty \quad for \;\; \gamma < 1.$$

2.4 Proof of the Main Results

Proof of Theorem 1.

Since $\lim_{x\to 0^+} \dfrac{1+x}{(1-x)^{q+1}} = 1$, there exists $\varepsilon_0 \in (0,1)$ be such that $\dfrac{1+\varepsilon_0}{(1-\varepsilon_0)^{q+1}} < 1+\varepsilon$, where ε is given by (A3).

Let $\tilde{z} = cz$, where $c = (l(1-\varepsilon_0)^{q+1})^{\frac{1}{p-1-q}}$ and z is defined in (A3). Then

$$-\Delta_p \tilde{z} = -c^{p-1}\Delta_p z = c^{p-1}\left(a^+(x)z^q - (1+\varepsilon)a^-(x)z^q\right)$$

$$= c^{p-1-q}\left(a^+(x)(cz)z^q - (1+\varepsilon)a^-(x)(cz)^q\right)$$

$$= l(1-\varepsilon_0)^{q+1}\left(a^+(x)\tilde{z}^q - (1+\varepsilon)a^-(x)\tilde{z}^q\right) \equiv h \text{ in } \Omega.$$

By (A1) and (A2), there exist constants $A, D > 0$ such that

$$l(1-\varepsilon_0)s^q \le f(s) \le l(1+\varepsilon_0)s^q \tag{2.8}$$

for $s \ge A$, and

$$|f(s)| \le \frac{D}{s^\alpha} \tag{2.9}$$

for $s < \dfrac{A}{1-\varepsilon_0}$. For $\lambda > 0$, let $\tilde{z}_\lambda \in C^1(\bar{\Omega})$ be the solution of

$$\begin{aligned} -\Delta_p \tilde{z}_\lambda &= \begin{cases} (1-\varepsilon_0)^{q+1} l\left(a^+(x)\tilde{z}^q - (1+\varepsilon)a^-(x)\tilde{z}^q\right) & \text{if } \tilde{z} \ge \dfrac{A}{(1-\varepsilon_0)\lambda^r} \\[2ex] -\dfrac{D||a||_\infty}{(1-\varepsilon_0)^\alpha \tilde{z}^\alpha} & \text{if } \tilde{z} < \dfrac{A}{(1-\varepsilon_0)\lambda^r} \end{cases} \equiv h_\lambda \\ \tilde{z}_\lambda &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where $r = \frac{1}{p-1-q}$. Note that the existence of $\tilde{z}_\lambda$ follows from Lemma A.

Indeed, since $\inf_\Omega \frac{\tilde{z}}{d} \equiv k > 0$ and

$$|a^+(x)\tilde{z}^q - (1+\varepsilon)a^-(x)\tilde{z}^q| \le (1+\varepsilon)||a||_\infty ||\tilde{z}||_\infty^q$$

for a.e. $x \in \Omega$, if follows that

$$|h_\lambda| \le l(1+\varepsilon)||a||_\infty ||\tilde{z}||_\infty^q + \frac{D_1}{d^\alpha} \le \frac{D_2}{d^\alpha},$$

where $D_1 = D\left(c(1-\varepsilon_0)k\right)^{-\alpha} ||a||_\infty$ and $D_2 = D_1 + l(1+\varepsilon)||a||_\infty ||\tilde{z}||_\infty^q ||d||_\infty$.

By the weak comparison principle [42, Lemma A2], $\tilde{z}_\lambda \le \tilde{z}$ in Ω . Since

$$\left\{x : \tilde{z}(x) < \frac{A}{(1-\varepsilon_0)\lambda^r}\right\} \subseteq \left\{x : d(x) < \frac{A}{(1-\varepsilon_0)k\lambda^r}\right\}$$

it follows that

$$\left|\left\{x : \tilde{z}(x) < \frac{A}{(1-\varepsilon_0)\lambda^r}\right\}\right| \to 0 \text{ as } \lambda \to \infty.$$

Consequently,

$$||h_\lambda - h||_1 \le \int_{\tilde{z} < \frac{A}{(1-\varepsilon_0)\lambda^r}} \left(l(1+\varepsilon)||a||_\infty + \frac{D||a||_\infty}{(1-\varepsilon_0)k^\alpha d^\alpha(x)} \right) dx \to 0$$

as $\lambda \to \infty$. Hence Lemma B gives $|\tilde{z}_\lambda - \tilde{z}|_1 \to 0$ as $\lambda \to \infty$, which implies

$$\tilde{z} \ge \tilde{z}_\lambda \ge (1-\varepsilon_0)\tilde{z} \text{ in } \Omega \tag{2.10}$$

for λ large enough, which we shall assume. Next, let $Z_\lambda = \lambda^r \tilde{z}_\lambda$. We shall verify that Z_λ is a subsolution of (2.1).

Indeed, when $\tilde{z} \ge \dfrac{A}{(1-\varepsilon_0)\lambda^r}$ we have $Z_\lambda \ge A$, from which (2.8) and (2.10) give

$$f(Z_\lambda) \le l(1+\varepsilon_0)Z_\lambda^q \le \lambda^{rq} l(1+\varepsilon_0)\tilde{z}^q$$

and

$$f(Z_\lambda) \ge \lambda^{rq} l(1-\varepsilon_0)^{q+1}\tilde{z}^q.$$

Thus

$$\lambda a(x) f(Z_\lambda) = \lambda\left(a^+(x)f(Z_\lambda) - a^-(x)f(Z_\lambda)\right)$$

$$\geq \lambda^{1+rq} l \left[(1-\varepsilon_0)^{q+1} a^+(x)\tilde{z}^q - (1+\varepsilon_0) a^-(x)\tilde{z}^q\right]$$

$$= \lambda^{1+rq} l \,(1-\varepsilon_0)^{q+1} \left(a^+(x)\tilde{z}^q - \frac{1+\varepsilon_0}{(1-\varepsilon_0)^{q+1}} a^-(x)\tilde{z}^q \right)$$

$$\geq \lambda^{r(p-1)} l \,(1-\varepsilon_0)^{q+1} \left(a^+(x)\tilde{z}^q - (1+\varepsilon) a^-(x)\tilde{z}^q \right) = -\Delta_p(\lambda^r \tilde{z}_\lambda) \tag{2.11}$$

by the choice of r and ε_0. Next, suppose $\tilde{z} < \dfrac{A}{(1-\varepsilon_0)\lambda^r}$. Then $Z_\lambda < \dfrac{A}{1-\varepsilon_0}$, from which (2.9) and (2.10) give

$$\lambda a(x) f(Z_\lambda) \geq -\frac{\lambda D||a||_\infty}{Z_\lambda^\alpha} \geq -\frac{\lambda^{1-r\alpha} D||a||_\infty}{(1-\varepsilon_0)^\alpha \tilde{z}^\alpha}$$

$$\geq -\frac{\lambda^{r(p-1)} D||a||_\infty}{(1-\varepsilon_0)^\alpha \tilde{z}^\alpha} = -\Delta_p(\lambda^r \tilde{z}_\lambda) \tag{2.12}$$

since $1 - r\alpha < r(p-1)$ and $\lambda > 1$.

Combining (2.11) and (2.12), we see that Z_λ is a subsolution of (2.1). Next, we shall a construct a supersolution of (2.1). Let $z_0 \in C^1(\bar{\Omega})$ be the solution of

$$\begin{cases} -\Delta_p z_0 = \dfrac{1}{d^\alpha(x)} \text{ in } \Omega, \\ z_0 = 0 \text{ on } \partial\Omega, \end{cases} \tag{2.13}$$

and let $M > 0$. Let $k_0 = \inf_\Omega \frac{z_0}{d} > 0$. Since f is continuous and $\sup_{(0, M||z_0||_\infty]} s^\alpha |f(s)| \equiv D_0 < \infty$ in view of (A2), it follows that

$$\lambda a(x) f(Mz_0) \leq \frac{\lambda ||a||_\infty D_0}{(Mz_0)^\alpha} \leq \frac{\lambda ||a||_\infty D_0}{(k_0 M)^\alpha d^\alpha(x)} \leq \frac{M^{p-1}}{d^\alpha(x)} = -\Delta_p(Mz_0)$$

in Ω if M is large enough so that $M^{p-1-\alpha} \geq \lambda ||a||_\infty k_0^{-\alpha} D_0$ i.e. Mz_0 is a supersolution of (2.1).

By increasing M if necessary, we can assume that $Mz_0 \geq Z_\lambda$ in Ω. Hence (2.1) has a solution $u_\lambda \in C^{1,\nu}(\bar{\Omega})$ with $u_\lambda \geq Z_\lambda$ in Ω. In particular, $u_\lambda(x) \to \infty$ as $\lambda \to \infty$ uniformly in compact subsets of Ω, which completes the proof of Theorem 1. ■

Proof of Theorem 2.

In view of [35, Theorem 5.1], the problem

$$\begin{cases} -\Delta_p w = \frac{1}{2}\, a_0(x) w^q \equiv h \ \text{ in } \Omega, \\ \qquad\quad w = 0 \text{ on } \partial\Omega \end{cases} \tag{2.14}$$

has a unique positive solution $w \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Note that $c\phi_1$ with c small was used as a subsolution, where ϕ_1 is a positive eigenfunction associated with the first eigenvalue of $-\Delta_p u = \lambda a_0(x)|u|^{p-2}u$ in Ω with Dirichlet boundary conditions (see [36]).

By Lemma A, $w \in C^{1,\nu}(\Omega)$ for some $\nu \in (0,1)$. Since

$$-\Delta_p w \le ||a_0||_\infty ||w||_\infty^q \quad \text{in } \Omega,$$

the weak comparison principle [42, Lemma A2] gives

$$w \le (||a_0||_\infty ||w||_\infty^q\ )^{\frac{1}{p-1}}\, \psi \ \text{ in } \Omega,$$

where ψ is the unique positive solution of

$$\begin{cases} -\Delta_p \psi = 1 \text{ in } \Omega, \\ \quad \psi = 0 \text{ on } \partial\Omega. \end{cases}$$

Consequently,

$$||w||_\infty \le \left(||a_0||_\infty ||\psi||_\infty^{p-1}\right)^{\frac{1}{p-1-q}}.$$

Let $w_1 \in C^1(\bar{\Omega})$ be the solution of

$$-\Delta_p w_1 = \begin{cases} \frac{1}{2} a_0(x) w^q & \text{in } A, \\ & \\ -(1+\varepsilon) a^-(x) w^q & \text{in } B, \end{cases} \equiv h_1,\ w_1 = 0 \text{ on } \partial\Omega, \tag{2.15}$$

where $A = \{x : a(x) > 0\}$ and $B = \Omega \backslash A$. Then $w_1 \leq w$ in Ω. In view of (2.4) and the fact that $a_0(x) = 0$ in B, we obtain

$$||h - h_1||_1 = \int_B \left(\frac{1}{2}a_0(x)w^q + (1+\varepsilon)a^-(x)w^q\right)dx$$

$$\leq (1+\varepsilon)\left(||a_0||_\infty ||\psi||_\infty^{p-1}\right)^{\frac{q}{p-1-q}} \int_B a^-(x)dx$$

Let $k > 0$ be such that $w \geq kd$ in Ω (see [24, Theorem 5]) and $k_0 = \left(1 - 2^{-1/q}\right)k$. By applying Lemma B to (2.14) and (2.15), we see that if $\int_B a^-(x)dx$ is small enough (depending on a_0) then

$$|w_1 - w|_1 \leq \varepsilon_0,$$

which implies

$$w_1 \geq w - k_0 d \geq \left(1 - \frac{k_0}{d}\right) w = 2^{-1/q} w \ \text{ in } \Omega.$$

Hence

$$-\Delta_p w_1 = \frac{1}{2}\, a_0(x)w^q \leq a_0(x)w_1^q \ ,$$

$$\leq a^+(x)w_1^q \ = a^+(x)w_1^q \ - (1+\varepsilon)a^-(x) \text{ in } A,$$

and

$$-\Delta_p w_1 = -(1+\varepsilon)a^-(x)w^q \quad \leq -(1+\varepsilon)a^-(x)w_1^q.$$

$$= a^+(x)w_1^q \ - (1+\varepsilon)a^-(x)w_1^q \ \text{ in } B.$$

Thus w_1 is a subsolution of (2.5). Let $M > 0$ be large enough so that

$$M^{p-1-q} \geq (1+\varepsilon)||a||_\infty ||\psi||_\infty^q.$$

Then

$$-\Delta_p(M\psi) = M^{p-1} \geq (a^+(x) - (1+\varepsilon)a^-(x))(M\psi)^q \ \text{ in } \Omega$$

i.e. $M\psi$ is a supersolution of (2.5). By increasing M if necessary we can assume that $w_1 \le M\psi$ in Ω. Hence (2.5) has a solution $w \in C^{1,\nu}(\bar{\Omega})$ with $w \ge w_1$ in Ω (see e.g. [9, Lemma A]), which completes the proof. ■

Proof of Theorem 3.

As in the proof of Theorem 1, let $\varepsilon_0 \in (0,1)$ be such that $\frac{1+\varepsilon_0}{1-\varepsilon_0} < 1+\varepsilon$, where ε is given by (B3). Let $\tilde{z}_i = c_i z_i$, where $c_i = (l_i(1-\varepsilon_0))^{\frac{1}{p_i-1}}$, $i = 1,2$. Then

$$-\Delta_p \tilde{z}_i = (1-\varepsilon_0) l_i \left(a_i^+(x) - (1+\varepsilon)a_i^-(x)\right) \equiv h_i \text{ in } \Omega.$$

By (B1), there exists a constant $A_0 > 0$ such that

$$l_i(1-\varepsilon_0) \le f_i(s) \le l_i(1+\varepsilon_0) \tag{2.16}$$

for $s \ge A_0, i = 1,2$, which, together with (B2), implies the existence of a constant $C_1 > 0$ such that

$$|f_i(s)| \le C_1 \left(\frac{1}{s^\alpha} + 1\right) \tag{2.17}$$

for $s > 0, i = 1,2$. For $\lambda > 0$, let $\tilde{z}_{i\lambda}$ be the solutions of

$$\begin{aligned} -\Delta_{p_i} \tilde{z}_{i\lambda} &= \begin{cases} (1-\varepsilon_0) l_i \left(a_i^+(x) - (1+\varepsilon)a_i^-(x)\right) & \text{if } \tilde{z}_j \ge \dfrac{A_0}{(1-\varepsilon_0)\lambda^{r_i}} \\ \\ -C_1 \|a_i\|_\infty \left(\dfrac{1}{(1-\varepsilon_0)^\alpha \tilde{z}_j^\alpha} + 1\right) & \text{if } \tilde{z}_j < \dfrac{A_0}{(1-\varepsilon_0)\lambda^{r_i}} \end{cases} \equiv h_{i\lambda}, \\ \tilde{z}_{i\lambda} &= 0 \text{ on } \partial\Omega, \end{aligned}$$

$i, j \in \{1,2\}$, $i \ne j$, where $r_i = \frac{1}{p_i-1}$. Note that $\tilde{z}_{i\lambda} \le \tilde{z}_i$ in Ω.

Since $||h_{i\lambda} - h_i||_1 \to 0$ as $\lambda \to \infty$, we obtain from Lemma B and the comparison principle that

$$\tilde{z}_i \ge \tilde{z}_{i\lambda} \ge (1-\varepsilon_0)\tilde{z}_i \quad \text{in } \Omega \tag{2.18}$$

for λ large, which we assume. Fix such a λ. Let $\bar{Z}_\lambda = (\lambda^{r_1}\tilde{z}_{1\lambda}, \lambda^{r_2}\tilde{z}_{2\lambda})$ and $\bar{Z} = (Mz_0, Mz_0)$ where z_0 is defined in (2.13) and $M > 1$ is a large constant so that $\bar{Z}_\lambda \leq \bar{Z}$ in Ω.

Let $E = C(\bar{\Omega}) \times C(\bar{\Omega})$ be equipped with norm $||(v_1, v_2)||_\infty = \max_{i=1,2} ||v_i||_\infty$ and note that E is a Banach space. Define

$$K = \{(v_1, v_2) \in E : \lambda^{r_i}\tilde{z}_{i\lambda} \leq v_i \leq Mz_0 \text{ in } \Omega, i = 1, 2\}.$$

Then K is a closed convex subset of E. For $v = (v_1, v_2) \in K$, define $Tv = u$, where $u = (u_1, u_2) \in E$ is the solution of

$$\begin{cases} -\Delta_{p_i} u_i = \lambda a_i(x) f_i(v_j) \text{ in } \Omega, \\ \qquad u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

where $i, j \in \{1, 2\}, i \neq j$. Note that the existence of u follows from Lemma A since in view of (2.15) and (2.16), we have

$$\begin{aligned} \lambda|a_i(x)f_i(v_j)| &\leq \lambda||a_i||_\infty \left(\frac{1}{v_j^\alpha} + 1\right) \leq \lambda||a_i||_\infty \left(\frac{1}{\lambda^{r_j^\alpha}\tilde{z}_{i\lambda}^\alpha} + 1\right) \\ &\leq \lambda||a_i||_\infty \left(\frac{1}{(\lambda^r(1-\varepsilon_0))^\alpha \tilde{z}_i^\alpha} + 1\right) \leq \frac{C}{d^\alpha(x)} \end{aligned} \tag{2.19}$$

for a.e. $x \in \Omega$, where C is a positive constant depending only on $\lambda, a_i, \varepsilon_0, z_i, i = 1, 2$.

We claim that $T : K \to K$. By (2.16), for $\tilde{z}_j \geq \dfrac{A_0}{(1-\varepsilon_0)\lambda^{r_j}}$ we have

$$v_j \geq \lambda^{r_j}\tilde{z}_{j\lambda} \geq \lambda^{r_j}(1-\varepsilon_0)\tilde{z}_j \geq A_0,$$

and therefore (2.14) together with the choice of ε_0 give

$$\lambda a_i(x) f_i(v_j) = \lambda(a_i^+(x) f_i(v_j) - a_i^-(x) f_i(v_j))$$

$$\geq \lambda l_i(\big(1-\varepsilon_0)a_i^+(x) - (1+\varepsilon_0)a_i^-(x)\big)$$

$$\geq \lambda(1-\varepsilon_0)l_i\left(a_i^+(x) - (1+\varepsilon)a_i^-(x)\right) = -\Delta_{p_i}(\lambda^{r_i}\tilde{z}_{i\lambda}).$$

On the other hand,

$$\lambda a_i(x)f_i(v_j) \geq -\lambda C_1||a_i||_\infty\left(\frac{1}{v_j^\alpha}+1\right) \geq -C_1||a_i||_\infty\left(\frac{\lambda^{1-r_j\alpha}}{(1-\varepsilon_0)^\alpha\tilde{z}_j^\alpha}+\lambda\right)$$

$$\geq -\lambda C_1||a_i||_\infty\left(\frac{1}{(1-\varepsilon_0)^\alpha\tilde{z}_j^\alpha}+1\right) = -\Delta_{p_i}(\lambda^{r_i}\tilde{z}_{i\lambda})$$

in $\left\{x : \tilde{z}_j < \frac{A}{(1-\varepsilon_0)\lambda^{r_j}}\right\}$. Hence

$$-\Delta_{p_i}u_i \geq -\Delta_{p_i}(\lambda^{r_i}\tilde{z}_{i\lambda}) \text{ in } \Omega,$$

and therefore $u_i \geq \lambda^{r_i}\tilde{z}_{i\lambda}$ in $\Omega, i = 1, 2$.

Let $k_0 > 0$ be such that $\tilde{z}_i \geq k_0 d$ in $\Omega,\ i = 1, 2$. By (2.15),

$$-\Delta_{p_i}u_i \leq \lambda C_1||a_i||_\infty\left(\frac{1}{v_j^\alpha}+1\right) \leq \lambda C_1||a_i||_\infty\left(\frac{1}{(1-\varepsilon_0)^\alpha\tilde{z}_j^\alpha}+1\right)$$

$$\leq \frac{\lambda D_i}{d^\alpha} \text{ in } \Omega,$$

where D_i is a constant depending on $C_1, ||a_i||_\infty, k_0, i = 1, 2$.

This implies $u_i \leq Mz_0$ in $\Omega, i = 1, 2$, where $M = \max\limits_{i=1,2}(\lambda D_i)^{\frac{1}{p_i-1}}$. Hence $T : K \to K$.

Next, we verify that T is continuous. Indeed, let $(v_{1n}, v_{2n}) \in K$ be such that $(v_{1n}, v_{2n}) \to (v_1, v_2) \in K$. Let $(u_{1n}, u_{2n}) = T(v_{1n}, v_{2n})$ and $(u_1, u_2) = T(v_1, v_2)$. Since

$$\lambda a_i(x)f_i(v_{jn}) \to \lambda a_i(x)f_i(v_j) \text{ a.e. in } \Omega,$$

and

$$\lambda|a_i(x)f_i(v_{jn})| \leq \frac{C}{d^\alpha(x)}$$

for $i \neq j$, it follows from the Lebesgue Dominated Convergence Theorem that

$$\lambda a_i(x) f_i(v_{jn}) \to \lambda a_i(x) f_i(v_j) \text{ in } L^1(\Omega).$$

Hence it follows from Lemma B that $u_{in} \to u_i$ in $C^1(\bar{\Omega})$ i.e. $(u_{1n}, u_{2n}) \to (u_1, u_2)$ in $C^1(\bar{\Omega})$. Hence T is continuous and since $T(K)$ is bounded in $C^1(\bar{\Omega})$, it follows that T is a compact operator. By the Schauder Fixed Point Theorem, T has a fixed point $u_\lambda = (u_{1\lambda}, u_{2\lambda}) \in K$. In particular, $u_{i\lambda}(x) \to \infty$ as $\lambda \to \infty$ uniformly in compact subsets of Ω, which completes the proof. ■

Proof of Theorem 4.

(i) Let u_λ be a positive solution of (2.1). Since $f(0) < 0$ and $\lim_{s\to\infty} f(s) = l \in (0, \infty)$, there exists a constant $c > 0$ such that $f(s) \leq c$ for $s > 0$.

Hence

$$-\Delta_p u_\lambda = \lambda a(x) f(u_\lambda) \leq \lambda c ||a||_\infty \text{ in } \Omega,$$

from which it follows that $u_\lambda \leq (\lambda c||a||_\infty)^{\frac{1}{p-1}} \psi$ in Ω, where ψ is defined in the proof of Theorem 2. Consequently, $||u_\lambda||_\infty \to 0$ as $\lambda \to 0^+$. Hence, if λ is sufficiently small, we have

$$(1 + \varepsilon_0) f(0) \leq f(u_\lambda) \leq (1 - \varepsilon_0) f(0) \text{ in } \Omega,$$

where $\varepsilon_0 = \frac{\varepsilon}{2+\varepsilon}$ and ε is given by (A3) with $q = 0$. This implies

$$-\Delta_p u_\lambda = \lambda (a^+(x) f(u_\lambda) - a^-(x) f(u_\lambda))$$

$$\leq \lambda f(0)((1 - \varepsilon_0) a^+(x) - (1 + \varepsilon_0) a^-(x))$$

$$= \lambda (1 - \varepsilon_0) f(0) \left(a^+(x) - (1 + \varepsilon) a^-(x)\right) \text{ in } \Omega,$$

from which the weak comparison principle gives

$$u_\lambda \leq -\left(\lambda(1-\varepsilon_0)|f(0)|\right)^{\frac{1}{p-1}} z < 0 \ \text{ in } \Omega,$$

a contradiction. Thus (2.1) has no positive solutions for $\lambda > 0$ small.

(ii) Let $u_\lambda = (u_{1\lambda}, u_{2\lambda})$ be a positive solution of (2.6). Since f_i is continuous at 0 with $f_i(0) < 0$ and $\lim_{x\to 0^+} f_j(x) < 0$, it follows from (B1) that there exists a constant $c_0 > 0$ such that $f_k(s) \leq c_0$ for $s > 0$, $k = 1, 2$. Since

$$-\Delta_{p_j} u_{j\lambda} = \lambda a_j(x) f_j(u_{i\lambda}) \geq \lambda c_0 ||a_j||_\infty \ \text{ in } \Omega,$$

it follows that $u_{j\lambda} \leq (\lambda c_0 ||a_j||_\infty)^{\frac{1}{p-1}} \psi_j$ in Ω, where ψ_j is the solution of

$$-\Delta_{p_j} \psi_j = 1 \text{ in } \Omega, \ \psi_j = 0 \text{ on } \partial\Omega.$$

In particular, $||u_{j\lambda}||_\infty \to 0$ as $\lambda \to 0^+$. Hence, as in the proof of part (i) above, for λ sufficiently small we get

$$(1+\varepsilon_0) f_i(0) \leq f_i(u_{j\lambda}) \leq (1-\varepsilon_0) f_i(0) \ \text{ in } \Omega,$$

where $\varepsilon_0 = \frac{\varepsilon}{2+\varepsilon}$ and ε is given by (B3). Consequently,

$$-\Delta_{p_i} u_{i\lambda} = \lambda \left(a_i^+(x) f_i(u_{j\lambda}) - a_i^-(x) f_i(u_{j\lambda})\right)$$

$$\leq \lambda f_i(0) \left((1-\varepsilon_0) a_i^+(x) - (1+\varepsilon_0) a_i^-(x)\right)$$

$$= \lambda(1-\varepsilon_0) f_i(0) \left(a_i^+(x) - (1+\varepsilon) a_i^-(x)\right) \ \text{ in } \Omega,$$

from which it follows that

$$u_{i\lambda} \leq -\left(\lambda(1-\varepsilon_0)|f_i(0)|\right)^{\frac{1}{p-1}} z_i \ < 0 \ \text{ in } \Omega.$$

Hence (2.6) has no positive solution for λ small, which completes the proof. ■

CHAPTER 3

P-LAPLACIAN SUPERLINEAR PROBLEM WITH NONLINEAR BOUNDARY CONDITIONS

3.1 Introduction

We prove the existence of positive radial solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda K(|x|)f(u) \text{ in } |x| > r_0, \\ \frac{\partial u}{\partial n} + \tilde{c}(u)u = 0 \text{ on } |x| = r_0, \;\; u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u),\ p > 1,\ \Omega = \{x \in \mathbb{R}^N : |x| > r_0 > 0\}, N > 2, f : (0,\infty) \to \mathbb{R}$ is superlinear at ∞ with possible singularity at 0, and λ is a small positive parameter. A nonexistence result is also established when f has semipositone structure at 0.

In this paper, we study the existence and nonexistence of positive radial solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda K(|x|)f(u) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} + \tilde{c}(u)u = 0 \text{ on } |x| = r_0, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases} \tag{3.1}$$

where $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2}\nabla u),\ N > p > 1,\ \Omega = \{x \in \mathbb{R}^N : |x| > r_0 > 0\}$, $K : [r_0,\infty) \to (0,\infty),\ \tilde{c} : [0,\infty) \to (0,\infty), f : (0,\infty) \to \mathbb{R}$ are continuous with

$r^{N+\mu}K(r)$ bounded for some $\mu > 0$, and λ is a positive parameter. Here n denotes the outer unit normal vector on $\partial\Omega$.

Let $r = |x|$ and $t = (r/r_0)^{\frac{p-N}{p-1}}$, then problem (3.1) becomes the ODE problem

$$\begin{cases} -(\phi(u'))' = \lambda h(t) f(u),\ t \in (0,1), \\ u(0) = 0,\ u'(1) + c(u(1))u(1) = 0, \end{cases} \tag{3.2}$$

where $\phi(z) = |z|^{p-2}z,\ h(t) = \left(\frac{p-1}{N-p} r_0\right)^p t^{\frac{p(N-1)}{p-N}} K(r_0 t^{\frac{p-1}{p-N}})$, and $c(s) = \frac{p-1}{N-p} r_0 \tilde{c}(s)$ (see Appendix A).

Note that $h : (0,1] \to (0,\infty)$ is continuous and could be singular at 0 if $\mu \in (0,(N-p)/(p-1))$.

We shall make the following assumptions:

(A1) $h : (0,1] \to (0,\infty)$ is of class C^1.

(A2) $c : [0,\infty) \to (0,\infty)$ is continuous.

(A3) There exist constants $\sigma, \gamma \geq 0$ with $\sigma + \gamma < 1$ such that

$$\limsup_{t\to 0^+} t^\sigma h(t) < \infty \text{ and } \limsup_{t\to 0^+} t^\gamma |f(t)| < \infty.$$

(A4) $f : (0,\infty) \to \mathbb{R}$ is continuous and $\lim\limits_{s\to\infty} \frac{f(s)}{s^{p-1}} = \infty$.

(A5) There exists a constant $\beta > 0$ such that $f(s)(s-\beta) > 0$ for $s > 0, s \neq \beta$.

(A6) There exists a constant $\nu \in [0,1)$ such that $\limsup\limits_{s\to 0^+} s^\nu f(s) < 0$.

Remark 1 *Note that (A6) and the assumption* $\limsup_{t\to 0^+} t^\gamma |f(t)| < \infty$ *in (A3) are different: the function* $f(t) = \frac{1}{\sqrt{t}}$ *satisfies (A3) but not (A6), while* $f(t) = -\frac{1}{t^2}$ *satisfies (A6) but not (A3).*

By a positive solution of (3.2), we mean a function $u \in C^1[0,1]$ with $\phi(u')$ absolutely continuous on $[0,1],\ u > 0$ on $(0,1)$ and satisfying (3.2).

3.2 Main Results

Theorem 5 *Let (A1)-(A4) hold. Then there exists a constant $\lambda_0 > 0$ such that (3.2) has a positive solution u_λ for $\lambda < \lambda_0$ with $\lim\limits_{\lambda \to 0^+} u_\lambda(t) = \infty$ for each $t \in (0,1]$ and the limit is uniform on compact subsets of $(0,1)$.*

Theorem 6 . *Let (A1)-(A6) hold and suppose h is decreasing. Then there exists a constant $\tilde{\lambda}_0 > 0$ such that (1.2) has no positive solution for $\lambda > \tilde{\lambda}_0$.*

Our results here allow the nonlinearity f to be singular at 0 with possible semipositone structure i.e. $\lim_{s \to 0^+} f(s) \in [-\infty, 0)$, and extend the results in [15], where $p = 2$, to the p-Laplace case with $p > 1$. Note that the proofs in [15] depend on the Green's function associated for the Laplace operator with Sturm-Liouiville type boundary conditions, which does not exist for $p \neq 2$. In particular, our existence result (Theorem 5) improves and extends (to the infinite semipositone case) a corresponding result in [38, Theorem 1.2], in which the existence of a positive solution to (3.2) was established for $\lambda > 0$ small under the following assumptions:

(B1) $f : [0,\infty) \to \mathbb{R}$ is continuous, nondecreasing with $f(0) < 0$.

(B2) There exist positive constants A, B and $q > p - 1$ such that

$$A(s^q - 1) \le f(s) \le B(s^q + 1)$$

for all $s \ge 0$

(B3) There exists a constant $\theta > p$ such that

$$sf(s) > \theta F(s)$$

for s large, where $F(s) = \int_0^s f$

(B4) $c\ :[0,\infty) \to (0,\infty)$ is continuous and

$$c(s) < \theta g(s)/s^p$$

for s large, where $f(s) = \int_0^s c(t)\phi(t)dt$ and θ is defined in (B3).

The results in [38], which provide an extension of the ones in [40] to the p-Laplace case, were established via the Mountain Pass Lemma in variational methods. Our approach here makes use of degree theory and comparison principles, which help avoiding the technical assumptions associated with variational methods in [38, 40]. In particular, our results when applied to the model case

$$\begin{cases} -(\phi(u'))' = \lambda t^{-\sigma}\left(au^{-\gamma} + u^q - 1\right) \\ \quad u(0) = 0,\ u'(1) + (u(1))^r = 0, \end{cases} \tag{3.3}$$

where $r > 1,\ a \in \mathbb{R},\ \sigma, \gamma$ are nonnegative constants with $\sigma + \gamma < 1$, $q > p - 1$, give the existence of a large positive solution for (3.3) when λ is sufficiently small and, if $\sigma > 1$ and $a < 0,$ the nonexistence of positive solutions to (3.3) when λ is sufficiently large (see Appendix B). We refer to [1, 6, 12, 14, 16, 22, 28, 32, 44, 49, 51] and the references therein for the literature on Laplacian and p-Laplacian superlinear problems with semipositone structure under Dirichlet boundary conditions.

3.3 Preliminary Results

We shall denote the norm in $L^p(0,1)$ and $C^1[0,1]$ by $||.||_p$ and $|.|_{C^1}$ respectively. Here $|u|_{C^1} = \max(||u||_\infty, ||u'||_\infty)$.

Throughout the paper, we shall use the following definitions:

$$p(t) = \min(t, 1-t),\ k_0(t) = \frac{h(t)}{p^\gamma(t)},\ m = 2^{-\left(\frac{2-p}{p-1}\right)^+},$$

$\tilde{v}(t) = \max(v(t), p(t))$ for $v \in C[0,1]$,

$\hat{g}(z) = \sup_{0\le s\le z} g(s)$ and $\check{g}(z) = \inf_{s\ge z} g(s)$ whenever they are finite.

We first state a comparison principle, whose proof is the same as in [6, Lemma 3.2].

Lemma A. *For $i = 1, 2$, let $h_i \in L^1(r, R)$ with $h_1 \le h_2$, and let u_i satisfy*

$$\begin{cases} (\phi(u_i'))' = h_i,\ r < t < R, \\ au_1(r) - bu_1'(r) \ge au_2(r) - bu_2'(r), \\ cu_1(R) + du_1'(R) \ge cu_2(R) + du_2'(R), \end{cases}$$

where a, b, c, d are nonnegative constants with $ac + ad + bc > 0$. Then

$$u_1 \ge u_2 \ \textit{on}\ [r, R].$$

In what follows, we shall use the following inequality

$$(x+y)^q \le 2^{(q-1)^+}(x^q + y^q) \text{ for } x, y \ge 0,\ q > 0. \tag{*}$$

where $z^+ = \max(z, 0)$.

Lemma 1 *Let $\lambda \ge 0$ and $k \in L^1(0,1)$ with $k \ge 0$. Let $v \in C^1[0,1]$ satisfy*

$$\begin{cases} (\phi(v'))' = \lambda k(t) \ \textit{in}\ (0,1), \\ v(0) = 0,\ v'(1) + \alpha v(1) = 0 \end{cases} \tag{3.4}$$

for some constant $\alpha > 0$. *Then* $||v||_\infty \leq \phi^{-1}(\lambda||k||_1)$ *and* $|v|_{C^1} \leq \phi^{-1}(2\lambda||k||_1)$.

Proof: Multiplying the equation in (3.4) by v and integrating on $[0, 1]$, we get

$$\int_0^1 |v'|^p dt + \phi(\alpha v(1))v(1) = -\lambda \int_0^1 kv dt. \tag{3.5}$$

Since $\phi(\alpha v(1))v(1) \geq 0$ and

$$|v(t)| = \left| \int_0^t v' \right| \leq \int_0^t |v'| \leq ||v'||_p$$

for $t \in [0, 1]$, it follows from (3.5) that

$$||v'||_p^p \leq \lambda||k||_1||v||_\infty \leq \lambda||k||_1||v'||_p.$$

Consequently, |

$$|v||_\infty \leq ||v'||_p \leq (\lambda||k||_1)^{1/(p-1)}$$

By integrating, we obtain

$$\phi(v') = C + \lambda \int_0^t k,$$

where $C = \phi(v'(0)$, which implies

$$v(t) = \int_0^t \phi^{-1} \left(C + \lambda \int_0^s k \right) ds.$$

Since $v'(1) + \alpha v(1) = 0$, it follows that

$$\phi^{-1} \left(C + \lambda \int_0^1 k \right) + \alpha \int_0^1 \phi^{-1} \left(C + \lambda \int_0^s k \right) ds = 0. \tag{3.6}$$

Since the left side of (3.6) is positive if $C > \lambda||k||_1$ and negative if $C < -\lambda||k||_1$, we deduce that $|C| \leq \lambda||k||_1$. Hence

$$|v'(t)| \leq \phi^{-1} \left(|C| + \lambda \int_0^t k \right) \leq \phi^{-1}(2\lambda||k||_1)$$

for $t \in [0,1]$, which completes the proof. ■

Lemma 2 *Let $\lambda \geq 0$ and $k \in L^1(0,1)$ with $k \geq 0$. Let $u \in C^1[0,1]$ satisfy*

$$\begin{cases} (\phi(u'))' \leq \lambda k(t) \ \ \textit{in}\ (0,1), \\ u(0) \geq 0,\ u'(1) + \alpha u(1) \geq 0 \end{cases} \tag{3.7}$$

for some constant $\alpha > 0$. Suppose $||u||_\infty > 2^{\left(\frac{2-p}{p-1}\right)^+} \phi^{-1}(\lambda ||k||_1)$ and $||u||_\infty = |u(\tau)|$. Then $u(\tau) > 0$ and

$$u(t) \geq (m||u||_\infty - \phi^{-1}(\lambda ||k||_1))p(t), \tag{3.8}$$

for $t \in [0,1]$.

Proof:. By the comparison principle (see e.g. [13, Lemma 3.2], $u \geq v$ on $[0,1]$, where v satisfies (3.4). If $u(\tau) \leq 0$ then

$$||u||_\infty = -u(\tau) \leq -v(\tau) \leq \phi^{-1}(\lambda ||k||_1)$$

in view of Lemma 1, a contradiction. Thus $u(\tau) > 0$. Let $z \in C^1[0,\tau]$ satisfy

$$\begin{cases} (\phi(z'))' = \lambda k(t) \ \ \text{in}\ (0,\tau), \\ z(0) = 0,\ z(\tau) = ||u||_\infty. \end{cases}$$

Suppose $\tau \in (0,1)$.Using Lemma A with $r = 0, R = \tau, a = c = 1, b = d = 0$, we see that

$$u \geq z \ \ \text{on}\ [0,\tau].$$

Note that z is given by

$$z(t) = \int_0^t \phi^{-1}\left(C + \lambda \int_0^s k\right) ds,$$

where $C = \phi(z'(0))$ is the unique number such that $z(\tau) = ||u||_\infty$.

If $z'(0) < 0$ then

$$\begin{aligned} z'(t) &= \phi^{-1}\left(\phi(z'(0)) + \lambda\int_0^t k\right) (2.6) \\ &\leq \phi^{-1}\left(\lambda\int_0^t k\right), \end{aligned} \tag{3.9}$$

for $t \in [0, \tau]$. Hence upon integrating on $(0, \tau)$, we get

$$||u||_\infty = z(\tau) \leq \phi^{-1}(\lambda||k||_1)$$

a contradiction. Thus $z'(0) \geq 0$, from which (3.9) and (*) with $q = (p-1)^{-1}$ give

$$z'(t) \leq 2^{\left(\frac{2-p}{p-1}\right)^+}\left(z'(0) + \phi^{-1}(\lambda||k||_1)\right)$$

for $t \in (0, \tau)$. Integrating this inequality on $(0, \tau)$ gives

$$||u||_\infty \leq 2^{\left(\frac{2-p}{p-1}\right)^+}(z'(0) + \phi^{-1}(\lambda||k||_1)),$$

which implies

$$z'(0) \geq m||u||_\infty - \phi^{-1}(\lambda||k||_1).$$

Since $z'(t) \geq z'(0)$ for $t \in (0, \tau)$ in view of (3.9), it follows that

$$u(t) \geq z(t) = \int_0^t z' \geq z'(0)t \geq (m||u||_\infty - \phi^{-1}(\lambda||k||_1))t \tag{3.10}$$

for $t \in [0, \tau]$. Next, let $w \in C^1[0, 1]$ be the solution of

$$\begin{cases} (\phi(w'))' = \lambda k(t) \text{ in } (\tau, 1), \\ w(\tau) = ||u||_\infty,\ w'(1) + \alpha w(1) = 0, \end{cases}$$

which is given by

$$w(t) = D + \int_t^1 \phi^{-1}\left(\phi(\alpha D) + \lambda\int_s^1 k\right) ds, \tag{3.11}$$

where $D = w(1)$ is the unique number such that $w(\tau) = ||u||_\infty$. Note that the existence and uniqueness of D follows from the fact that the function

$$H(D) = D + \int_t^1 \phi^{-1}\left(\phi(\alpha D) + \lambda \int_s^1 k\right) ds$$

is increasing and continuous in $D \in \mathbb{R}$ with $\lim_{D\to-\infty} H(D) = -\infty$ and $\lim_{D\to\infty} H(D) = \infty$.

Using Lemma A with $r = \tau, R = 1, a = 1, b = 0, c = \alpha, d = 1$, we see that

$$u \geq w \quad \text{on } [\tau, 1].$$

If $D \leq 0$ then (3.11) gives $w(\tau) \leq \phi^{-1}(\lambda||k||_1)$, a contradiction. Hence $w(1) > 0$ from which (3.11) and (*) imply

$$w(\tau) \leq w(1) + 2^{\left(\frac{2-p}{p-1}\right)^+}\left(\alpha w(1) + \phi^{-1}(\lambda||k||_1)\right)(1-\tau)$$

$$\leq w(1)\left(1 + m^{-1}\alpha(1-\tau)\right) + m^{-1}\phi^{-1}(\lambda||k||_1).$$

Consequently,

$$w(1) \geq \frac{||u||_\infty - m^{-1}\phi^{-1}(\lambda||k||_1)}{1 + m^{-1}\alpha(1-\tau)}. \tag{3.12}$$

Since $k \geq 0$ in $(0, 1)$, it follows from (3.11) and (3.12) that

$$u(t) \geq w(t) \geq w(1) + \alpha w(1)(1-t) = (1 + \alpha(1-t))w(1)$$

$$\geq \frac{1 + \alpha(1-t)}{1 + m^{-1}\alpha(1-\tau)}\left(||u||_\infty - m^{-1}\phi^{-1}(\lambda||k||_1)\right)$$

$$\geq (m||u||_\infty - \phi^{-1}(\lambda||k||_1))(1-t), \tag{3.13}$$

for $t \in [\tau, 1]$, where we have used the inequality $\frac{1+\alpha(1-t)}{1+m^{-1}\alpha(1-\tau)} \geq m(1-t)$ for $t \in [0, 1]$. Combining (3.10) and (3.13), we obtain (3.8).

If $\tau = 1$ (resp. $\tau = 0$) then it follows from (3.10) (resp.(3.13) and the assumption $m||u||_\infty \geq \phi^{-1}(\lambda||k||_1)$ that (3.8) holds, which completes the proof. ■

Remark 2 *Since* $\frac{1+\alpha(1-t)}{1+m^{-1}\alpha(1-\tau)} \geq \frac{1}{1+m^{-1}\alpha}$, *we also get the estimate*

$$u(t) \geq \frac{||u||_\infty - m^{-1}\phi^{-1}(\lambda||k||_1)}{1+m^{-1}\alpha}$$

for $t \in [\tau, 1]$ *in addition to (3.13).*

Let $\lambda > 0$. For $v \in C[0,1]$, define $T_\lambda v = u$ to be the (unique) solution of

$$\begin{cases} -(\phi(u'))' = \lambda h(t) f(\tilde{v}),\ t \in (0,1), \\ u(0) = 0,\ u'(1) + c(|v(1)|)u(1) = 0. \end{cases}$$

Note that u is given by

$$u(t) = \int_0^t \phi^{-1}\left(C + \lambda \int_s^1 hf(\tilde{v})\right) ds,$$

where C is the unique number such that $u'(1) + c(|v(1)|)u(1) = 0$.

Note that $C = \phi(u'(1)) = -\phi\{c(|v(1)|)u(1)\}$.

Lemma 3 *Let (A1)-(A3) hold. Then* $T_\lambda : C[0,1] \to C[0,1]$ *is completely continuous.*

Proof: We first show that T_λ maps bounded sets in $C[0,1]$ into bounded sets in $C^1[0,1]$ and hence relatively compact subsets in $C[0,1]$. To this end, let $M > 1$ be such that $||v||_\infty \leq M$. Then $||\tilde{v}||_\infty \leq M$. Since $\limsup_{t\to 0^+} t^\gamma |f(t)| < \infty$, there exists a constant $\tilde{M} > 0$ depending on M such that

$$|h(t)f(\tilde{v})| \leq \frac{\tilde{M}h(t)}{\tilde{v}^\gamma(t)} \leq \frac{\tilde{M}h(t)}{p^\gamma(t)} = \tilde{M}k_0(t)$$

for $t \in (0,1)$. Since $k_0 \in L^1(0,1)$ in view of (A1) and (A3), it follows from Lemma 1 that

$$|u|_{C^1} \leq \phi^{-1}\left(2\lambda ||hf(\tilde{v})||_1\right) \leq \phi^{-1}(2\lambda \tilde{M} ||k_0||_1)$$

Next, we verify that T_λ is continuous. Let $(v_n) \subset C[0,1]$ be such that $v_n \to v$ in $C[0,1]$ and let $u_n = T_\lambda v_n, u = T_\lambda v$.

Let (u_{n_k}) be a subsequence of (u_n). Since (u_{n_k}) is bounded in $C^1[0,1]$, there exists $\tilde{u} \in C[0,1]$ and a subsequence $(u_{n_{k_j}})$ of (u_{n_k}) such that $u_{n_{k_j}} \to \tilde{u}$ in $C[0,1]$.

Letting $j \to \infty$ in

$$u_{n_{k_j}}(t) = \int_0^t \phi^{-1}\left(-\phi\left\{c(|v_{n_{k_j}}(1)|)u_{n_{k_j}}(1)\right\} + \lambda \int_s^1 hf(\tilde{v}_{n_{k_j}})\right) ds$$

and using the Lebesgue dominated convergence theorem, we obtain

$$\tilde{u}(t) = \int_0^t \phi^{-1}\left(-\phi(c\{|v(1)|)\tilde{u}(1)\} + \lambda \int_s^1 hf(\tilde{v})\right) ds$$

for $t \in [0,1]$ and so $\tilde{u} \equiv u$. Thus $u_n \to u$ in $C[0,1]$ i.e. T_λ is continuous. Hence T_λ is completely continuous by the Ascoli-Arzela's Theorem.

Let $F(z) = \int_0^z f(s)ds$. By (A2) and (A3), there exists a number $\theta > \beta$ such that $F(z) < 0$ for $0 < z < \theta$ and $F(z) > 0$ for $z > \theta$. ■

Lemma 4 *Let (A1)-(A6) hold and suppose h is decreasing. Let u be a positive solution of (3.2). Then there exists $t_0 \in (0,1)$ such that u is increasing on $[0,t_0)$ and decreasing on $(t_0,1]$ with $u(t_0) \geq \theta$.*

Proof: Using the equation in (3.2) and (A5), we see that

$$(\phi(u'))' = -\lambda h(t) f(u) > 0 \tag{3.14}$$

for t near 0. Hence $\phi(u'(t)) > \phi(u'(0)) \geq 0$ i.e. $u'(t) > 0$ for t near 0. If $u' > 0$ on $(0,1)$ then $u(1) \geq u(0) = 0$ and $u'(1) \geq 0$. Hence $u(1) = u'(1) = 0$ in view of the boundary condition for u at 1. Hence it follows from (3.14) that u' is increasing for t near 1 and $u'(t) < u'(1) = 0$ for t near 1, a contradiction. Thus u' has a zero in $(0,1)$. Let $t_0 \in (0,1)$ be the first zero of u'.

Using (A3) and $u(t) \leq ||u'||_\infty t$ for $t \in (0,1)$, it follows that there exists a constant $C > 0$ such that

$$h(t)|F(u(t))| \leq h(t)\int_0^{u(t)} |f(z)|dz \leq \frac{C}{t^\sigma}\int_0^{u(t)} z^{-\gamma}dz = \frac{Cu^{1-\gamma}(t)}{(1-\gamma)t^\sigma} \leq C_1 t^{1-\gamma-\sigma}$$

for $t \in (0,1)$, where $C_1 = C(1-\gamma)^{-1}||u'||_\infty$. Consequently, $\lim_{t\to 0^+} h(t)F(u(t) = 0$.

Suppose $u(t_0) < \theta$. Then $h'F(u) \geq 0$ on $[0,t_0]$. Multiplying the equation in (3.2) by u' and integrating on $[0,t_0]$, we get

$$0 \leq \left(1 - \frac{1}{p}\right)|u'(0)|^p = \lambda h(t_0)F(u(t_0)) - \lambda\int_0^{t_0} h'(t)F(u)dt$$

$$\leq \lambda h(t_0)F(u(t_0)) < 0,$$

a contradiction. Thus $u(t_0) \geq \theta$. Next, we verify that u is decreasing on $(t_0,1)$. Suppose not. Then there exist $t_1, t_2 \in (t_0,1)$ with $t_1 < t_2$ such that $u(t_1) \leq u(t_2)$. Since $(\phi(u'))'(t_0) < 0$, it follows that u is decreasing for t near $t_0, t > t_0$. Hence u achieves its minimum value on $[t_0,t_2]$ at some $t^* \in (t_0,t_2)$. This implies

$$\lambda h(t^*)f(u(t^*)) = -(\phi(u'))'(t^*) \leq 0$$

i.e. $0 < u(t^*) \leq \beta$ and so $F(u(t^*)) < 0$. Since $F(u(t_0)) \geq 0$, there exists $\hat{t} \in [t_0,t^*)$ be such that $F(u) < 0$ on $(\hat{t},t^*]$ and $F(u(\hat{t})) = 0$.

Multiplying the equation in (3.2) by u' and integrating on $[\hat{t}, t^*]$ gives

$$0 \leq \left(1 - \frac{1}{p}\right) |u'(\hat{t})|^p = \lambda h(t^*) F(u(t^*)) - \lambda \int_{\hat{t}}^{t^*} h'(t) F(u) dt < 0,$$

a contradiction. Hence u is increasing on $(0, t_0)$ and decreasing on $(t_0, 1)$, which completes the proof.

By Lemma 1, for each $\mu \in (0, \theta]$ there exists a unique $t_\mu \leq t_0$ such that $u(t_\mu) = \mu$. ■

Lemma 5 *Let* $\rho = \frac{\beta+\theta}{2}$*.There exists a constant* $c > 0$ *such that any positive solution* u *of (1.2) satisfies*

$$u(t) \geq c\lambda^{\frac{1}{p-1}} \left(\int_0^t h^{1/p}\right)^{\frac{p}{p-1}} \tag{3.15}$$

for $t \in [0, t_\rho]$*. In particular,* $t_\rho \to 0$ *as* $\lambda \to \infty$*.*

Proof: Note that $h'F(u) \geq 0$ on $(0, t_\theta]$. Hence it follows from multiplying the equation in (3.2) by u' and integrating on $[0, s]$, $s \in (0, t_\theta]$, that

$$\left(1 - \frac{1}{p}\right) (u'(s))^p = \left(1 - \frac{1}{p}\right) (u'(0))^p - \lambda h(s) F(u) + \lambda \int_0^s h'(\tau) F(u) d\tau$$

$$\geq -\lambda h(s) F(u).$$

Hence

$$\frac{u'(s)}{(-F(u))^{1/p}} \geq (\lambda h(s))^{1/p},$$

which implies

$$\int_0^{u(t)} \frac{dz}{(-F(z))^{1/p}} = \int_0^t \frac{u'(s)}{(-F(u))^{1/p}} ds \geq \lambda^{1/p} \int_0^t h^{1/p}(s) ds \tag{3.16}$$

for $t \in (0, t_\theta]$. By (A6), there exist constants $\delta \in (0, \beta)$ and $\eta > 0$ such that

$$f(z) \leq -\frac{\eta}{z^\nu} \leq -\eta_0$$

for $z \in (0,\delta)$, where $\eta_0 = \eta\delta^{-\nu}$. Hence $-F(z) \geq \eta_0 z$ for $z \in [0,\delta)$ and since $\tilde{\eta}_0 \equiv \min_{[\delta,\rho]}(-F) > 0$, it follows that

$$-F(z) \geq \eta_1 z \quad \text{for } z \in [0,\rho], \tag{3.17}$$

where $\eta_1 = \min(\eta_0, \tilde{\eta}_0)$. Using (3.17) in (3.16), we obtain

$$u^{1-\frac{1}{p}}(t) \geq (1-1/p)(\eta_1\lambda)^{1/p} \int_0^t h^{1/p}(s)ds$$

for $t \in (0, t_\rho]$ i.e. (3.15) holds with $c = \left[(1-1/p)\eta_1^{1/p}\right]^{\frac{p}{p-1}}$. In particular, when $t = t_\rho$,

$$\int_0^{t_\rho} h^{1/p}(s)ds \leq (\beta/c)^{\frac{p-1}{p}}\lambda^{-1/p} \to 0 \text{ as } \lambda \to \infty, \tag{3.18}$$

which implies $t_\rho \to 0$ as $\lambda \to \infty$.

■

3.4 Proof of the Main Results

Let E be the Banach space $C[0,1]$ equipped with the usual norm $||u||_\infty$.

Proof of Theorem 5.

Let $a > 1$ be such that $f(z) > 0$ for $z \geq a$. Define

$$f_0(z) = \begin{cases} f(z) \text{ if } 0 < z < a, \\ 0 \quad \text{if } z \geq a \end{cases}, \quad f_1(z) = \begin{cases} 0 & \text{if } 0 < z < a, \\ f(z) & \text{if } z \geq a \end{cases}.$$

Then $f = f_0 + f_1$ on $(0,\infty)$. Note that $f_1 \geq 0$ and $f_1 \leq \hat{f}_1$.

Since $f_0(z) = 0$ for $z \geq a$ and $\limsup_{t\to 0^+} t^\gamma |f_0(t)| = \limsup_{t\to 0^+} t^\gamma |f(t)| < \infty$, there exists a positive constant b such that

$$|f_0(z)| \leq \frac{b}{z^\gamma}$$

for $z > 0$. Hence

$$|f(z)| \leq |f_0(z)| + f_1(z) \leq \frac{b}{z^\gamma} + \hat{f}_1(z) \tag{3.19}$$

for $z > 0$. Since $f = f_0$ on $(0, a)$ and $f \geq 0$ on $[a, \infty)$,

$$f(z) \geq -\frac{b}{z^\gamma} \tag{3.20}$$

for $z > 0$. Let $\lambda < \dfrac{\phi(a)}{2(b||k_0||_1 + \hat{f}_1(a)||h||_1)}$. We claim that (i) *If* $u \in E$ *satisfies* $u = \theta T_\lambda u$ *for some* $\theta \in (0,1]$ *then* $||u||_\infty \neq a$. Indeed, let $u \in E$ be such that $u = \theta T_\lambda u$ for some $\theta \in (0,1]$. Then $u/\theta = T_\lambda u$ and therefore u satisfies

$$\begin{cases} -(\phi(u'))' = \lambda \theta^{p-1} h(t) f(\tilde{u}),\ t \in (0,1), \\ u(0) = 0,\ u'(1) + \alpha u(1) = 0, \end{cases}$$

where $\alpha = c(|u(1)|)$. Hence

$$u(t) = \int_0^t \phi^{-1}\left(-\phi(\alpha u(1)) + \lambda\theta^{p-1}\int_s^1 hf(\tilde{u})\right) ds.$$

Let $t^* \in [0,1]$ be such that

$$\begin{aligned} u(1) &= \int_0^1 \phi^{-1}\left(-\phi(\alpha u(1)) + \lambda\theta^{p-1}\int_s^1 hf(\tilde{u})\right) ds \\ &= \phi^{-1}\left(-\phi(\alpha u(1)) + \lambda\theta^{p-1}\int_{t^*}^1 hf(\tilde{u})\right). \end{aligned}$$

Then

$$\phi(\alpha u(1)) = \frac{\lambda\theta^{p-1}\phi(\alpha)}{1+\phi(\alpha)}\int_{t^*}^1 hf(\tilde{u}),$$

which implies

$$u(t) = \int_0^t \phi^{-1}\left(\lambda\theta^{p-1}\left(\int_s^1 hf(\tilde{u}) - \frac{\phi(\alpha)}{1+\phi(\alpha)}\int_{t^*}^1 hf(\tilde{u})\right) ds\right). \tag{3.21}$$

Suppose $||u||_\infty = a$. Then $||\tilde{u}||_\infty \le \max(||u||_\infty, 1) = a$ and (3.19) gives

$$h(\tau)|f(\tilde{u}(\tau)| \le \frac{bh(\tau)}{\tilde{u}^\gamma} + h(\tau)\hat{f}_1(\tilde{u}) \le \frac{bh(\tau)}{p^\gamma(\tau)} + h(\tau)\hat{f}_1(||\tilde{u}||_\infty) \tag{3.22}$$

$$\le bk_0(\tau) + \hat{f}_1(a)h(\tau)$$

for $\tau \in (0, 1)$. Hence

$$\int_0^1 h|f(\tilde{u})| + \frac{\phi(\alpha)}{1+\phi(\alpha)}\int_0^1 h|f(\tilde{u})| \le 2(b||k_0||_1 + \hat{f}_1(a)||h||_1),$$

which together with (3.21) imply

$$a = ||u||_\infty \le \phi^{-1}\left(2\lambda\left(b||k_0||_1 + \hat{f}_1(a)||h||_1\right)\right).$$

Consequently,

$$\frac{\phi(a)}{2(b||k_0||_1 + \hat{f}_1(a)||h||_1)} \le \lambda,$$

a contradiction with the choice of λ. Thus $||u||_\infty \ne a$, as claimed.

Next, we show (ii) *There exists* $R_\lambda > a$ *such that if* $u = T_\lambda u + \xi,\ \xi \ge 0$, *then* $||u||_\infty \ne R_\lambda$.

Let $u \in E$ satisfy $u = T_\lambda u + \xi$ for some $\xi \ge 0$.

Then $v \equiv u - \xi = T_\lambda u$ satisfies

$$\begin{cases} -(\phi(v'))' = \lambda h(t)f(\tilde{u}),\ t \in (0,1), \\ v(0) = 0,\ v'(1) + \alpha v(1) = 0, \end{cases} \tag{3.23}$$

where $\alpha = c(|u(1)|)$. Hence

$$\begin{cases} -(\phi(u'))' = \lambda h(t)f(\tilde{u}),\ t \in (0,1), \\ u(0) = \xi \ge 0,\ u'(1) + \alpha u(1) = \alpha\xi \ge 0. \end{cases} \tag{3.24}$$

Let $R_\lambda > 0$ be large enough so that $R_\lambda > m^{-1}\max\left(2\phi^{-1}(\lambda||k||_1), 8a\right)$ and

$$\frac{\check{f}\left(\frac{m}{8}R_\lambda\right)}{\phi(4R_\lambda)} > \frac{2}{\lambda h_0}\ , \ \ \check{f}\left(\frac{mR_\lambda}{8}\right) > \frac{2||k||_1}{h_0}, \tag{3.25}$$

where $k(t) = bk_0(t),\ h_0 = \min\left(\int_{1/4}^{1/2} h, \int_{1/2}^{3/4} h\right)$. Note that the existence of R_λ follows from (A4). We claim that $||u||_\infty \neq R_\lambda$. Suppose on the contrary that $||u||_\infty = R_\lambda$.

Using (3.20), we obtain

$$\lambda h(t) f(\tilde{u}) \geq -\lambda b \frac{h(t)}{\tilde{u}^\gamma(t)} \geq -\lambda k(t), \tag{3.26}$$

which, together with (3.24), Lemma 2, and the choice of R_λ gives

$$u(t) \geq (m||u||_\infty - \phi^{-1}(\lambda||k||_1))p(t) \geq \frac{m}{2}||u||_\infty p(t)$$

for $t \in [0,1]$. In particular,

$$u(t) \geq \frac{m}{8}||u||_\infty > a \ \text{ for } t \in [1/4, 3/4]\,, \tag{3.27}$$

again by the choice of R_λ. Suppose $||v||_\infty = |v(t_0)|$ for some $t_0 \in [0,1]$. Then $||v||_\infty > 0$ for otherwise (3.23) implies $f(\tilde{u}) = 0$ for $t \in (0,1)$, a contradiction with $f(\tilde{u}(1/4)) = f(u(1/4)) > 0$ in view of (3.27) and the choice of a.

If $t_0 = 1$ then v^2 has a maximum value at 1, and hence $0 \leq (v^2)'(1) = 2v(1)v'(1) = -2\alpha v^2(1) < 0$, a contradiction. Thus $t_0 \in (0,1)$ and so $v'(t_0) = 0$. Since

$$v'(t) = \phi^{-1}\left(-\phi(\alpha v(1)) + \lambda \int_t^1 hf(\tilde{u})\right),$$

we deduce that $\phi(\alpha v(1)) = \lambda \int_{t_0}^1 hf(\tilde{u})$ and therefore

$$v'(t) = \phi^{-1}\left(\lambda \int_t^{t_0} hf(\tilde{u})\right)$$

for $t \in (0,1)$. We shall divide into two cases.

Case 1 $t_0 \geq 1/2$.

By integrating, we deduce that

$$v(1/4) = \int_0^{1/4} \phi^{-1}\left(\lambda \int_s^{t_0} hf(\tilde{u})\right) ds \tag{3.28}$$

Using (3.26) and (3.27), we get

$$\int_s^{1/4} hf(\tilde{u}) + \int_{1/2}^{t_0} hf(\tilde{u}) \geq -\left(\int_s^{1/4} k + \int_{1/2}^{t_0} k\right) \geq -||k||_1 \tag{3.29}$$

and

$$\int_{1/4}^{1/2} hf(\tilde{u}) = \int_{1/4}^{1/2} hf(u) \geq h_0 \check{f}\left(\frac{m}{8}||u||_\infty\right). \tag{3.30}$$

For $s < 1/4$, it follows from (3.25), (3.29) and (3.30) that

$$\lambda \int_s^{t_0} hf(\tilde{u}) = \lambda\left(\int_s^{1/4} hf(\tilde{u}) + \int_{1/2}^{t_0} hf(\tilde{u})\right) + \lambda \int_{1/4}^{1/2} hf(\tilde{u})$$

$$\geq \lambda\left(h_0 \check{f}\left(\frac{m}{8}||u||_\infty\right) - ||k||_1\right) \geq \frac{\lambda h_0}{2}\check{f}\left(\frac{m}{8}||u||_\infty\right)$$

for $s \in [0, 1/4]$. This, together with (3.28), gives

$$||u||_\infty \geq v(1/4) \geq \frac{1}{4}\phi^{-1}\left(\frac{\lambda h_0}{2}\check{f}\left(\frac{m}{8}||u||_\infty\right)\right).$$

Consequently,

$$\frac{\check{f}\left(\frac{m}{8}||u||_\infty\right)}{\phi(4||u||_\infty)} \leq \frac{2}{\lambda h_0}, \tag{3.31}$$

a contradiction with (3.25).

Case 2. $t_0 < 1/2$.

Using the arguments in case 1, we obtain

$$\lambda \int_{t_0}^{1} hf(\tilde{u}) = \lambda \left(\int_{t_0}^{1/2} hf(\tilde{u}) + \int_{3/4}^{1} hf(\tilde{u}) \right) + \lambda \int_{1/2}^{3/4} hf(\tilde{u})$$
$$\geq \frac{\lambda h_0}{2} \check{f} \left(\frac{m}{8} ||u||_\infty \right) > 0,$$

which implies

$$v'(1) = -\phi^{-1} \left(\lambda \int_{t_0}^{1} hf(\tilde{u}) \right) < 0.$$

Hence $v(1) = -(1/\alpha)v'(1) > 0$. Let $w \in C^1[0,1]$ be the solution of

$$\begin{cases} -(\phi(w'))' = \lambda h(t) f(\tilde{u}), \ t \in (0,1), \\ w(0) = 0 = w(1). \end{cases}$$

Then $v \geq w$ on $[0,1]$ by Lemma A with $r = 0, R = 1, \ a = c = 1, b = d = 0$.

Suppose $||w||_\infty = |w(T)|$ for some $T \in (0,1)$. Then $w'(T) = 0$ and

$$w'(s)) = \phi^{-1} \left(\lambda \int_{s}^{T} hf(\tilde{u}) \right)$$

for $s \in [0,1]$. Hence, since $w(0) = 0 = w(1)$, it follows that

$$w(t) = \int_{t}^{1} \phi^{-1} \left(\lambda \int_{T}^{s} hf(\tilde{u}) \right) ds = \int_{0}^{t} \phi^{-1} \left(\lambda \int_{s}^{T} hf(\tilde{u}) \right) ds$$

for $t \in [0,1]$. If $T \leq 1/2$ then for $s > 3/4$, we get

$$\lambda \int_{T}^{s} hf(\tilde{u}) = \lambda \left(\int_{T}^{1/2} hf(\tilde{u}) + \int_{3/4}^{s} hf(\tilde{u}) \right) + \lambda \int_{1/2}^{3/4} hf(\tilde{u})$$
$$\geq \frac{\lambda h_0}{2} \check{f} \left(\frac{m}{8} ||u||_\infty \right).$$

Consequently,

$$||u||_\infty \geq u(3/4) \geq w(3/4) = \int_{3/4}^{1} \phi^{-1} \left(\lambda \int_{T}^{s} hf(\tilde{u}) \right) ds$$

$$\geq \frac{1}{4}\phi^{-1}\left(\frac{\lambda h_0}{2}\check{f}\left(\frac{m}{8}||u||_\infty\right)\right),$$

i.e. (3.31) holds, a contradiction. If $T > 1/2$ then for $s < 1/4$, we get

$$\lambda \int_s^T hf(\tilde{u}) = \lambda \left(\int_s^{1/4} hf(\tilde{u}) + \int_{1/2}^s hf(\tilde{u})\right) + \lambda \int_{1/4}^{1/2} hf(\tilde{u})$$
$$\geq \frac{\lambda h_0}{2}\check{f}\left(\frac{m}{8}||u||_\infty\right).$$

Consequently,

$$u(1/4) \geq w(1/4) = \int_0^{1/4} \phi^{-1}\left(\lambda \int_s^T hf(\tilde{u})\right) ds$$
$$\geq \frac{1}{4}\phi^{-1}\left(\frac{\lambda h_0}{2}\check{f}\left(\frac{m}{8}||u||_\infty\right)\right),$$

a contradiction. Hence we reach a contradiction in either case and therefore $||u||_\infty \neq R_\lambda$, which proves (ii).

From (i) and (ii), it follows that (see [21, Theorem 12.3]

$$\deg(I - T_\lambda, B(0, a), 0) = 1 \text{ and } \deg(I - T_\lambda, B(0, R_\lambda), 0) = 0,$$

where $B(0, r)$ denotes the open ball centered at 0 with radius r in E.

Hence $\deg(I - T_\lambda, B(0, R_\lambda)\backslash\overline{B(0, a)}, 0) = -1$ and so there exists $u_\lambda \in E$ with $||u_\lambda||_\infty > a$ such that $u_\lambda = T_\lambda u_\lambda$. We verify next that $||u_\lambda||_\infty \to \infty$ as $\lambda \to 0^+$. Using (3.4) and the fact that $||u_\lambda||_\infty > a$, we deduce that $||\tilde{u}||_\infty \leq ||u||_\infty$ and

$$h(\tau)|f(\tilde{u}_\lambda(\tau))| \leq bk_0(\tau) + h(\tau)\hat{f}_1(||u_\lambda||_\infty)$$

for $\tau \in (0, 1)$, from which Lemma 1 gives

$$||u_\lambda||_\infty \leq \phi^{-1}\left(\lambda(b||k_0||_1 + ||h||_1\hat{f}_1(||u_\lambda||_\infty))\right).$$

Consequently,

$$\frac{b||k_0||_1 + ||h||_1 \hat{f}_1(||u_\lambda||_\infty)}{\phi(||u_\lambda||_\infty)} \geq \frac{1}{\lambda}. \tag{3.32}$$

We verify that $||u_\lambda||_\infty \to \infty$ as $\lambda \to 0^+$. Indeed, if $||u_\lambda||_\infty \not\to \infty$ as $\lambda \to 0^+$ then there exist a constant $M > a$ and a sequence (λ_n) with $\lambda_n > 0, \lambda_n \to 0$ such that $||u_{\lambda_n||_\infty} \in (a, M)$ for all n. Thus

$$\frac{1}{\lambda_n} \leq \frac{b||k_0||_1 + ||h||_1 \hat{f}_1(||u_{\lambda_n}||_\infty)}{\phi(||u_{\lambda_n}||_\infty)} \leq \frac{b||k_0||_1 + ||h||_1 \hat{f}_1(M)}{\phi(a)},$$

a contradiction for n large.

Hence for λ sufficiently small, Lemma 2 gives

$$u_\lambda(t) \geq (m||u_\lambda||_\infty - \phi^{-1}(\lambda||k||_1)p(t) \geq p(t)$$

for $t \in (0, 1)$. Thus $\tilde{u}_\lambda = u_\lambda$ i.e. u_λ is a positive solution of (3.2). Clearly, $u_\lambda(t) \to \infty$ as $\lambda \to 0^+$ uniformly for t in compact subsets of $(0, 1)$. By Remark 2,

$$u_\lambda(1)(1 + m^{-1}c(u_\lambda(1)) \geq ||u_\lambda||_\infty - m^{-1}\phi^{-1}(\lambda||k||_1),$$

and hence $u_\lambda(1) \to \infty$ as $\lambda \to 0^+$, which completes the proof of Theorem 5 ■

Proof of Theorem 6.

We first modify the proof of [40, Lemma 2.3] to show that $u(1) > \rho$ for $\lambda >> 1$.

Suppose $u(1) \leq \rho$. Let t_0 be defined in Lemma 4 and let $\tilde{t}_\theta > t_0$ be such that $u(\tilde{t}_\theta) = \theta$. Then

$$\lambda h'(t)F(u) \geq 0 \text{ on } [\tilde{t}_\theta, 1],$$

and so

$$[(1 - 1/p)|u'(1)|^p + \lambda h(t)F(u)]' = \lambda h'(t)F(u) \geq 0$$

on $[\tilde{t}_\theta, 1]$. Since $F(\theta) = 0$, we deduce that

$$(1 - 1/p)|u'(1)|^p + \lambda h(1)F(u(1)) \geq 0,$$

from which (3.35) and the boundary condition for u at 1 imply

$$(1 - 1/p)\,(c(u(1))u(1))^p = (1 - 1/p)|u'(1)|^p$$

$$\geq -\lambda h(1)F(u(1)) \geq \lambda \eta_1 h(1)u(1).$$

Hence

$$u(1)^{p-1} \geq \frac{\lambda \eta_1 h(1) p}{(p-1) \sup\limits_{0 \leq s \leq \rho} c^p(s)}$$

i.e. $u(1) \to \infty$ as $\lambda \to \infty$, a contradiction. Thus $u(1) > \rho$ for $\lambda >> 1$.

Let λ be large enough so that $u(1) > \rho$ and $t_\rho < 1/8$. Then $u > \rho$ on $[1/8, 1]$, and therefore

$$-(\phi(u'))' = \lambda h(t) f(u) > 0 \text{ on } [1/8, 1]. \tag{3.33}$$

Thus u is concave on $[1/8, 1]$ and since $u(1/8) > 0, u(1) > 0$, we obtain

$$u(t) \geq (8/7)||u||_\infty \min(t - 1/8, 1 - t) \text{ for } t \in [1/4, 1].$$

In particular,

$$u(t) \geq \frac{||u||_\infty}{7} \text{ for } t \in [1/4, 3/4] \tag{3.34}$$

follows. Let $u_0 \in C^1[0, 1]$ be the solution of

$$\begin{cases} -(\phi(u_0'))' = \lambda h(t) f(u) \text{ in } (1/8, 1), \\ u_0(1/8) = 0 = u_0(1). \end{cases}$$

Then $u \geq u_0$ on $[1/8, 1]$ by Lemma A with $r = 1/8, R = 1,\ a = c = 1, b = d$.

Since

$$h(t)f(u) \geq h(1) \inf_{s \geq \rho} f(s) \equiv M \quad \text{for } t \in [1/8, 1],$$

it follows that

$$-(\phi(u_0'))' \geq \lambda M \text{ in } (1/8, 1),$$

from which the weak comparison principle implies $u_0 \geq (\lambda M)^{\frac{1}{p-1}} z_0$ on $[1/8, 1]$, where $z_0 \in C^1[0,1]$ is the solution of

$$\begin{cases} -(\phi(z_0'))' = 1 \text{ in } (1/8, 1), \\ z_0(1/8) = 0 = z_0(1). \end{cases}$$

Hence

$$||u||_\infty \geq ||u_0||_\infty \geq (\lambda M)^{\frac{1}{p-1}} ||z_0||_\infty \to \infty$$

as $\lambda \to \infty$. Let $\hat{t} \in (1/8, 1)$ be such that $u_0'(\hat{t}) = 0$. If $\hat{t} > 1/2$ then it follows from (3.34) that

$$\begin{aligned} u(1/4) &\geq u_0(1/4) = \int_{1/8}^{1/4} \phi^{-1}\left(\lambda \int_s^{\hat{t}} hf(u)\right) ds \\ &\geq \frac{1}{8}\phi^{-1}\left(\lambda \int_{1/4}^{1/2} hf(u)\right) \geq \frac{1}{8}\phi^{-1}\left(\lambda h_0 \check{f}\left(\frac{||u||_\infty}{7}\right)\right), \end{aligned}$$

where $\check{f}$ and h_0 are defined in (ii) of the proof of Theorem 5. Consequently,

$$\frac{\check{f}\left(||u||_\infty/7\right)}{\phi(8||u||_\infty)} \leq \frac{1}{\lambda h_0}. \tag{3.35}$$

On the other hand, if $\hat{t} \leq 1/2$ then we have

$$\begin{aligned} u(3/4) &\geq u_0(3/4) = \int_{3/4}^{1} \phi^{-1}\left(\lambda \int_{\hat{t}}^{s} hf(u)\right) ds \\ &\geq \frac{1}{4}\phi^{-1}\left(\lambda \int_{1/2}^{3/4} hf(u)\right) \geq \frac{1}{4}\phi^{-1}\left(\lambda h_0 \check{f}\left(\frac{||u||_\infty}{7}\right)\right) \end{aligned}$$

i.e. (3.35) holds. Thus (3.35) holds in either case, which is a contradiction for λ large since $||u||_\infty \to \infty$ as $\lambda \to \infty$ and $\lim_{s\to\infty} \frac{\check{f}(s/7)}{\phi(8s)} = \infty$. Hence (3.2) has no positive solution for λ large, which completes the proof of Theorem 6 ■

CHAPTER 4

FUTURE WORK

1. Study the existence and nonexistence of positive solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda a(x) f(u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

when $f : (0, \infty) \to \mathbb{R}$ is p-superlinear at ∞ i.e. $\lim\limits_{u\to\infty} \frac{f(u)}{u^{p-1}} = \infty$, and related system

$$\begin{cases} -\Delta_{p_i} u_i = \lambda f_i(u_j) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases}$$

where $i, j \in \{1, 2\}$ with $i \neq j$ and $f_i : (0, \infty) \to \mathbb{R}$ are p_i-superlinear at ∞.

Note that the case when f and f_i are p and p_i-sublinear respectively at ∞ was considered in Chapter 2.

2. Study the existence and nonexistence of positive solutions to the problem

$$\begin{cases} -\Delta_p u = \lambda K(|x|) f(u) \text{ in } \Omega, \\ \frac{\partial u}{\partial n} + \tilde{c}(u)u = 0 \text{ on } |x| = r_0, \\ u(x) \to 0 \text{ as } |x| \to \infty, \end{cases}$$

when $f : (0, \infty) \to \mathbb{R}$ is p-sublinear at ∞, and determine whether the problem has non-radial positive solutions. Note that the radial case when f is p-superlinear at ∞ respectively was considered in Chapter 3.

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APPENDIX A

VERIFICATION OF SOME STATEMENTS IN CHAPTER 3

A. *The transformation $r = |x|$ and $t = (r/r_0)^{\frac{p-N}{p-1}}$ reduces problem (1.1) to the ODE problem (1.2).*

We have $\nabla u = u'r)\frac{x}{r}$ and so

$$|\nabla u|^{p-2}\nabla u = |u'(r)|^{p-2}u'(r)\frac{x}{r} = \phi(u_r)\frac{x}{r}.$$

Thus

$$\Delta_p u = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\nabla u\right) = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\phi(u_r)\frac{x_i}{r}\right) =$$

$$\sum_{i=1}^{N}\left[(\phi(u_r)_r\left(\frac{x_i}{r}\right)^2 + \phi(u_r)\left(\frac{1}{r} - \frac{x_i}{r^3}\right)\right]$$

$$= (\phi(u_r)_r + \frac{N-1}{r}\phi(u_r) = -\lambda K(r)f(u). \quad (1)$$

Multiplying (1) by the integrating factor r^{N-1} gives

$$-(\phi\left(r^{\frac{N-1}{p-1}}u_r\right)_r = -(r^{N-1}\phi(u_r))_r = \lambda K(r)f(u). \quad (2)$$

Let $\delta = \frac{p-N}{p-1} < 0$. Then $t = (r/r_0)^{\delta},\ r = r_0 t^{1/\delta}$, and

$$u_t = (r_0/\delta)u_r t^{1/\delta - 1} = u_r(r_0/\delta)(r/r_0)^{1-\delta}$$

$$= \frac{r_0^{\delta}}{\delta}r^{1-\delta}u_r = \frac{r_0^{\delta}}{\delta}r^{\frac{N-1}{p-1}}u_r.$$

Hence (2) gives

$$-(\phi(u_t))_t = -\phi(r_0^{\delta}/\delta)\left(\phi\left(r^{\frac{N-1}{p-1}}u_r\right)\right)_t =$$

$$-\phi(r_0^{\delta}/\delta)\left(\phi\left(r^{\frac{N-1}{p-1}}u_r\right)\right)_r (r_0/\delta)t^{1/\delta-1} = \lambda\phi(r_0^{\delta}/\delta)(r_0/\delta)r^{N-1}K(r)t^{1/\delta-1}f(u) =$$

$$\lambda\phi(r_0^{\delta}/\delta)(r_0/\delta)\left(r_0 t^{1/\delta}\right)^{N-1}t^{1/\delta-1}K(r)f(u) = \lambda\frac{r_0^{\delta(p-1)+N}}{|\delta|^{p-2}\delta^2}K(r)t^{\frac{N}{\delta}-1}f(u) =$$

$$\lambda\left(\frac{r_0}{|\delta|}\right)^p K(r)t^{\frac{N}{\delta}-1}f(u)=\lambda\left(\frac{r_0(p-1)}{N-p}\right)^p K\left(r_0 t^{\frac{p-1}{p-N}}\right)t^{\frac{(N-1)p}{p-N}}=\lambda h(t)f(u),$$

where $h(t)=\left(\frac{r_0(p-1)}{N-p}\right)^p K\left(r_0 t^{\frac{p-1}{p-N}}\right)t^{\frac{(N-1)p}{p-N}}$.

Since $\lim_{r\to\infty}u(r)=0$ and $p<N$, it follows that $u(0)=0$. Finally, since $n=-\frac{x}{r_0}$ on $|x|=r_0$, it follows that

$$\frac{\partial u}{\partial n}=\nabla u.n=-u_r(r_0)\frac{x}{r_0}\cdot\frac{x}{r_0}=-u_r(r_0)\ \text{ on } |x|=r_0,$$

and since $u'(1)=u_t(1)=-\frac{r_0}{|\delta|}u_r(r_0)$, the boundary condition $\frac{\partial u}{\partial n}+\tilde{c}(u)u=0$ on $|x|=r_0$ becomes

$$u'(1)+\tilde{c}u(1)u(1)=0,$$

where $c(s)=\frac{p-1}{N-p}r_0\tilde{c}(s)$.

B. *The problem (1.3)* i.e.

$$\begin{cases} -(\phi(u'))'=\lambda t^{-\sigma}\left(au^{-\gamma}+u^q-1\right) \\ u(0)=0,\ u'(1)+(u(1))^r=0, \end{cases}$$

where $r>1,\ a\in\mathbb{R},\ \sigma,\gamma$ *are nonnegative constants with* $\sigma+\gamma<1$, $q>p-1$ *has a large positive solution for* λ *is sufficiently small and, if* $a\le 0$, *has no positive solutions for* λ *large.*

Clearly $h(t)=t^{-\sigma}$ satisfies (A1) and $c(s)=s^r$ satisfies (A2). Since $\lim_{t\to 0^+}t^\sigma h(t)=1$ and

$$\limsup_{t\to 0^+}t^\gamma|f(t)|=\limsup_{t\to 0^+}t^\gamma|at^{-\gamma}+t^q-1|\le|a|+1,$$

we see that (A3) hold. Since $q>p-1$, (A4) holds. Hence (1.3) has a positive solution u_λ for $\lambda>0$ small with $\lim\limits_{\lambda\to 0^+}u_\lambda(t)=\infty$ *for each* $t\in(0,1]$ *and the limit is uniform on* *compact subsets of* $(0,1)$. If $a \leq 0$ then f is increasing on $(0,\infty)$ with $\lim_{s\to 0^+} f(s) \in [-\infty, 0)$ and $\lim_{s\to\infty} f(s) = \infty$. Hence (A5) holds and since

$$\lim_{s\to 0^+} s^\gamma f(s) = \lim_{s\to 0^+} s^\gamma (as^{-\gamma} + s^q - 1) = a - 1 < 0,$$

it follows that (A6) holds. Since h is decreasing, it follows from Theorem 1.2 that (1.3) has no positive solutions for λ large.

C. *Under the assumption (A4),* $\lim_{z\to\infty} \frac{\check{f}(z)}{z^{p-1}} = \infty$.

Let $K > 0$. By (A4), there exists $z_0 > 0$ such that $f(s) > Ks^{p-1}$ for $s \geq z_0$. Hence, for $z \geq z_0$,

$$f(s) \geq Ks^{p-1} \geq Kz^{p-1} \text{ for } s \geq z,$$

which implies $\check{f}(z) = \inf_{z\leq s} f(s) \geq Kz^{p-1}$.