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Classes of reaction diffusion equations with nonlinear boundary conditions

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CLASSES OF REACTION DIFFUSION EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

By

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CLASSES OF REACTION DIFFUSION EQUATIONS WITH NONLINEAR

BOUNDARY CONDITIONS

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We study positive solutions to classes of steady state reaction diffusion equations that arise naturally in applications. In particular, we study models arising from population dynamics and combustion theory.

The main focus of this dissertation is the mathematical analysis of a challenging new class of problems when a certain nonlinear boundary condition is satisfied. In particular, we establish existence and multiplicity results by making use of the Quadrature method, the method of sub-super solutions, and degree theory. The results in this dissertation provide a significant contribution towards the analysis of elliptic boundary value problems with nonlinear boundary conditions.

Key words: Nonlinear boundary conditions, sub-super solutions, positive solutions

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LIST OF SYMBOLS, ABBREVIATIONS AND NOMENCLATURE

Ω bounded domain of $\mathbb{R}^n$.

$\partial\Omega$ boundary of Ω.

Ω_δ $:= \{x \in \Omega | d(x, \partial\Omega) < \delta\}$.

$C((0,\infty))$ is the space of continuous real-valued functions on $(0,\infty)$.

$C(\Omega)$ is the space of continuous real-valued functions on Ω.

$C^m(\Omega)$ is the space of continuously m-times differentiable functions on Ω.

$C^\infty(\Omega)$ $= \bigcap_{k=0}^{\infty} C^k(\Omega)$.

$C_0^\infty(\Omega)$ is the space of functions in $C^\infty(\Omega)$ with compact support in Ω.

$W^{m,p}(\Omega)$ is the Sobolev space of order m for $1 \leq p \leq \infty$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Δu the Laplacian of u, i.e., $\Delta u = u_{x_1x_1} + u_{x_2x_2} + ... + u_{x_nx_n}$.

CHAPTER 1

INTRODUCTION

Reaction diffusion equations have proved to be invaluable in modeling physical phenomena in areas such as biology, physics, chemistry, and ecology (see [8], [18], [42], [44], and [58]). Since the 1960's a rich study of reaction diffusion equations has yielded many mathematically and physically interesting results. The purpose of this thesis is to initiate a study of the structure of positive solutions for nonlinear elliptic equations when a certain nonlinear boundary condition is fulfilled. In particular, we explore the positive solutions of

$$-\Delta u = \frac{1}{d}\widetilde{f}(x,u); \quad x \in \Omega \tag{1.1}$$

$$d\alpha(x,u)\frac{\partial u}{\partial \eta} + [1-\alpha(x,u)]\,u = 0; \quad x \in \partial\Omega \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with $n \geq 1$, Δ is the Laplace operator, d is the diffusion coefficient, $\widetilde{f}(x,u) : \Omega \times [0,\infty) \longrightarrow \mathbb{R}$ is a C^1 function, $\frac{\partial u}{\partial \eta}$ is the outward normal derivative, and $\alpha(x,u) : \overline{\Omega} \times \mathbb{R} \longrightarrow [0,1]$ is a C^1 function nondecreasing in u.

The prototypical reaction diffusion equation used to describe spatiotemporal phenomena is of the form

$$u_t = d\Delta u + \widetilde{f}(x,u) \tag{1.3}$$

where $u(t, x)$ can represent temperature, population density, or in general the quantity of a substance, $d > 0$ is the diffusion coefficient, Δu is the Laplacian of u with respect to the variable x, and $\widetilde{f}(x, u)$ is the reaction term which can depend on the heterogenous environment. In the context of population dynamics, $\widetilde{f}(x, u)$ is usually represented in the form

$$\widetilde{f}(x, u) = u\bar{f}(x, u) \tag{1.4}$$

where $\bar{f}(x, u)$ represents the per capita growth rate. Skellam first studied such ecology models in his pioneering work [57]. Various reaction diffusion biological models have been studied by [30], previously, though the most classic example is Fisher's equation (see [19]) with $\bar{f}(x, u) = (1 - u)$. Skellam was the first to use the logistic growth rate $\bar{f}(x, u) = a(x) - b(x)u$ in population dynamics to model the crowding effect. However, a general logistic type model can be described by a declining growth rate per capita function, such that $\widetilde{f}(x, u)$ is decreasing with respect to u.

Notwithstanding, several authors have witnessed empirically an increase in the per capita growth rate at low densities. Odum [43] was the first to recognize this phenomenon as the Allee principle which is now called the Allee effect (see [2], [16], [37], [40], [46], [47], [54], and [55]). The Allee effect can be either weak or strong. For a strong Allee effect the per capita growth rate is negative at low population densities, which imposes a threshold on the population that must be overcome in order to grow. Whereas, if the per capita growth rate is positive for low population densities the Allee effect is weak. Weak Allee effect has no such population threshold. Many factors are thought to induce an Allee effect in population dynamics including: predator saturation (see [15]), less efficient feeding at low densities (see [67] and [68]), shortage of mates (see [26] and [61]), lack of effective pollination (see [22]), cooperative behaviors (see [69]), and reduced effectiveness of vigilance and anti-predator defenses (see [29] and [33]), among others.

Additionally, most ecological systems have some form of predation or harvesting of the population. In systems where harvesting is well regulated, such as fishery management, constant yield harvesting is preferred instead of density dependent harvesting. Constant yield harvesting is not dependent upon the population density, u, or time, t. The addition of such a harvesting term to (1.3) leads to:

$$u_t = d\Delta u + \widetilde{f}(x,u) - ch(x); \quad \Omega \tag{1.5}$$

where the parameter $c \geq 0$ represents the level of harvesting, $h(x) \geq 0$ (not equivalently zero) for $x \in \Omega$ represents the intrinsic properties of the domain with $h(x) = 0$ for $x \in \partial\Omega$ and $||h||_\infty = 1$. Here, $ch(x)$ is understood as the rate of the harvesting distribution. Harvesting is typically not allowed on the boundary of the patch. In the literature, (1.5) is known as a semipositone problem. Determining the structure of positive solutions for such problems is known to be challenging (see [6], [39], and [45]).

Typically in the literature, linear boundary conditions have been employed in population models, e.g. Dirichlet boundary condition ($u(x) = 0; \partial\Omega$), Neumann boundary condition ($\frac{\partial u}{\partial \eta} = 0; \partial\Omega$), and Robin boundary condition, a linear combination of the two aforementioned boundary conditions. These types of linear boundary conditions make the assumption that emigration at the boundary is not dependent on the population density at the boundary. Recently however, several authors have reported density dependent emigration at the boundary. Density dependent dispersal can be positive (organisms disperse and emigrate more with higher densities and less with lower densities) or negative (organisms disperse and emigrate less with higher densities and more with lower densities). Positive density dependent dispersal is one of the most common generalizations in Ecology, i.e. most animals exhibit positive density dispersal. Recently, however the use of such generalizations has been cautioned by Paivinen et. al. in [48] after several authors observed negative density dispersal in various organisms. Negative density dispersal has been reported in the black-headed gull *Larus ridibundus* (see [25]), cassin's auklet *Ptychoramphus aleuticus* (see [52]), great tit *Parus major* (see [21]), bighorn sheep *Ovis canadensis* (see [38]), roe deer *Capreolus capreolus* (see [62] and [63]), banner-tailed kangaroo rat *Dipodomys spectabilis* (see [28]), and the Glanville fritillary butterfly *Melitaea cinxi* (see [34]) among others. Several mechanisms have been proposed as a source of negative density dependent dispersal including, range position (in which the density of organisms decreases while moving along a gradient from the center of the species distribution range toward its edge), niche breadth (where a particular organism that has the ability to use a wider range of resources is assumed to be widespread and more abundant), density dependent habitat selection (in particular when organisms tend to occupy more habitats when density is low), and dispersal ability (especially when organisms differ in their ability to disperse which can reduce density but increase distribution) (see [48]).

In particular, conspecific attraction has also been shown to induce negative density dependent dispersal by Kuussaari et al. who observed that Glanville fritillary butterflies emigrated out of low density areas and Danielson et al. who reported a tendency for individuals to be more attracted to areas with conspecifics, see [14], [34], and [59]. Cantrell and Cosner proposed the following nonlinear boundary condition to model conspecific attraction occurring on the boundary of a patch (see [8], [9], and [10]),

$$d\alpha(x,u)\frac{\partial u}{\partial \eta} \quad + \quad [1-\alpha(x,u)]\,u = 0; \quad \partial\Omega \tag{1.6}$$

where d is the diffusion coefficient, $\frac{\partial u}{\partial \eta}$ is the outward normal derivative, and $\alpha(x,u)$: $\overline{\Omega} \times \mathbb{R} \longrightarrow [0,1]$ is a nondecreasing C^1 function. If $\alpha(x,u) \equiv 0$, then (1.6) becomes the Dirichlet boundary condition and all organisms leave the patch upon reaching the boundary. For the case when $\alpha(x,u) \equiv 1$, (1.6) becomes the Neumann boundary condition implying that all organisms remain on the boundary when reached. If $\alpha(x,u) \in (0,1)$ only a fraction of the organisms will remain on the boundary when reached. Using this boundary condition, one is lead to the study of the reaction diffusion equation:

$$u_t \quad = \quad d\Delta u + \widetilde{f}(x,u); \quad x \in \Omega, t > 0, \tag{1.7}$$

$$d\alpha(x,u)\frac{\partial u}{\partial \eta} \quad + \quad [1-\alpha(x,u)]\,u = 0; \quad x \in \partial\Omega, t > 0, \tag{1.8}$$

$$u(0,x) \quad = \quad u_0(x); \quad x \in \Omega. \tag{1.9}$$

Study of the steady states of (1.7) - (1.9) is crucial to fully understanding its dynamics. In particular, we are interested in the structure of the positive solutions of

$$-\Delta u \quad = \quad \frac{1}{d}\widetilde{f}(x,u); \quad x \in \Omega \tag{1.10}$$

$$d\alpha(x,u)\frac{\partial u}{\partial \eta} \quad + \quad [1-\alpha(x,u)]\,u = 0; \quad x \in \partial\Omega. \tag{1.11}$$

In the context of combustion theory, (1.10) with Dirichlet boundary conditions has been extensively employed, here $u(x)$ represents temperature. The motivating example for our study arises naturally from combustion theory (see [4], [17], [24], [31], [32], [49], [50], [53], [60], [65], and [66]) where $\widetilde{f}(u)$ takes the form:

$$\widetilde{f}(u) = \lambda e^{\frac{\beta u}{\beta + u}}$$

with positive parameters β and λ. Dirichlet boundary conditions dictate that the flow of heat on the boundary is fixed regardless of boundary temperature. However, heat flow through a surface usually depends on the value of the temperature at the surface itself (see [5]). With this in mind, we are interested in the structure of positive solutions of (1.10) - (1.11) where the aforementioned nonlinear boundary condition is satisfied. When $\alpha(x, u) \equiv 0$ (Dirichlet boundary condition case) there is already a very rich history in the literature about positive solutions of (1.10) - (1.11). In particular, when $d = 1$, $\widetilde{f}(u) = \lambda e^{\frac{\beta u}{\beta + u}}$, and $\beta \gg 1$ the bifurcation diagram of positive solutions is known to be S-shaped (see [7] and [56]), yielding a range of the parameter, $\lambda > 0$, where there are exactly three positive solutions, see Figure 1.1.

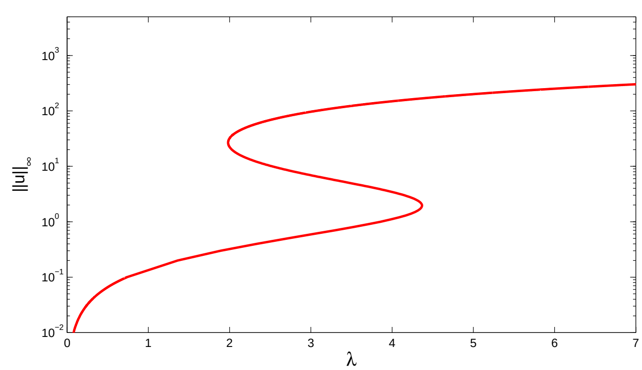

Figure 1.1

S-shaped bifurcation curve.

The remainder of this chapter will be concerned with the discussion of our results. We first present several higher dimension results for these models. In one-dimension, the complete evolution of the bifurcation curve can be determined for these models. Hence, we conclude this thesis by providing two examples of such complete results. In Chapter 2, we will present some preliminary results necessary to our proofs. For the higher dimensional logistic equation and Strong Allee effect equation Theorems 3 - 5 and Theorems 8 - 10 will be proved in Chapters 3 and 4, respectively. In Chapter 5, proofs of Theorems 12 - 14 pertaining to our higher dimensional combustion model will be furnished. Proofs of Theorems 16 - 19 pertaining to the one-dimensional logistic equation will be presented in Chapter 6. Chapter 7 will be devoted to proofs of our one-dimensional combustion model Theorems 29 - 33. Finally, we will discuss conclusions and future directions in Chapter 8.

1.1 Logistic equation in higher dimensions

This section is devoted to the study of the logistic equation with constant yield harvesting in higher dimensional domains, $\Omega \in \mathbb{R}^n;\ n > 1$. The $\alpha(x,u)$'s of biological importance are those that are zero at $u = 0$ and increase to one as $u \to \infty$. With this in mind we will be interested in the study of positive steady state solutions of (1.7) - (1.9) when

$$\alpha(x,u) = \alpha(u) := \frac{u}{u + g(u)}; \quad \partial\Omega$$

where $g \in C^1([0,\infty),[\delta,\infty))$ for some $\delta > 0$, $\frac{g(u)}{u}$ is decreasing, and tends to 0 as $u \to \infty$. Therefore, we study the model

$$
\begin{aligned}
-\Delta u &= au - bu^2 - ch(x) =: f(x,u); \quad \Omega &(1.12)\\
u\left[\frac{\partial u}{\partial \eta} + g(u)\right] &= 0; \quad \partial\Omega &(1.13)
\end{aligned}
$$

where $a > b > 0$, $c \geq 0$, $h(x) \geq 0$ for $x \in \Omega$, $h(x) = 0$ for $x \in \partial\Omega$, and $||h||_\infty = 1$. Here we have that $d = 1$. It is obvious that the positive solutions of

$$
\begin{aligned}
-\Delta u &= au - bu^2 - ch(x); \quad \Omega &(1.14)\\
u &= 0; \quad \partial\Omega &(1.15)
\end{aligned}
$$

and

$$
\begin{aligned}
-\Delta u &= au - bu^2 - ch(x); \quad \Omega &(1.16)\\
\frac{\partial u}{\partial \eta} &= -g(u); \quad \partial\Omega &(1.17)
\end{aligned}
$$

are positive solutions of (1.12) - (1.13).

For (1.14) - (1.15) the following results are known from [45]:

Theorem 1 (See [45])

If $a \leq \lambda_1$ then (1.14) - (1.15) has no positive solution for any $c \geq 0$, where λ_1 is the principal eigenvalue for of $-\Delta$ with Dirichlet boundary conditions.

Theorem 2 (See [45])

If $a > \lambda_1$ and $b > 0$ then there exists a $c_0 = c_0(\Omega, a, b) > 0$ such that for $0 \leq c < c_0$, (1.14) - (1.15) has a positive solution. Further, this solution, $u(x)$, is such that $u(x) \geq ch(x)$ for $x \in \overline{\Omega}$.

The further statement in Theorem 2 is known as an obstacle problem and simply states that the population density is greater than the amount to be harvested throughout the patch.

Now, we present our main results in the form of the following analogous theorems for (1.16) - (1.17).

Theorem 3

There is a constant, $\tilde{a} = \tilde{a}(\Omega, b, N) > 0$, such that if $a < \tilde{a}$ then (1.16) - (1.17) has no positive solution for any $c \geq 0$, where $N := \inf_{u\in[0,\infty)}\{g(u)\}$.

Theorem 4

There exists a constant, $\overline{a} = \overline{a}(\Omega, b) > 0$, such that if $a > \overline{a}$ then there is a $c_1 = c_1(\Omega, a, b) > 0$ such that if $0 \leq c < c_1$ then (1.16) - (1.17) has a strict positive solution, $u > 0;\ \overline{\Omega}$. Moreover, this solution, $u(x)$, satisfies $u(x) \geq ch(x)$ for $x \in \overline{\Omega}$.

We prove Theorem 4 by the method of sub-super solutions. Construction of a super solution is straightforward, while construction of a positive subsolution is difficult since (1.16) - (1.17) is a semipositone problem. In [45] the authors make use of the Anti-Maximum Principle (see [12]) to overcome this difficulty for (1.14) - (1.15). To construct the crucial subsolution for (1.16) - (1.17) we need to do further delicate analysis to accommodate our boundary condition, (1.17). To conclude the section we state a multiplicity result for (1.12) - (1.13) obtained by combining Theorems 2 and 4, see Figure 1.2.

Theorem 5

If $a > \max\{\overline{a}, \lambda_1\}$ and $0 \leq c < \min\{c_0, c_1\}$ then (1.12) - (1.13) has at least two positive solutions, $u_i;\ i = 1, 2$ with $u_1(x) = 0;\ \partial\Omega$ while $u_2(x) > 0;\ \partial\Omega$. Further, the solutions satisfy $u_i(x) \geq ch(x); i = 1, 2$ for $x \in \overline{\Omega}$.

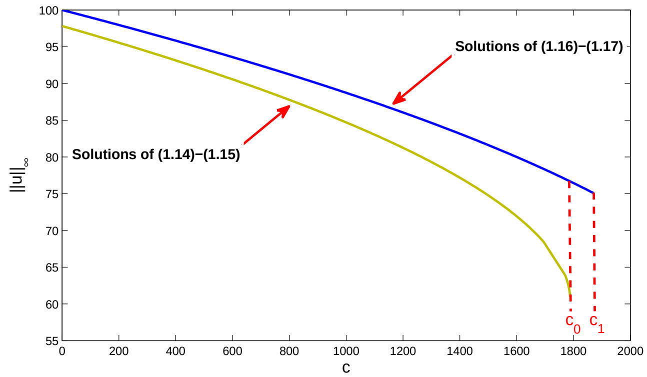

Figure 1.2

Typical bifurcation curve of (1.12) - (1.13).

1.2 Strong Allee effect in higher dimensions

This section is concerned with presenting results for a strong Allee effect model with nonconstant coefficients and constant yield harvesting. In particular, we consider the case when $d = 1$, $n > 1$, $\Omega \subset \mathbb{R}^n$, and

$$\alpha(x,u) = \alpha(u) := \frac{u}{u + g(u)}; \quad \partial\Omega$$

where $g \in C^1([0,\infty),[\delta,\infty))$ for some $\delta > 0$, $\frac{g(u)}{u}$ is decreasing, and tends to 0 as $u \to \infty$. Thus, we examine the model

$$-\Delta u = a(x)u + b(x)u^2 - m(x)u^3 - ch(x); \quad \Omega \tag{1.18}$$

$$u\left[\frac{\partial u}{\partial \eta} + g(u)\right] = 0; \quad \partial\Omega, \tag{1.19}$$

here a, b, m are C^β (Holder continuous) functions such that $b(x), m(x)$ are strictly positive on $\overline{\Omega}$ with $a(x)$ negative at least for some $x \in \Omega$ (strong Allee effect), $c \geq 0$ is the harvesting parameter, $h(x) \geq 0$ for $x \in \Omega$, $h(x) = 0$ for $x \in \partial\Omega$, and $||h||_\infty = 1$. It is straightforward that the positive solutions of

$$-\Delta u = a(x)u + b(x)u^2 - m(x)u^3 - ch(x); \quad \Omega \tag{1.20}$$

$$u = 0; \quad \partial\Omega \tag{1.21}$$

and

$$-\Delta u = a(x)u + b(x)u^2 - m(x)u^3 - ch(x); \quad \Omega \tag{1.22}$$

$$\frac{\partial u}{\partial \eta} = -g(u); \quad \partial\Omega \tag{1.23}$$

are also positive solutions of (1.18) - (1.19).

Define $a_0,\ a_1,\ b_0,\ b_1,\ m_0$, and m_1 as $a_0 := -\inf_{x\in\overline{\Omega}} a(x)$, $a_1 := \sup_{x\in\overline{\Omega}} a(x)$, $b_0 := \inf_{x\in\overline{\Omega}} b(x)$, $b_1 := \sup_{x\in\overline{\Omega}} b(x)$, $m_0 := \inf_{x\in\overline{\Omega}} m(x)$, and $m_1 := \sup_{x\in\overline{\Omega}} m(x)$. Ali, Shivaji, and Wampler analyzed the structure of positive solutions to strong Allee growth rate models with Dirichlet boundary conditions ($u = 0;\ \Omega$) in [1]. Specifically, they proved the following theorems for (1.20) - (1.21).

Theorem 6 (see [1])

(a) If $a_1 < \lambda_1$ and $b_1 \leq 2\sqrt{m_0(\lambda_1 - a_1)}$ then (1.20) - (1.21) has no positive solution for any $c \geq 0$, where λ_1 is the principal eigenvalue of $-\Delta\phi_1 = \lambda_1\phi_1;\ \Omega$, $\phi_1 = 0;\ \partial\Omega$.

(b) For $c > 0$ large, (1.20) - (1.21) has no positive solution.

Theorem 7 (see [1])

There exists a $\tilde{b}_0 := \tilde{b}_0(\Omega, a_0, m_1) > 0$ such that if $b_0 \geq \tilde{b}_0$ then (1.20) - (1.21) has a positive solution for c less than or equal to some $c_0 := c_0(\Omega, a_0, b_0, m_1) > 0$.

We extend these results by proving the following analogous Theorems for (1.22) - (1.23):

Theorem 8

(a) If $a_1 < \mu_1$ and $b_1 \leq 2\sqrt{m_0(\mu_1 - a_1)}$ then (1.22) - (1.23) has no positive solution with $||u||_\infty < \delta$ for any $c \geq 0$, where μ_1 is the principal eigenvalue of $-\Delta\psi_1 = \mu_1\psi_1;\ \Omega,\ \frac{\partial\psi_1}{\partial\eta} = -\psi_1;\ \partial\Omega.$

(b) For $c > 0$ large, (1.22) - (1.23) has no positive solution.

Theorem 9

There exists a $\bar{b}_0 := \bar{b}_0(\Omega, a_0, m_1) > 0$ such that if $b_0 \geq \bar{b}_0$ then (1.18) - (1.19) has a positive solution for c less than or equal to some $c_1 := c_1(\Omega, a_0, b_0, m_1) > 0.$

Again, finding a positive solution to (1.22) - (1.23) is a nontrivial task, due to its semipositone nature. When studying the logistic growth model, in order to construct the crucial subsolution we make use of the Anti-Maximum Principle (see [12]) and the fact that the logistic growth function increases for small u. However, in the strong Allee case, the growth function decreases initially, preventing the use of that argument. For the Dirichlet boundary condition case with strong Allee growth rate, Ali, Shivaji, and Wampler avoided use of the Anti-Maximum Principle by employing the subsolution, $\psi = R\phi_1^2$, where $\phi_1 > 0;\ \Omega$ is a principal eigenfunction of $-\Delta\phi_1 = \lambda_1\phi_1;\ \Omega,\ \phi_1 = 0;\ \partial\Omega$ and $R > 0$ is carefully chosen (see [1]). To construct the crucial subsolution for (1.22) - (1.23) we use a similar approach along with further delicate analysis to accommodate the boundary condition, (1.23). Finally, by combining Theorems 7 and 9 we obtain the following multiplicity result for (1.22) - (1.23):

Theorem 10

If $b_0 \geq \max\{\tilde{b}_0, \bar{b}_0\}$ and $0 \leq c \leq \min\{c_0, c_1\}$ then (1.22) - (1.23) has at least two positive solutions, $u_i;\ i = 1, 2$ with $u_1(x) = 0;\ \partial\Omega$ while $u_2(x) > 0;\ \partial\Omega$.

1.3 Combustion model in an annulus in higher dimensions

In this section, we consider (1.10) - (1.11) when $f : [0, \infty) \to (0, \infty)$ is a continuous function, Ω is an annulus in $\mathbb{R}^n$ with $n > 1$, i.e. $\Omega = \{x \in \mathbb{R}^n | R_1 < |x| < R_2,\ 0 < R_1 < R_2\}$, $d = 1$, and

$$\alpha(x, u) := \begin{cases} 0 & \text{if } |x| = R_1 \\ \frac{u}{u+1} & \text{if } |x| = R_2. \end{cases}$$

Hence, we study the structure of positive radially symmetric solutions for

$$-\Delta u = \lambda f(u); \quad \Omega \tag{1.24}$$

$$u = 0 \quad \text{if } |x| = R_1 \tag{1.25}$$

$$u\left[\frac{\partial u}{\partial \eta} + 1\right] = 0 \quad \text{if } |x| = R_2 \tag{1.26}$$

where $\lambda > 0$. Thus to obtain positive solutions for (1.24) - (1.26) we consider

$$-\Delta u = \lambda f(u); \quad \Omega \tag{1.27}$$

$$u = 0; \quad \partial\Omega \tag{1.28}$$

and

$$-\Delta u = \lambda f(u); \quad \Omega \tag{1.29}$$

$$u = 0 \quad \text{if } |x| = R_1 \tag{1.30}$$

$$\frac{\partial u}{\partial \eta} = -1 \quad \text{if } |x| = R_2. \tag{1.31}$$

Under the following assumptions on $f(u)$:

(H_1) $\lim_{u\to\infty} \frac{f(u)}{u} = 0$

(H_2) $M := \inf_{u\in[0,\infty)}\{f(u)\} > 0$

the existence of positive radial solutions of (1.27) - (1.28) follows from [7] and [64] in the following theorem:

Theorem 11 (see [7] and [64])

If (H_1) holds then (1.27) - (1.28) has a positive radial solution for all $\lambda > 0$.

Now we consider radial solutions to the problem (1.29) - (1.31). Let

$$m = -\int_{R_1}^{R_2} \frac{1}{\tau^{n-1}} d\tau. \tag{1.32}$$

By applying consecutive changes of variables, $r = |x|$, $s = -\int_r^{R_2} \frac{1}{\tau^{n-1}} d\tau$, $t = \frac{m-s}{m}$ and $z(t) = u(r) = u(|x|)$, (1.29) - (1.31) is equivalently transformed into the problem

$$-z''(t) = \lambda h(t) f(z(t)); \quad t \in (0,1) \tag{1.33}$$

$$z(0) = 0 \tag{1.34}$$

$$z'(1) = -b, \tag{1.35}$$

where

$$b = -mR_2^{n-1} > 0,$$

$$h(t) = m^2[r(m(1-t))]^{2(n-1)}. \tag{1.36}$$

Note that $h : [0,1] \to [0,\infty)$ is continuous function. For details about this transformation, see [36]. Now we present the existence of a positive solution of (1.33) - (1.35) in the following theorem.

Theorem 12

If (H_1) and (H_2) both hold then (1.33) - (1.35) has a positive solution for all $\lambda > \frac{b}{M\int_0^1 sh(s)ds}$, where b and $h(t)$ are defined as in (1.36).

Further, if we additionally assume that

$(H_3)\quad N := \sup_{u\in[0,\infty)} \{f(u)\} < \infty$

then we can show non-existence for $\lambda \ll 1$.

Theorem 13

If (H_3) holds then (1.33) - (1.35) has no positive solution for any $\lambda < \frac{b}{N\int_0^1 sh(s)ds}$, where b and $h(t)$ are defined as in (1.36).

Finally, by combining Theorem 11 and Theorem 12, we have following multiplicity result for (1.24) - (1.26).

Theorem 14

Assume (H_1) and (H_2). Then

(1) If $0 < \lambda \leq \frac{b}{M\int_0^1 sh(s)ds}$, then (1.24) - (1.26) has a positive radial solution.

(2) If $\lambda > \frac{b}{M\int_0^1 sh(s)ds}$, then (1.24) - (1.26) has at least two positive radial solutions,

where b and $h(t)$ are defined as in (1.36).

1.4 Logistic equation in one-dimension

In this section, we consider the logistic equation with constant yield harvesting in one-dimension. In particular, we consider the case when $d = 1$, $n = 1$, $\Omega = (0, 1)$, $h(x) \equiv 1$, and

$$\alpha(x, u) = \alpha(u) := \frac{u}{u+1}; \quad \partial\Omega.$$

Hence, we study,

$$-u'' = au - bu^2 - c := f(u); \quad x \in (0,1) \tag{1.37}$$

$$[-u'(0)+1]\,u(0) = 0 \tag{1.38}$$

$$[u'(1)+1]\,u(1) = 0, \tag{1.39}$$

where $a > b > 0$ and $c \geq 0$. It is easy to see that analyzing the structure of positive solutions of (1.37) - (1.39) is equivalent to analyzing the four boundary value problems,

$$-u'' = au - bu^2 - c; \quad x \in (0,1) \tag{1.40}$$

$$u(0) = 0 \tag{1.41}$$

$$u(1) = 0, \tag{1.42}$$

$$-u'' = au - bu^2 - c; \quad x \in (0,1) \tag{1.43}$$

$$u(0) = 0 \tag{1.44}$$

$$u'(1) = -1, \tag{1.45}$$

$$-u'' = au - bu^2 - c; \quad x \in (0,1) \tag{1.46}$$

$$u'(0) = 1 \tag{1.47}$$

$$u(1) = 0, \tag{1.48}$$

and

$$-u'' = au - bu^2 - c; \quad x \in (0,1) \tag{1.49}$$

$$u'(0) = 1 \tag{1.50}$$

$$u'(1) = -1. \tag{1.51}$$

Thus, the positive solutions of these four boundary value problems are the positive solutions of (1.37) - (1.39). Notice that if $u(x)$ is a solution of (1.43) - (1.45) then $v(x) := u(1-x)$ is also a solution of (1.46) - (1.48). Therefore, it suffices to only consider (1.40) - (1.42), (1.43) - (1.45), and (1.49) - (1.51).

For (1.40) - (1.42) the structure of positive solutions has been well studied in [13] and [45]. The following Theorem can be proved by implementing the Quadrature method of [35]:

Theorem 15 (see [13] and [45])

(1) If $a \leq \lambda_1$ then (1.40) - (1.42) has no positive solution for any $c \geq 0$.

(2) If $\lambda_1 < a < \lambda^*$ (some $\lambda^* > \lambda_1$) then there exists a $c_0(a,b) > 0$ such that if

   (a) $0 \leq c < c_0$ then (1.40) - (1.42) has 2 positive solutions.
   (b) $c = c_0$ then (1.40) - (1.42) has a unique positive solution.
   (c) $c > c_0$ then (1.40) - (1.42) has no positive solution.

(3) If $a > \lambda^*$ then there exist $c_0(a,b), \tilde{c}(a,b) > 0$ such that if

   (a) $\tilde{c} < c < c_0$ then (1.40) - (1.42) has 2 positive solutions.
   (b) $0 \leq c < \tilde{c}$ or $c = c_0$ then (1.40) - (1.42) has a unique positive solution.
   (c) $c > c_0$ then (1.40) - (1.42) has no positive solution.

Here, λ_1 is the principal eigenvalue of $-\Delta$ with Dirichlet boundary conditions.

See Section 6.4.1 for an evolution of the bifurcation curve of (1.40) - (1.42).

By adapting the Quadrature method in [35], we are able to prove several theorems for (1.43) - (1.45).

Theorem 16

If $c > \hat{c}(a,b)$ (some $\hat{c}(a,b) > 0$) then (1.43) - (1.45) has no positive solution.

Theorem 17

If $a \leq a_0$ (some $a_0 > 0$) then (1.43) - (1.45) has no positive solution for any $c \geq 0$.

Theorem 18

If $a > a_0$ then there is a $c^*(a,b) > 0$ such that for $c \geq c^*$ (1.43) - (1.45) has no positive solution.

The main result for (1.43) - (1.45) is contained in the next theorem.

Theorem 19

If $a > a_0$ and $c < c^*(a,b)$ then there is a unique $r(a,b,c) \in (\theta_1, \ell_2)$ such that $F(r) = \frac{1}{2}$ and

$$G_1(\rho) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

is well defined for all $\rho \in [r, \ell_2)$ where $q < \rho$ is the unique solution of $F(\rho) = F(q) + \frac{1}{2}$. Moreover, (1.43) - (1.45) has a positive solution, $u(x)$, with $\rho = ||u||_\infty$ if and only if $G_1(\rho) = \sqrt{2}$ for some $\rho \in [r, \ell_2)$. Here, $\ell_i, \theta_i > 0;\ i = 1, 2$ are the zeros of $f(x)$ and $F(x)$, respectively and $F(x) := \int_0^x f(t)dt$.

See Section 6.4.2 for the evolution of the bifurcation diagram for (1.43) - (1.45) and (1.46) - (1.48). Our computational results indicate the following theorems:

Theorem 20

For $b = 1$, if $a < a_4$ (for $a_4 \approx 5.0407$) then (1.43) - (1.45) and (1.46) - (1.48) have no positive solution for any $c \geq 0$.

Theorem 21

If $b = 1$ then $c_0(a) \to c^*(a)$ as $a \to \infty$. Furthermore, $\rho \to \ell_2$ as $a \to \infty$ where $u(x)$ is a positive solution to (1.43) - (1.45) or (1.46) - (1.48) with $||u||_\infty = \rho$ and $c_0(a) > 0$ is such that (1.43) - (1.45) and (1.46) - (1.48) have no positive solution for $c > c_0(a)$.

Through a similar adaptation of the Quadrature method, we obtain analogous results for (1.49) - (1.51).

Theorem 22

If $a \leq a_0$ then (1.49) - (1.51) has no positive solution for any $c \geq 0$.

Theorem 23

If $a > a_0$ and $c \geq c^*$ then (1.49) - (1.51) has no positive solution.

The main result for (1.49) - (1.51) is given in the next theorem.

Theorem 24

If $a > a_0$ and $c < c^*(a, b)$ then there is a unique $r(a, b, c) \in (\theta_1, \ell_2)$ such that $F(r) = \frac{1}{2}$ and

$$G_2(\rho) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} - 2\int_0^q \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

is well defined for all $\rho \in [r, \ell_2)$ where $q < \rho$ is the unique solution of $F(\rho) = F(q) + \frac{1}{2}$. Moreover, (1.49) - (1.51) has a positive solution, $u(x)$, with $\rho = ||u||_\infty$ if and only if $G_2(\rho) = \sqrt{2}$ for some $\rho \in [r, \ell_2)$. Here, $\ell_i, \theta_i > 0$; $i = 1, 2$ are the zeros of $f(x)$ and $F(x)$, respectively and $F(x) := \int_0^x f(t)dt$.

Also, Section 6.4.3 contains the evolution of the bifurcation curve for (1.49) - (1.51). Our computational results indicate the following theorems:

Theorem 25

For $b = 1$, if $a < a_1$ (for $a_1 \approx 2.8324$) then (1.49) - (1.51) has no positive solution for any $c \geq 0$.

Theorem 26

If $b = 1$ then $c_0(a) \to c^*(a)$ as $a \to \infty$. Furthermore, $\rho \to \ell_2$ as $a \to \infty$ where $u(x)$ is a positive solution to (1.49) - (1.51) with $||u||_\infty = \rho$ and $c_0(a) > 0$ is such that (1.49) - (1.51) has no positive solution for $c > c_0(a)$.

Comparison of nonexistence Theorems 15 - 18, 22, and 23 yields the following nonexistence result for (1.37) - (1.39).

Theorem 27

If $a \leq min\,[a_0, \lambda_1]$ then (1.37) - (1.39) has no positive solution for any $c \geq 0$.

See Section 6.4.4 for the complete evolution of the bifurcation curve for (1.37) - (1.39).

1.5 Combustion model in one-dimension

In this section we consider a combustion model in one-dimension. Thus, we study (1.10) - (1.11) for the case when $n = 1$, $d = 1$, $\Omega = (0, 1)$, $f(u) = \Lambda e^{\frac{\beta u}{\beta + u}}$, $\beta, \lambda > 0$, and

$$\alpha(x, u) := \begin{cases} 0; & x = 0 \\ \\ \frac{u}{u+1}; & x = 1. \end{cases}$$

Therefore, we study the nonlinear boundary value problem

$$-u'' = \lambda e^{\frac{\beta u}{\beta + u}}; \quad x \in (0, 1) \tag{1.52}$$

$$u(0) = 0 \tag{1.53}$$

$$\frac{u(1)}{u(1)+1}[u'(1)] + [1 - \frac{u(1)}{u(1)+1}]u(1) = 0 \tag{1.54}$$

Clearly, studying (1.52) - (1.54) is equivalent to analyzing the two boundary value problems

$$
\begin{aligned}
-u'' &= \lambda e^{\frac{\beta u}{\beta+u}}; \quad x \in (0,1) && (1.55)\\
u(0) &= 0 && (1.56)\\
u(1) &= 0 && (1.57)
\end{aligned}
$$

and

$$
\begin{aligned}
-u'' &= \lambda e^{\frac{\beta u}{\beta+u}}; \quad x \in (0,1) && (1.58)\\
u(0) &= 0 && (1.59)\\
u'(1) &= -1. && (1.60)
\end{aligned}
$$

In particular, the positive solutions of (1.55) - (1.57) and (1.58) - (1.60) are the positive solutions of (1.52) - (1.54).

The structure of positive solutions for (1.55) - (1.57) is well known for one-dimension, as well as higher dimensions. It has been studied by authors such as [7] and [11], among others. By implementing the Quadrature method developed by Laetsch in [35] we analyze the structure of positive solutions via the following theorems.

Theorem 28 (See [7])

For $\beta > 0$ (1.55) - (1.57) has a positive solution for every $\lambda > 0$.

Theorem 29 (See [7])

Let $\beta > 0$, then (1.55) - (1.57) has a positive solution, $u(x)$, with $||u||_\infty = \rho$ if and only if $G_3(\rho) := \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} = \sqrt{\lambda}$ for some $\lambda > 0$.

In addition, Brown, Ibrahim, and Shivaji showed the existence of a S-shaped bifurcation curve when $\beta \gg 1$ for (1.55) - (1.57), see Figure 1.1. The evolution of the bifurcation curve is given in Section 7.5.1 for (1.55) - (1.57). Our focus here is to present extensions of their results by proving analogous theorems for (1.58) - (1.60). By adaptation of the Quadrature method, we obtain:

Theorem 30

For $\beta > 0$, (1.58) - (1.60) has no positive solution with $||u||_\infty < \rho_0(\beta)$ (some $\rho_0(\beta) > 0$). Moreover, for every $\beta > 0$ there exists a $\lambda_1 > 0$ such that (1.58) - (1.60) has no positive solution for any $\lambda < \lambda_1$.

Theorem 31

Let $\beta > 0$, then (1.58) - (1.60) has a positive solution, $u(x)$, with $||u||_\infty = \rho \in S(\beta) := [\rho_0(\beta), \infty) \Leftrightarrow G_4(\rho, q(\rho)) := \frac{1}{2[F(\rho)-F(q(\rho))]} = \lambda$ for some $\lambda > 0$ where $q = q(\rho) \in [0, \rho)$ is the unique solution of $\widetilde{G_4}(\rho, q(\rho)) := 2\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} - \int_0^{q(\rho)} \frac{ds}{\sqrt{F(\rho)-F(s)}} - \frac{1}{\sqrt{F(\rho)-F(q(\rho))}} = 0$.

In addition, we list two theorems that give some global characteristics of the bifurcation curve of (1.58) - (1.60).

Theorem 32

For every $\beta > 0$, $G_4(\rho, q(\rho)) \leq [G_3(\rho)]^2$ for all $\rho > \rho_0(\beta)$. Moreover, equality is achieved if and only if $\rho = \rho_0(\beta)$ in which case, $q(\rho) = 0$.

Theorem 33

Let $\beta > 0$. If $\rho \geq \rho_0(\beta)$, $q(\rho)$, and λ are as in Theorem 31 with $G_4(\rho, q(\rho)) = \lambda$ then

(a) $q(\rho) \to \rho$ as $\rho \to \infty$.

(b) $\lambda \to \infty$ as $\rho \to \infty$;

(c) $\lambda \geq \frac{2\rho}{e^{2\beta}}$.

See Section 7.5.2 for the evolution of the bifurcation curve for (1.58) - (1.60). We note the existence of a S-shaped bifurcation curve for (1.58) - (1.60) when $\beta \gg 1$. Further, by combining the results from (1.55) - (1.57) and (1.58) - (1.60) we present the complete evolution of the bifurcation curve of (1.52) - (1.54) in Section 7.5.3. We also show that for β large enough, (1.52) - (1.54) has a double S-shaped bifurcation curve with exactly 6 positive solutions for a certain range of λ, see Figure 1.3.

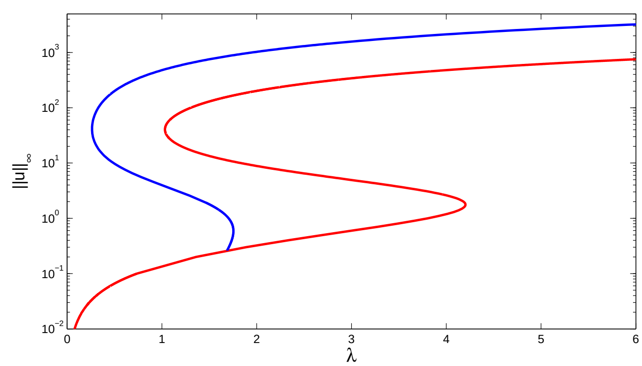

Figure 1.3

Double S-shaped bifurcation curve.

CHAPTER 2

PRELIMINARIES

We will discuss some preliminary results in this chapter that will be used to establish some of our results. In particular, we will discuss maximum principles, anti-maximum principles, comparison principles, degree theory, the Quadrature method, and the method of sub-super solutions for both classical Dirichlet boundary conditions and nonlinear boundary conditions.

2.1 Linear elliptic boundary value problems

Consider the following Laplacian equation:

$$-\Delta u = f(x); \quad \Omega \tag{2.1}$$

$$u = 0; \quad \partial\Omega, \tag{2.2}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary, $\partial\Omega$. Let $C^{m+r}(\Omega), 0 < r < 1$ be the space of m-times continuously differentiable functions whose m^{th} derivatives are Hölder continuous on Ω with Hölder exponent, r. We shall consider classical solutions of (2.1) - (2.2), that is $C^2(\Omega) \bigcap C^1(\overline{\Omega})$ functions satisfying (2.1) - (2.2) pointwise. Let $f \in C^\alpha(\overline{\Omega})$ with $\alpha = 0$ if $n = 1$ and $0 < \alpha < 1$ if $n \geq 2$. Then it is well known that (2.1) - (2.2) has a solution $u = Kf$ where $K : C^\alpha(\overline{\Omega}) \to C^{2+\alpha}(\overline{\Omega})$ is a solution operator whose kernel is the Green's function $G(x, y)$ for (2.1) - (2.2), i.e., $Kf(x) := \int_\Omega G(x, y)f(y)dy$.

2.2 Maximum, anti-maximum and comparison principles

Here we recall the classical maximum principle, the Hopf maximum principle, the anti-maximum principle, and two comparison principles.

Lemma 1 (Maximum principle, see [20] and [51])

Let $\Delta u \geq 0$ in Ω. If u attains its maximum at any interior point in Ω, then $u = M$ in Ω.

Lemma 2 (Hopf Maximum Principle, see [20] and [51])

Let $\Delta u \geq 0$ in Ω. Suppose that $u \leq M$ in Ω and that $u = M$ at some $p \in \partial\Omega$. Then $\frac{\partial u}{\partial \eta} > 0$ at $p \in \partial\Omega$ unless $u = M$. Here $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative.

Lemma 3 (Anti-Maximum Principle, See [12])

There exists a $\delta = \delta(\Omega) > 0$ such that for $\alpha \in (\lambda_1, \lambda_1 + \delta)$ the problem

$$\begin{aligned} -\Delta z_\alpha - \alpha z_\alpha &= -1; \quad \Omega \\ z_\alpha &= 0; \quad \partial\Omega \end{aligned}$$

has a $C^1(\overline{\Omega})$ solution, z_α, such that $z_\alpha > 0$ in Ω and $\frac{\partial z_\alpha}{\partial \eta} < 0$ on $\partial\Omega$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions.

Lemma 4 (Weak comparison principle)

Assume that $\Delta u \geq \Delta v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\overline{\Omega}$.

Lemma 5 (Strong comparison principle)

Assume that $\Delta u > \Delta v$ in Ω and $u = v$ on $\partial\Omega$. Then $u < v$ in Ω and $\frac{\partial u}{\partial \eta} > \frac{\partial v}{\partial \eta}$ on $\partial\Omega$.

2.3 Degree theory

Suppose we have a mapping, $f \in C^1(\overline{D}, \mathbb{R}^n)$ where $D \subset \mathbb{R}^n$ is open and bounded, $p \in \mathbb{R}^n$ such that $f \neq p$; $\partial\Omega$, $D \cup f^{-1}(p)$ is finite, and the Jacobian matrix, $f'(x)$, is non-singular at these points. Then the degree of f at p relative to D is defined by

$$d(f, D, p) := \sum_{x \in f^{-1}(p) \cap D} Sign|f'(x)| \tag{2.3}$$

where $|f'(x)|$ is the determinant of the Jacobian matrix.

The above degree can be extended to a function defined on a Banach space, X. Define $\chi(u) := u - T(u)$ where $T \in C(\overline{D}, E)$ is completely continuous (compact) with $D \subset E$ bounded and open. If $p \in E$ and $p \notin \chi(\partial D)$ then $d(\chi, D, p)$ can be defined by approximating χ with mappings over finite dimensional spaces. This extended degree is known as Leray-Schuder degree (see [41]).

2.4 Quadrature method

This section is concerned with recalling the Quadrature method of Laetsch (see [35]). Consider

$$-u'' = \lambda f(u); \quad (0,1) \tag{2.4}$$

$$u(0) = 0 \tag{2.5}$$

$$u(1) = 0, \tag{2.6}$$

where $f : [0, \infty) \longrightarrow (0, \infty)$ is a continuous function and $\lambda > 0$. We will now state and prove the main theorem of this section.

Theorem 34 (See [35])

(2.4) - (2.6) has a positive solution, $u(x)$, with $||u||_\infty = \rho$ if and only if $G_0(\rho) := \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} = \sqrt{\lambda}$ where $F(u) = \int_0^u f(s)ds$, the primitive of $f(u)$.

Proof: ($\Rightarrow$:) Suppose $u(x)$ is a positive solution to (2.4) - (2.6) with $||u||_\infty = \rho$. First note that (2.4) is an autonomous differential equation. Thus, if there exists a $x_0 \in (0,1)$ such that $u'(x_0) = 0$ then both $v(x) := u(x_0 + x)$ and $w(x) := u(x_0 - x)$ satisfy the initial value problem,

$$\begin{aligned} -z'' &= \lambda f(z) \\ z(0) &= u(x_0) \\ z'(0) &= 0, \end{aligned} \tag{2.7}$$

for all $x \in [0, d)$ where $d = \min\{x_0, 1 - x_0\}$. By Picard's Existence and Uniqueness Theorem, $u(x_0 + x) \equiv u(x_0 - x)$. Hence, $u(x)$ must be symmetric about $x_0 = \frac{1}{2}$ and $u'(x) \geq 0;\ x \in [0, x_0]$ while $u'(x) \leq 0;\ x \in [x_0, 1]$. Now, multiplying (2.4) by $u'(x)$ yields,

$$-\left[\frac{[u'(x)]^2}{2}\right]' = \lambda[F(u(x))]'. \tag{2.8}$$

Integrating throughout (2.8) from x to $\frac{1}{2}$ we have,

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2\lambda}; \quad x \in [0, \frac{1}{2}). \tag{2.9}$$

Integration of (2.9) from 0 to x gives,

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x; \quad x \in [0, \frac{1}{2}]. \tag{2.10}$$

Using the fact that $u(\frac{1}{2}) = \rho$, (2.10) becomes,

$$G_0(\rho) := \sqrt{2}\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{\lambda}. \tag{2.11}$$

($\Leftarrow$:) Suppose: there exist $\lambda, \rho \in (0, \infty)$ such that $G_0(\rho) = \sqrt{\lambda}$. Now, define $u : [0, \frac{1}{2}) \to \mathbb{R}$ by

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x. \tag{2.12}$$

We will show that $u(x)$ is a positive solution of (2.4). It follows that the left-hand side of (2.12) is a differentiable function of u which is strictly increasing from 0 to $\frac{1}{2}$ as u increases from 0 to ρ. Hence, for each $x \in [0, \frac{1}{2})$ there exists a unique $u(x)$ that satisfies

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x. \tag{2.13}$$

By the Implicit Function Theorem, $u(x)$ is differentiable as a function of x. Differentiating (2.13), we have

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]}; \quad x \in [0, \frac{1}{2}]. \tag{2.14}$$

Simplifying (2.14) gives,

$$-\frac{[u'(x)]^2}{2} = \lambda[F(u(x)) - F(\rho)]; \quad x \in [0, \frac{1}{2}]. \tag{2.15}$$

Differentiating (2.15), we have

$$-u''(x) = f(u(x)).$$

Thus, $u(x)$ satisfies the differential equation in (2.4). Also, it is clear that $u(0) = 0$. Finally, defining $u(x)$ as a symmetric function on (0, 1) gives a positive solution to (2.4) - (2.6) with $||u||_\infty = \rho$ and $u(0) = 0 = u(1)$. ■

Remark 1 (see [7]) $G_0(\rho)$ is well defined and the included improper integral is convergent since $f(\rho) > 0$ and $F(u)$ is strictly increasing. Moreover, $G_0(\rho)$ is a continuous and differentiable function.

2.5 Sub-super solutions for Dirichlet boundary conditions

In this section we discuss the classical method of sub-super solutions. Consider

$$-\Delta u = f(x,u); \quad \Omega \tag{2.16}$$

$$u = 0; \quad \partial\Omega \tag{2.17}$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function.

By a sub-solution of (2.16) - (2.17) we mean a function $\psi \in C^2(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta\psi &\leq f(x,\psi); \quad \Omega \\ \psi &\leq 0; \quad \partial\Omega \end{aligned}$$

and by a super-solution of (2.16) - (2.17) we mean a function $Z \in C^2(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta Z &\geq f(x,Z); \quad \Omega \\ Z &\geq 0; \quad \partial\Omega. \end{aligned}$$

Then we have the following result:

Lemma 6 (See [3])

Suppose there exists a sub-solution, ψ, and a super-solution, Z, for the problem (2.16) - (2.17) satisfying $\psi \leq Z$ in Ω, then there exists a solution u such that $\psi \leq u \leq Z$.

2.6 Sub-super solutions for nonlinear boundary conditions

In this section we present a method of sub-super solutions for boundary value problems with nonlinear boundary conditions. Consider

$$-\Delta u = f(x,u); \quad \Omega \tag{2.18}$$

$$\frac{\partial u}{\partial \eta} = -g(u); \quad \partial\Omega \tag{2.19}$$

where $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and g is a continuously differentiable function.

We define a sub-solution of (2.18) - (2.19) as a function $\psi \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta\psi &\leq f(x,\psi); \quad \Omega \\ \frac{\partial \psi}{\partial \eta} &\leq -g(\psi); \quad \partial\Omega \end{aligned}$$

and a super-solution of (2.18) - (2.19) as a function $Z \in C^2(\Omega) \cap C^1(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta Z &\geq f(x,Z); \quad \Omega \\ \frac{\partial Z}{\partial \eta} &\geq -g(Z); \quad \partial\Omega. \end{aligned}$$

From [27] we have the following result:

Lemma 7 (See [27])

If there exist a sub-solution, ψ, and a super-solution, Z, for the problem (2.18) - (2.19) with $\psi \leq Z$ in Ω, then (2.18) - (2.19) has a solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ such that $\psi \leq u \leq Z$ on Ω.

CHAPTER 3

LOGISTIC EQUATION IN HIGHER DIMENSIONS

In this chapter we will prove Theorems 3-5. We prove Theorem 3 in Section 3.1, Theorem 4 in Section 3.2, and Theorem 5 in Section 3.3.

3.1 Proof of Theorem 3

Let $\mu_1 > 0$ be the first eigenvalue and $\phi > 0$ be a corresponding positive eigenfunction of

$$\begin{aligned} -\Delta\phi &= \mu_1\phi; \quad \Omega \\ \frac{\partial\phi}{\partial\eta} &= -\phi; \quad \partial\Omega. \end{aligned} \tag{3.1}$$

Define $\tilde{a} = \tilde{a}(\Omega, b, N) = \min\{\mu_1, Nb\}$. For $a < \tilde{a}$, multiplying (1.16) - (1.17) by ϕ, and integrating over Ω, we obtain

$$\int_\Omega \phi(-\Delta u)\, dx = \int_\Omega au\phi - bu^2\phi - ch(x)\phi\, dx.$$

We know

$$\begin{aligned} \int_\Omega \phi(-\Delta u)\, dx &= \int_\Omega u(-\Delta\phi)\, dx + \int_{\partial\Omega} \frac{\partial\phi}{\partial\eta}u - \frac{\partial u}{\partial\eta}\phi\, ds \\ &= \int_\Omega \mu_1\phi u\, dx + \int_{\partial\Omega} -\phi u + \phi g(u)\, ds. \end{aligned}$$

Thus we have

$$\int_{\partial\Omega} -\phi u + \phi g(u)\, ds = \int_\Omega (a-\mu_1)u\phi - bu^2\phi - ch(x)\phi\, dx. \tag{3.2}$$

Since $a < \mu_1$, we can see that the right hand side of (3.2) is negative. By the maximum principle, we know that $\|u\|_\infty \leq \frac{a}{b} < N$ which gives,

$$\int_{\partial\Omega} -\phi u + \phi g(u)\, ds = \int_{\partial\Omega} (g(u) - u)\phi\, ds > 0,$$

and by this contradiction Theorem 3 is proven.

■

3.2 Proof of Theorem 4

We prove this theorem by using the method of sub-super solutions. To construct the sub-solution, we recall the Anti-Maximum Principle from [12] in the following form. There exists a $\delta_1 = \delta_1(\Omega) > 0$ such that the solution z_λ of

$$\begin{aligned} -\Delta z - \lambda z &= -1; \quad \Omega \\ z &= 0; \quad \partial\Omega. \end{aligned} \tag{3.3}$$

for $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ is positive in Ω and $\frac{\partial z_\lambda}{\partial \eta} < 0$ on $\partial\Omega$. Let $\alpha_\lambda = \|z_\lambda\|_\infty$, $m_\lambda = \inf\{m | \frac{\partial(m z_\lambda)}{\partial \eta} \leq -g(0) - 1\}$, and $\overline{a} = \overline{a}(\Omega, b) = \inf_{\lambda \in (\lambda_1, \lambda_1 + \delta_1)} \max\{2\lambda, 2b m_\lambda \alpha_\lambda\}$. For $a > \overline{a}$, we can choose $\lambda^* \in (\lambda_1, \lambda_1 + \delta_1)$ such that $a > \max\{2\lambda^*, 2b m_{\lambda^*} \alpha_{\lambda^*}\}$. Let $K_1 = \inf\{K | K z_{\lambda^*} \geq h(x)\}$. Define

$$D := \sup_{K \geq \max\{1, K_1\}} \frac{(a - \lambda^*) K \alpha_{\lambda^*} + (K - 1)}{b (K \alpha_{\lambda^*})^2}$$

and let $\widetilde{K} \geq \max\{1, K_1\}$ be such that $D = \frac{(a-\lambda^*)\widetilde{K}\alpha_{\lambda^*} + (\widetilde{K} - 1)}{b(\widetilde{K}\alpha_{\lambda^*})^2}$. First, we state and prove an important lemma:

Lemma 8

If $a > \max\{2\lambda^*, 2bm_{\lambda^*}\alpha_{\lambda^*}\}$, then

$$\frac{m_{\lambda^*}}{\widetilde{K}} < \min\{D, \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}\}.$$

To prove this lemma, we note that $\frac{m_{\lambda^*}}{\widetilde{K}} < \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}$ follows immediately from $a > 2bm_{\lambda^*}\alpha_{\lambda^*}$.

Now, since $a > 2\lambda^*$, $a > 2bm_{\lambda^*}\alpha_{\lambda^*}$, and $\widetilde{K} \geq 1$ the following are true:

$$\begin{aligned}
\frac{a}{2} - \lambda^* &> 0 \\
a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - \frac{a}{2}\alpha_{\lambda^*} &> 0 \\
a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - m_{\lambda^*}b\alpha_{\lambda^*}^2 + 1 &> 1 \\
\left[a\alpha_{\lambda^*} - \lambda^*\alpha_{\lambda^*} - m_{\lambda^*}b\alpha_{\lambda^*}^2 + 1\right]\widetilde{K} &> 1 \\
(a - \lambda^*)\widetilde{K}\alpha_{\lambda^*} + \widetilde{K} - 1 &> m_{\lambda^*}b\alpha_{\lambda^*}^2\widetilde{K}.
\end{aligned}$$

Hence,

$$D = \frac{(a - \lambda^*)\widetilde{K}\alpha_{\lambda^*} + (\widetilde{K} - 1)}{b(\widetilde{K}\alpha_{\lambda^*})^2} > \frac{m_{\lambda^*}}{\widetilde{K}}$$

which proves Lemma 8.

Next, let $c_1 := \min\{D, \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}\}$. Now for $c < c_1$ there exists a d_c such that

$$\max\{c, \frac{m_{\lambda^*}}{\widetilde{K}}\} < d_c < \min\{D, \frac{a}{2\widetilde{K}\alpha_{\lambda^*}b}\}.$$

Define $\psi = \widetilde{K}d_c z_{\lambda^*}$. We know

$$-\Delta\psi = \widetilde{K}d_c(-\Delta z_{\lambda^*}) = \widetilde{K}d_c(\lambda^* z_{\lambda^*}) - \widetilde{K}d_c.$$

Thus if we prove

$$(a - \lambda^*)\widetilde{K}z_{\lambda^*} - bd_c(\widetilde{K}z_{\lambda^*})^2 + \widetilde{K} - 1 \geq 0, \tag{3.4}$$

then

$$
\begin{aligned}
-\Delta\psi &= \widetilde{K}d_c(\lambda^* z_{\lambda^*}) - \widetilde{K}d_c \\
&\leq a(\widetilde{K}d_c z_{\lambda^*}) - b(\widetilde{K}d_c z_{\lambda^*})^2 - d_c \\
&\leq a(\widetilde{K}d_c z_{\lambda^*}) - b(\widetilde{K}d_c z_{\lambda^*})^2 - d_c h(x) \\
&= a\psi - b\psi^2 - d_c h(x)
\end{aligned}
$$

and on $\partial\Omega$

$$\frac{\partial\psi}{\partial\eta} = \widetilde{K}d_c\frac{\partial z_{\lambda^*}}{\partial\eta} < m_{\lambda^*}\frac{\partial z_{\lambda^*}}{\partial\eta} \leq -g(0) - 1.$$

To establish (3.4) we show $H(y) := (a-\lambda^*)y - bd_c y^2 + (\widetilde{K}-1) \geq 0$ for all $y \in [0, \widetilde{K}\alpha_{\lambda^*}]$. Since $a > \lambda^*$, $\widetilde{K} \geq 1$, and $H''(y) \leq 0$ it suffices to show that $H(\widetilde{K}\alpha_{\lambda^*}) = (a-\lambda^*)\widetilde{K}\alpha_{\lambda}^* - bd_c(\widetilde{K}\alpha_{\lambda^*})^2 + (\widetilde{K}-1) \geq 0$. This easily follows from the fact that $d_c < D$. Figure 3.1 illustrates $H(y)$. To construct a subsolution $\overline{\psi} > 0;\ \overline{\Omega}$, let $\overline{f}(x,u) = au - bu^2 - d_c h(x)$. Then $\overline{f}$ is increasing with respect to u on $[0, \frac{a}{2b}]$ for each x. Since $\widetilde{K}d_c\alpha_{\lambda^*} < \frac{a}{2b}$, there is an $\varepsilon > 0$ such that $\widetilde{K}d_c\alpha_{\lambda^*} + \varepsilon < \frac{a}{2b}$ and $g(\epsilon) \leq g(0) + 1$.

Now define $\overline{\psi} = \psi + \varepsilon$, then $\|\overline{\psi}\|_\infty = \widetilde{K}d_c\alpha_{\lambda^*} + \varepsilon$. Also,

$$
\begin{aligned}
-\Delta\overline{\psi} = -\Delta\psi \leq \overline{f}(x,\psi) < \overline{f}(x,\psi+\varepsilon) &= \overline{f}(x,\overline{\psi}) \\
&= a\overline{\psi} - b\overline{\psi}^2 - d_c h(x) \leq a\overline{\psi} - b\overline{\psi}^2 - ch(x)
\end{aligned}
$$

and on $\partial\Omega$:

$$\frac{\partial\overline{\psi}}{\partial\eta} = \frac{\partial\psi}{\partial\eta} \leq -g(0) - 1 \leq -g(\epsilon) = -g(\overline{\psi}).$$

Hence, $\overline{\psi}$ is a subsolution of (1.16) - (1.17) and clearly $\overline{\psi} > 0;\ \overline{\Omega}$.

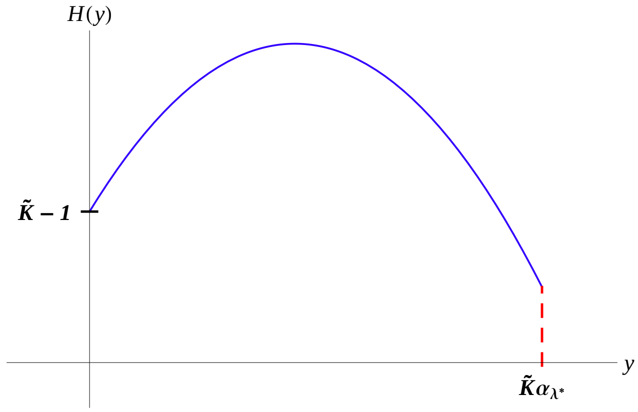

Figure 3.1

Graph of $H(y)$.

Now we choose a large constant $M = M(c)$ such that $aM - bM^2 - ch(x) \leq 0$ and $M \geq \overline{\psi}(x)$ for $x \in \overline{\Omega}$. Then $Z := M$ is a super solution of (1.16) - (1.17) with $Z \geq \overline{\psi}$. Thus there is a strict positive solution, $u(x) > 0$; $\overline{\Omega}$, such that $\overline{\psi} \leq u \leq Z$ which is clearly not a solution of (1.14) - (1.15) (solutions of (1.14) - (1.15) have to satisfy Dirichlet boundary conditions). Also since $\widetilde{K} z_\lambda^*(x) \geq h(x)$ on $\overline{\Omega}$ and $c < d_c$ it is easy to see that

$$\psi = \widetilde{K} d_c z_\lambda^* \geq ch(x)$$

and hence, $u(x) \geq ch(x)$; $\overline{\Omega}$. Thus, Theorem 4 is proven.

■

3.3 Proof of Theorem 5

Let $a > \max\{\overline{a}, \lambda_1\}$ and $0 \leq c < \min\{c_0, c_1\}$. By Theorem 2, (1.12) - (1.13) has a positive solution, $u_1(x)$ with $u_1(x) = 0$; $\partial\Omega$. Also, Theorem 4 implies that (1.12) - (1.13) has a second positive solution, $u_2(x)$, with $u_2(x) > 0$; $\partial\Omega$. Clearly, $u_1(x) \neq u_2(x)$ and

$u_i(x) \geq ch(x);\ \overline{\Omega}$ for $i = 1, 2.$

■

CHAPTER 4

STRONG ALLEE EFFECT IN HIGHER DIMENSIONS

In this chapter we present the proofs of Theorems 8 - 10. We prove Theorem 8 in Section 4.1, Theorem 9 in Section 4.2, and Theorem 10 in Section 4.3.

4.1 Proof of Theorem 8

We prove Theorem 8(a) by contradiction. Let μ_1 be the principal eigenvalue of

$$\begin{aligned} -\Delta\psi_1 &= \mu_1\psi_1; \quad \Omega \\ \frac{\partial\psi_1}{\partial\eta} &= -\psi_1; \quad \partial\Omega \end{aligned}$$

and $\psi_1 > 0;\ \Omega$ be a corresponding eigenfunction. Suppose that u is a positive solution of (1.22) - (1.23). Thus, by Green's identity we know

$$\int_\Omega \psi_1(-\Delta u)dx = \int_\Omega u(-\Delta\psi_1)dx + \int_{\partial\Omega}\left[\frac{\partial\psi_1}{\partial\eta}u - \frac{\partial u}{\partial\eta}\psi_1\right]ds \tag{4.1}$$

or equivalently,

$$\int_\Omega \psi_1\left[a(x)u + b(x)u^2 - m(x)u^3 - ch(x)\right]dx - \int_\Omega\left[\mu_1\psi_1 u\right]dx = \int_{\partial\Omega}\psi_1\left[g(u) - u\right]ds. \tag{4.2}$$

Now, letting $\hat{f}(s) := a_1 + b_1 s - m_0 s^2$ we have

$$a(x)u + b(x)u^2 - m(x)u^3 - ch(x) \leq u\hat{f}(u) - ch(x) \text{ for } u \geq 0. \tag{4.3}$$

Hence if $\hat{f}_1 := \sup_{[0,\infty)} \hat{f}(s) = \frac{b_1^2+4a_1m_0}{4m_0} \leq \mu_1$ or equivalently, $a_1 < \mu_1$ and $b_1 \leq 2\sqrt{m_0(\mu_1 - a_1)}$ then

$$\begin{aligned}\int_\Omega \psi_1 \left[a(x)u + b(x)u^2 - m(x)u^3 - ch(x)\right] dx - \int_\Omega \left[\mu_1\psi_1 u\right] dx \leq \\ \int_\Omega \left[\psi_1 u\hat{f}(u) - \mu_1\psi_1 u\right] dx - c\int_\Omega [h(x)\psi_1]dx \leq \\ \int_\Omega \psi_1 u\left[\hat{f}_1 - \mu_1\right] dx \leq 0. \qquad (4.4)\end{aligned}$$

Also, if $||u||_\infty < \delta$ then we have,

$$\int_\Omega \psi_1 \left[g(u) - u\right] dx \geq \int_\Omega \psi_1 \left[\delta - ||u||_\infty\right] dx > 0. \qquad (4.5)$$

Therefore, (4.4) and (4.5) together contradict (4.2).

To prove Theorem 8(b) note that from (4.1) we have,

$$\int_\Omega \left[\psi_1 \Delta u\right] dx = -\mu_1 \int_\Omega \left[\psi_1 u\right] dx + \int_{\partial\Omega} \psi_1 \left[u - g(u)\right] dx. \qquad (4.6)$$

By the maximum principle, every solution, u, of (1.22) - (1.23) (for any $c \geq 0$) must be such that $u \leq \widetilde{M}$ where $\widetilde{M} := \frac{b_1+\sqrt{b_1^2+4m_0a_1}}{2m_0}$ is the positive zero of $\hat{f}(s)$. Thus we have,

$$\begin{aligned} c\int_\Omega \left[h(x)\psi_1(x)\right] dx &= \int_\Omega \left[\psi_1(\Delta u)\right] dx + \int_\Omega \psi_1 \left[a(x)u + b(x)u^2 - m(x)u^3\right] dx \\ &\leq -\mu_1 \int_\Omega \left[\psi_1 u\right] dx + \int_{\partial\Omega} \psi_1 \left[u - g(u)\right] dx + \hat{f}_1 \int_\Omega \psi_1 u dx \\ &\leq \int_{\partial\Omega} \psi_1 \left[\widetilde{M} - \delta\right] dx + \hat{f}_1\widetilde{M} \int_\Omega \psi_1 dx. \qquad (4.7)\end{aligned}$$

But, (4.7) cannot hold for large c-values thus proving Theorem 8(b).

■

4.2 Proof of Theorem 9

Let λ_1 be the principal eigenvalue of

$$\begin{aligned} -\Delta\phi_1 &= \lambda_1\phi_1; \quad \Omega \\ \phi_1 &= 0; \quad \partial\Omega \end{aligned}$$

and $\phi_1 > 0;\ \Omega$ be the corresponding eigenfunction with $||\phi_1||_\infty = 1$. It is known that $\frac{\partial\phi_1}{\partial\eta} < 0;\ \partial\Omega$ where η is the unit outward normal. Thus, there exist $\tau, \epsilon > 0$ and $\mu \in (0,1)$ such that

$$\begin{aligned} |\nabla\phi_1|^2 - \lambda_1\left[\phi_1^2 + \frac{\sqrt{2}-1}{2}\phi_1\right] &\geq \tau; \quad \overline{\Omega}_\epsilon, \\ \phi_1 &\geq \mu; \quad \Omega - \overline{\Omega}_\epsilon \end{aligned}$$

where $\overline{\Omega}_\epsilon := \{x \in \Omega \mid d(x, \partial\Omega) < \epsilon\}$. Also, define $H(s) := s(-a_0 + b_0 s - m_1 s^2)$. If $b_0 > b_0^{(1)} := 2\sqrt{a_0 m_1}$ then the zeros of $H(s)$ are $0, R_1 := \frac{b_0 - \sqrt{b_0^2 - 4a_0 m_1}}{2m_1}$, and $R_2 := \frac{b_0 + \sqrt{b_0^2 - 4a_0 m_1}}{2m_1}$ and $H(s)$ can be factored as $H(s) = -m_1 s(s - R_1)(s - R_2)$. Let $r_1 := \frac{b_0 - \sqrt{b_0^2 - 3a_0 m_1}}{3m_1}$ denote the first positive zero of $H'(s)$. Since $H(s)$ is convex on $(0, \frac{b_0}{3m_1})$ and $r_1 < \frac{b_0}{3m_1}$ we have

$$\beta := -\inf_{s \in [0, R_2]} H(s) < a_0\left[\frac{b_0 - \sqrt{b_0^2 - 3a_0 m_1}}{3m_1}\right] = a_0 r_1, \tag{4.8}$$

see Figure 4.1. Now for fixed a_0 and m_1 we show that

$$\frac{\beta}{R_2} \to 0 \text{ as } b_0 \to \infty \tag{4.9}$$

by noting that

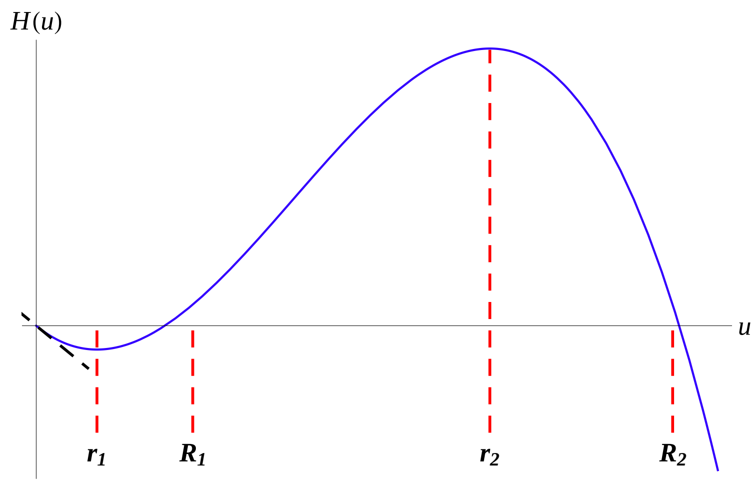

Figure 4.1

Graph of $H(s)$.

$$\begin{aligned}\frac{\beta}{R_2} &< \frac{\frac{a_0\left[b_0-\sqrt{b_0^2-3a_0m_1}\right]}{3m_1}}{\frac{b_0+\sqrt{b_0^2-4a_0m_1}}{2m_1}} = \frac{2a_0\left[b_0-\sqrt{b_0^2-3a_0m_1}\right]}{3\left[b_0+\sqrt{b_0^2-4a_0m_1}\right]}\\ &= \frac{2a_0^2m_1}{\left[b_0+\sqrt{b_0^2-4a_0m_1}\right]\left[b_0+\sqrt{b_0^2-3a_0m_1}\right]} \to 0\end{aligned}$$

as $b_0 \to \infty$. Next, we see that

$$\frac{R_2}{R_1} = \frac{b_0+\sqrt{b_0^2-4a_0m_1}}{b_0-\sqrt{b_0^2-4a_0m_1}} = \frac{\left[b_0+\sqrt{b_0^2-4a_0m_1}\right]^2}{4a_0m_1}$$

and hence we have:

$$\frac{R_2}{R_1} \to \infty \text{ as } b_0 \to \infty. \tag{4.10}$$

Define $K_\mu := \inf_{s\in\left[\frac{R_2(2\mu+\sqrt{2}-1)^2}{8},\frac{R_2(\sqrt{2}+1)^2}{8}\right]} H(s)$. We remark that from (4.10) there exists a $b_0^{(2)} := b_0^{(2)}(\Omega, a_0, m_1)$ such that for $b_0 \geq b_0^{(2)}$,

$$\left[\frac{R_2\left(2\mu+\sqrt{2}-1\right)^2}{8}, \frac{R_2\left(\sqrt{2}+1\right)^2}{8}\right] \subseteq (R_1, R_2)$$

and thus $K_\mu > 0$. Also,

$$\frac{K_\mu}{R_2} = \frac{\min\left\{H\left(\frac{R_2\left(2\mu+\sqrt{2}-1\right)^2}{8}\right), H\left(\frac{R_2\left(\sqrt{2}+1\right)^2}{8}\right)\right\}}{R_2} =$$

$$\min\{m_1 \frac{R_2\left(2\mu+\sqrt{2}-1\right)^2}{8}\left[\frac{R_2\left(2\mu+\sqrt{2}-1\right)^2}{8} - R_1\right]\left[1 - \frac{\left(2\mu+\sqrt{2}-1\right)^2}{8}\right],$$
$$m_1 \frac{R_2\left(\sqrt{2}+1\right)^2}{8}\left[\frac{R_2\left(\sqrt{2}+1\right)^2}{8} - R_1\right]\left[1 - \frac{\left(\sqrt{2}+1\right)^2}{8}\right]\}$$

and since clearly $R_2 \to \infty$ as $b_0 \to \infty$ we have that

$$\frac{K_\mu}{R_2} \to \infty \text{ as } b_0 \to \infty \tag{4.11}$$

Now, define $\psi := \frac{R_2}{2}\left(\phi_1 + \frac{\sqrt{2}-1}{2}\right)^2$. We will show that ψ is a subsolution of (1.22) - (1.23). Let $\widetilde{N} := \min_{x\in\partial\Omega}\left|\frac{\partial\phi_1}{\partial\eta}\right|$. Since $\frac{g(s)}{s} \downarrow 0$ as $s \to \infty$, $\phi_1 = 0;\ \partial\Omega$, and $R_2 \to \infty$ as $b_0 \to \infty$ for fixed a_0 and m_1, there exists a $b_0^{(3)} := b_0^{(3)}(\Omega, a_0, m_1)$ such that if $b_0 \geq b_0^{(3)}$ then on $\partial\Omega$ we have,

$$\begin{aligned}
\frac{\partial\psi}{\partial\eta} &= R_2\left(\phi_1 + \frac{\sqrt{2}-1}{2}\right)\frac{\partial\phi_1}{\partial\eta} \\
&= \frac{R_2}{2}\left(\sqrt{2}-1\right)\frac{\partial\phi_1}{\partial\eta} \\
&\leq -\frac{R_2}{2}\left(\sqrt{2}-1\right)\widetilde{N} \\
&\leq -g\left(\frac{R_2\left(\sqrt{2}-1\right)^2}{8}\right); \quad \partial\Omega.
\end{aligned} \tag{4.12}$$

From (4.9) there exists a $b_0^{(4)} := b_0^{(4)}(\Omega, a_0, m_1)$ such that if $b_0 \geq b_0^{(4)}$ then

$$\frac{\beta}{R_2} < \tau.$$

Thus for $x \in \overline{\Omega}_\epsilon$,

$$\begin{aligned}
-\Delta\psi &= -\text{div}(\nabla\psi) \\
&= -\text{div}\left[R_2\left(\phi_1 + \frac{\sqrt{2}-1}{2}\right)\nabla\phi_1\right] \\
&= R_2\left[\lambda_1\left(\phi_1^2 + \frac{\sqrt{2}-1}{2}\phi_1\right) - |\nabla\phi_1|^2\right] \\
&\leq -R_2\tau \\
&\leq -\beta - c \\
&\leq -a_0\psi + b_0\psi^2 - m_1\psi^3 - ch(x) \\
&\leq a(x)\psi + b(x)\psi^2 - m(x)\psi^3 - ch(x); \quad \overline{\Omega}_\epsilon
\end{aligned} \tag{4.13}$$

for $c \leq R_2\tau - \beta$. Also, (4.11) implies that there exists a $b_0^{(5)} := b_0^{(5)}(\Omega, a_0, m_1)$ such that if $b_0 \geq b_0^{(5)}$ then

$$\frac{K_\mu}{R_2} > \frac{\sqrt{2}+1}{2}\lambda_1.$$

Next, for $x \in \Omega - \overline{\Omega}_\epsilon$,

$$\begin{aligned}
-\Delta\psi &= R_2\left[\lambda_1\left(\phi_1^2 + \frac{\sqrt{2}-1}{2}\phi_1\right) - |\nabla\phi_1|^2\right] \\
&\leq R_2\lambda_1\left(\frac{\sqrt{2}+1}{2}\right) \\
&\leq K_\mu - c \\
&\leq -a_0\psi + b_0\psi^2 - m_1\psi^3 - ch(x) \\
&\leq a(x)\psi + b(x)\psi^2 - m(x)\psi^3 - ch(x); \quad \Omega - \overline{\Omega}_\epsilon
\end{aligned} \tag{4.14}$$

for $c \leq K_\mu - R_2\lambda_1\left(\frac{\sqrt{2}+1}{2}\right)$. Hence, if $b_0 \geq \bar{b}_0(\Omega, a_0, m_1) := \max\left\{b_0^{(i)} \mid i = 1,2,3,4,5\right\}$ and $c \leq c_1(\Omega, a_0, b_0, m_1) := \min\left\{K_\mu - R_2\lambda_1\left(\frac{\sqrt{2}+1}{2}\right), R_2\tau - \beta\right\}$ then from (4.13) and (4.14) we have

$$-\Delta\psi \;\; \leq \;\; a(x)\psi + b(x)\psi^2 - m(x)\psi^3 - ch(x); \quad \Omega$$

making ψ a subsolution for (1.22) - (1.23). Now, define $Z := R_2$. This implies that,

$$-\Delta Z = 0 \geq a(x)Z + b(x)Z^2 - m(x)Z^3 - ch(x); \quad \Omega$$
$$\frac{\partial Z}{\partial \eta} = 0 \geq -\delta \geq -g(Z); \quad \partial\Omega$$

making Z a supersolution for (1.22) - (1.23). Finally, $||\psi||_\infty \leq \frac{R_2}{8}\left(\sqrt{2}+1\right)^2 \leq R_2$ giving $\psi \leq Z;\ \Omega$ and thus a positive solution, u, to (1.22) - (1.23). Moreover, since $\frac{R_2}{8}\left(\sqrt{2}+1\right)^2 > 0$ we have $u \geq \psi > 0;\ \overline{\Omega}$ proving the theorem.

■

Remark 2 We note that this subsolution construction argument can be employed in the logistic growth case to provide an alternate proof of the Theorem 4 without having to make use of the Anti-Maximum Principle.

4.3 Proof of Theorem 10

Let $b > \max\left\{\tilde{b}_0, \bar{b}_0\right\}$ and $0 \leq c \leq \min\{c_0, c_1\}$. By Theorem 7, (1.18) - (1.19) has a positive solution, $u_1(x)$, with $u_1(x) = 0;\ \partial\Omega$. Also, Theorem 9 implies that (1.18) - (1.19) has a second positive solution, $u_2(x)$, with $u_2(x) > 0;\ \partial\Omega$. Clearly, $u_1(x) \neq u_2(x)$ for $i = 1, 2$.

■

CHAPTER 5

COMBUSTION MODEL IN AN ANNULUS

This chapter is devoted to the presentation of proofs of Theorems 12 - 14. We prove Theorem 12 in Section 5.1, Theorem 13 in Section 5.2, and Theorem 14 in Section 5.3.

5.1 Proof of Theorem 12

Recall the consecutive change of variables we employ,

$$m = -\int_{R_1}^{R_2} \frac{1}{\tau^{n-1}} d\tau, \tag{5.1}$$

$r = |x|$, $s = -\int_r^{R_2} \frac{1}{\tau^{n-1}} d\tau$, $t = \frac{m-s}{m}$ and $z(t) = u(r) = u(|x|)$, by which (1.29) - (1.31) is equivalently transformed into (1.33) - (1.35), namely:

$$\begin{aligned} -z''(t) &= \lambda h(t) f(z(t)); \quad t \in (0,1) \\ z(0) &= 0 \\ z'(1) &= -b, \end{aligned}$$

where

$$\begin{aligned} b &= -mR_2^{n-1} > 0, \\ h(t) &= m^2 [r(m(1-t))]^{2(n-1)}. \end{aligned} \tag{5.2}$$

We prove the existence of a positive solution of (1.33) - (1.35) by using the fixed point index in a cone. This fixed point index is equivalent to the Leray-Schauder degree which means if K is a cone in a Banach space E, $\mathcal{O}$ bounded open in E, $0 \in \mathcal{O}$, and $T : K \cap \overline{\mathcal{O}} \to K$ is completely continuous then

$$i(T, K \cap \mathcal{O}, K) = deg(id - T \circ r, r^{-1}(K \cap \mathcal{O}), 0),$$

where $r : E \to K$ is an arbitrary retraction (see [3]). We recall the following lemma from [23]:

Lemma 9 (See [23])

Let E be a Banach space, K a cone in E, and $\mathcal{O}$ bounded open in E. Let $0 \in \mathcal{O}$ and $T : K \cap \overline{\mathcal{O}} \to K$ be completely continuous. Suppose that $Tx \neq \nu x$, for all $x \in K \cap \partial\mathcal{O}$ and all $\nu \geq 1$. Then

$$i(T, K \cap \mathcal{O}, K) = 1.$$

Now, define $T_\lambda : C[0, 1] \to C[0, 1]$ by

$$T_\lambda z(t) = -bt + \lambda \int_0^1 G(t, s)h(s)f(z(s))ds,$$

where

$$G(t, s) = \begin{cases} t & 0 \leq t \leq s \leq 1 \\ s & 0 \leq s \leq t \leq 1. \end{cases}$$

Then $T_\lambda : C[0, 1] \to C[0, 1]$ is completely continuous and u is a solution of (1.33) - (1.35) if and only if u is a fixed point of T_λ, i.e. $T_\lambda u = u$.

Next, define $K := \{z \in C[0,1] | z(t) \geq 0,\ t \in [0,1]$ and z is concave$\}$, then K is a cone in $C[0,1]$. Also if $\lambda > \frac{b}{M\int_0^1 sh(s)ds}$, then $T_\lambda(K) \subset K$. In fact, if $z \in K$, then it is easy to show that $T_\lambda z(0) = 0$ and $T_\lambda z''(t) \leq 0$ for $t \in [0,1]$. Also if $\lambda > \frac{b}{M\int_0^1 sh(s)ds}$, we have

$$\begin{aligned} T_\lambda z(1) &= -b + \lambda \int_0^1 sh(s) f(z(s)) ds \\ &\geq -b + \lambda M \int_0^1 sh(s) ds > 0. \end{aligned}$$

Thus $T_\lambda(z) \in K$.

Define $\tilde{f}(z) := \max_{t\in[0,z]} f(t)$. Then $f(z) \leq \tilde{f}(z)$, $\tilde{f}$ is nondecreasing, and from (H_1), we have

$$\lim_{z\to\infty} \frac{\tilde{f}(z)}{z} = 0. \tag{5.3}$$

Fix $\rho_\lambda \in (0, \frac{1}{\lambda\int_0^1 sh(s)ds})$. From (5.3), there is $m_\lambda > 0$ such that

$$\tilde{f}(z) \leq \rho_\lambda z \qquad \text{for all } z \geq m_\lambda. \tag{5.4}$$

Let $\mathcal{O}_\lambda := \{z \in C[0,1] | \ \|z\|_\infty < m_\lambda\}$. Then $\mathcal{O}_\lambda$ is bounded and open in $C[0,1]$, $0 \in \mathcal{O}_\lambda$, and $T_\lambda : K \cap \overline{\mathcal{O}_\lambda} \to K$ is completely continuous. If $z \in K \cap \partial\mathcal{O}_\lambda$, then from monotonicity of $\tilde{f}$ and (5.4) we have

$$\begin{aligned} T_\lambda z(t) &\leq \lambda \int_0^1 sh(s) \tilde{f}(z(s)) ds \\ &\leq \lambda \tilde{f}(m_\lambda) \int_0^1 sh(s) ds \\ &\leq \lambda \rho_\lambda m_\lambda \int_0^1 sh(s) ds \\ &< m_\lambda = \|z\|_\infty. \end{aligned}$$

Thus $T_\lambda z \neq \nu z$ for all $\nu \geq 1$. Now applying Lemma 9, we have

$$i(T_\lambda, K \cap \mathcal{O}_\lambda, K) = 1,$$

which means T_λ has a fixed point in $K \cap \mathcal{O}_\lambda$. Thus Theorem 12 is proven.

■

5.2 Proof of Theorem 13

Assume (H_1), (H_2), and (H_3) hold. Now, suppose u_λ is a positive solution of (1.33) - (1.35). Thus,

$$\begin{aligned} u_\lambda(t) = T_\lambda u_\lambda(t) &= -bt + \lambda \int_0^1 G(t,s)h(s)f(u_\lambda(s))ds \\ &\leq -bt + \lambda N \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

Letting $t = 1$ gives,

$$u_\lambda(1) \leq -b + \lambda N \int_0^1 sh(s)ds.$$

Since $u_\lambda(t)$ is positive, we have

$$-b + \lambda N \int_0^1 sh(s)ds \geq 0$$

or equivalently,

$$\lambda \geq \frac{b}{N \int_0^1 sh(s)ds}.$$

Hence, the theorem is proved.

■

5.3 Proof of Theorem 14

Suppose (H_1) and (H_2) both hold. By Theorem 11 (1.24) - (1.26) has a positive radial solution, $u_1(x)$, with $u_1(x) = 0;\ \partial\Omega$. From Theorem 12 (1.24) - (1.26) has a second radial solution, $u_2(x)$, with $u_2(x) > 0;\ |x| = R_2$. Thus, $u_1(x) \neq u_2(x)$ providing (1.24) - (1.26) with two distinct positive radial solutions.

■

CHAPTER 6

LOGISTIC EQUATION IN ONE-DIMENSION

In this chapter we present the proofs of Theorem 19 in Section 6.1, Theorem 16 in Section 6.2, and Theorems 17 and 18 in Section 6.3. To close the chapter, we present the complete evolution of the bifurcation curve for (1.37) - (1.39) in Section 6.4.

6.1 Proof of Theorem 19

Define $F(u) = \int_0^u f(s)ds$, the primitive of $f(u)$. Since (1.43) is an autonomous differential equation, if $u(x)$ is a positive solution of (1.43) with $u'(x_0) = 0$ for some $x_0 \in (0,1)$ then $v(x) := u(x_0 - x)$ and $w(x) := u(x_0 + x)$ both satisfy the initial value problem,

$$-z'' = f(z) \tag{6.1}$$

$$z(0) = u(x_0) \tag{6.2}$$

$$z'(0) = 0 \tag{6.3}$$

for all $x \in [0, d)$ where $d = min\{x_0, 1 - x_0\}$. As a result of Picard's existence and uniqueness theorem, $u(x_0 - x) \equiv u(x_0 + x)$. Thus, if we assume that $u(x)$ is a positive solution of (1.43) - (1.45) then it is symmetric around x_0 with $\rho := ||u||_\infty = u(x_0)$. This implies that $u'(x_0) = 0$, $u'(x) > 0;\ [0, x_0)$, and $u'(x) < 0;\ (x_0, 1]$. Using symmetry about x_0, the boundary conditions (1.44) - (1.45), and the sign of u'' given by $f(u)$ we see that positive solutions of (1.43) - (1.45) must resemble Figure 6.1 where $\rho = ||u||_\infty$ and

$q = u(1)$. This implies that $\ell_1 < \rho < \ell_2$ and $0 \leq q < \rho$ where ℓ_i, $i = 1, 2$ are the zeros

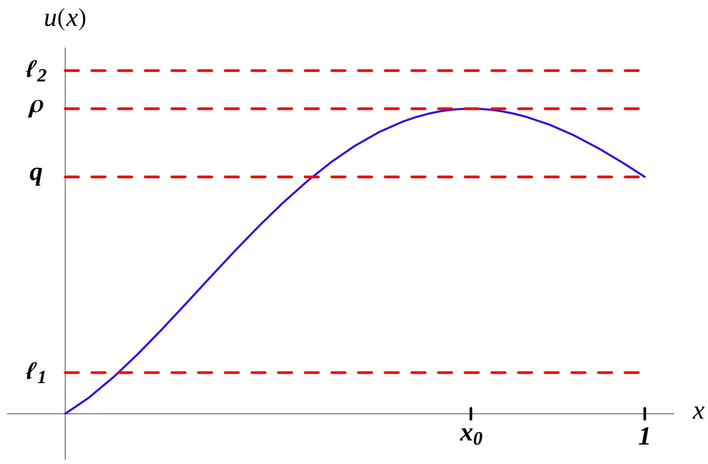

Figure 6.1

Typical positive solution of (1.43) - (1.45).

of $f(u)$. Let $a, b > 0$ s.t. $a > a_0$ and $c \in [0, c^*(a, b))$. $(\Rightarrow:)$ Suppose $u(x)$ is a positive solution to (1.43) - (1.45). Multiplying (1.43) by u' gives

$$-u'u'' = f(u)u' \tag{6.4}$$

Integration of (6.4) with respect to x gives,

$$-\left(\frac{[u'(x)]^2}{2}\right) = F(u(x)) + K. \tag{6.5}$$

Substituting $x = 1$ and $x = x_0$ into (6.5) yields,

$$-K = F(q) + \frac{1}{2} \tag{6.6}$$

$$K = -F(\rho). \tag{6.7}$$

Combining (6.6) and (6.7), we have

$$F(\rho) = F(q) + \frac{1}{2}. \tag{6.8}$$

Substituting (6.7) into (6.5) yields,

$$-\left(\frac{[u'(x)]^2}{2}\right) \;=\; F(u(x)) - F(\rho). \tag{6.9}$$

Now, solving for u' in (6.9) gives,

$$u'(x) \;=\; \sqrt{2}\sqrt{F(\rho) - F(u(x))}; \quad x \in [0, x_0] \tag{6.10}$$

$$u'(x) \;=\; -\sqrt{2}\sqrt{F(\rho) - F(u(x))}; \quad x \in [x_0, 1]. \tag{6.11}$$

Integrating (6.10) and (6.11) with respect to x and using a change of variables, we have

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; \sqrt{2}x; \quad x \in [0, x_0] \tag{6.12}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; -\sqrt{2}(x - x_0); \quad x \in [x_0, 1]. \tag{6.13}$$

Substitution of $x = x_0$ into (6.12) and $x = 1$ into (6.13) gives

$$\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; \sqrt{2}x_0 \tag{6.14}$$

$$\int_\rho^{q} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; -\sqrt{2}(1 - x_0). \tag{6.15}$$

Finally, subtracting (6.15) from (6.14), yields

$$\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} + \int_q^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; \sqrt{2}, \tag{6.16}$$

or equivalently,

$$G_1(\rho) := 2\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} - \int_0^{q} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; \sqrt{2}. \tag{6.17}$$

($\Leftarrow$:) Suppose $G_1(\rho) = \sqrt{2}$ for some $\rho \in [r, \ell_2)$. Define $u(x) : (0, 1) \to \mathbb{R}$ by

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; \sqrt{2}x; \quad x \in [0, x_0] \tag{6.18}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;=\; -\sqrt{2}(x - x_0); \quad x \in [x_0, 1]. \tag{6.19}$$

Now, we show that $u(x)$ is a positive solution to (1.43) - (1.45). It is easy to see that the turning point is given by $x_0 = \frac{1}{\sqrt{2}}\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$. The function, $\int_0^u \frac{ds}{\sqrt{F(\rho)-F(s)}}$, is a differentiable function of u which is strictly increasing from 0 to x_0 as u increases from 0 to ρ. Thus, for each $x \in [0, x_0]$, there is a unique $u(x)$ such that

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2}x \tag{6.20}$$

Moreover, by the Implicit Function theorem, u is differentiable with respect to x. Differentiating (6.20) gives,

$$u'(x) = \sqrt{2[F(\rho) - F(u)]}; \quad x \in [0, x_0]$$

Similarly, u is a decreasing function of x for $x \in [x_0, 1]$ which yields,

$$u'(x) = -\sqrt{2[F(\rho) - F(u)]}; \quad x \in [x_0, 1]$$

This implies,

$$\frac{-(u')^2}{2} = F(\rho) - F(u(x)).$$

Differentiating again, we have,

$$-u''(x) = f(u(x))$$

Thus, $u(x)$ satisfies (1.43). Now, from our assumption, $G_1(\rho) = \sqrt{2}$, it follows that $u(0) = 0$ and $u(1) = q(\rho)$. Since $F(\rho) = H(q(\rho)) = F(q) + \frac{1}{2}$, we have that $u'(1) =$

$-\sqrt{2[F(\rho) - F(q)]} = -1$. Hence, the boundary conditions (1.44) and (1.45) are both satisfied.

■

6.2 Proof of Theorem 16

We note that in order for $\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}}$ in $G_1(\rho, q)$ to be well defined, $F(\rho) > F(s)$ for all $s \in [0, \rho)$. Moreover, the improper integral is convergent if $f(\rho) > 0$. Thus, for such a positive solution to exist, $f(u)$ and $F(u)$ must resemble Figures 6.2 and 6.3, respectively. Here, μ_1, ℓ_i, and θ_i are the zeros of $f'(u)$, $f(u)$, and $F(u)$, respectively for $i = 1, 2$.

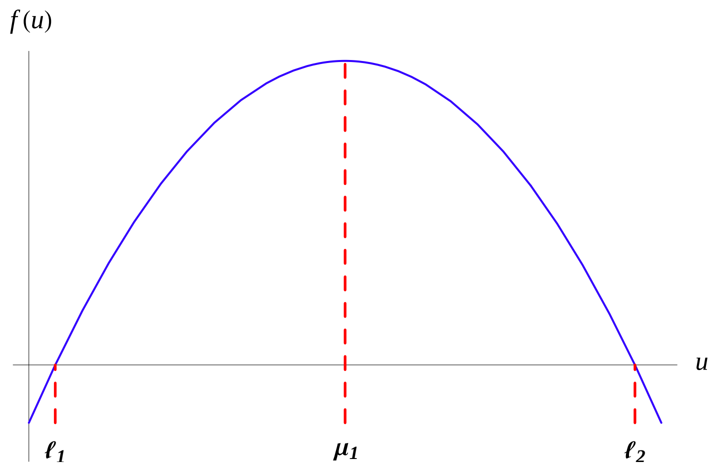

Figure 6.2

Graph of $f(u)$.

From these figures, we note that if $\rho \in (\theta_1, \ell_2)$ then both of these conditions hold and the integrals in (6.17) are well defined. It is easy to see that to ensure $\theta_1 > 0$ and $\ell_2 > 0$ we must have $c \leq c_1 := \frac{3a^2}{16b}$ and $c \leq c_2 := \frac{a^2}{4b}$, respectively. Thus, if $c > \hat{c}(a, b)$ then (1.43) -

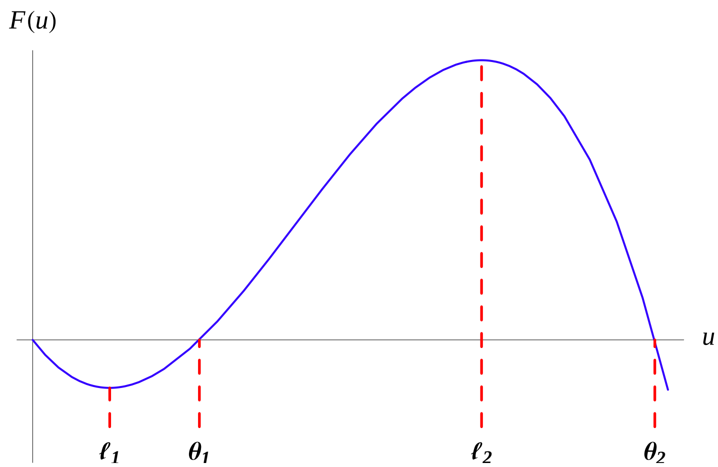

Figure 6.3

Graph of $F(u)$.

(1.45) has no positive solution, where $\hat{c}(a,b) = min\,\{c_1, c_2\} = \frac{3a^2}{16b}$.

■

6.3 Proof of Theorems 17 and 18

Recalling the proof of Theorem 19 from Section 6.1, since $x_0 \in (0,1)$ is fixed for each $\rho > 0$, we need a unique $q < \rho$ corresponding to each ρ-value such that (6.8) is satisfied. Otherwise, uniqueness of solutions to the initial value problem, (6.1) - (6.3), would be violated. Let

$$H(x) := F(x) + \frac{1}{2}.$$

It follows that $H'(x) = -bx^2 + ax - c$, $H(0) = \frac{1}{2}$, and $H'(0) = -c < 0$. In order for a unique $q < \rho$ to exist such that $H(q) = F(\rho)$, $H(x)$ must have the structure illustrated in Figure 6.4 where $H'(\ell_2) = 0$. So, for such a unique $q < \rho$ to exist $F(\rho) > \frac{1}{2}$. Since $\rho \in (\theta_1, \ell_2)$, for this to be true we will need $H(\ell_2) > \frac{1}{2}$. In fact, if

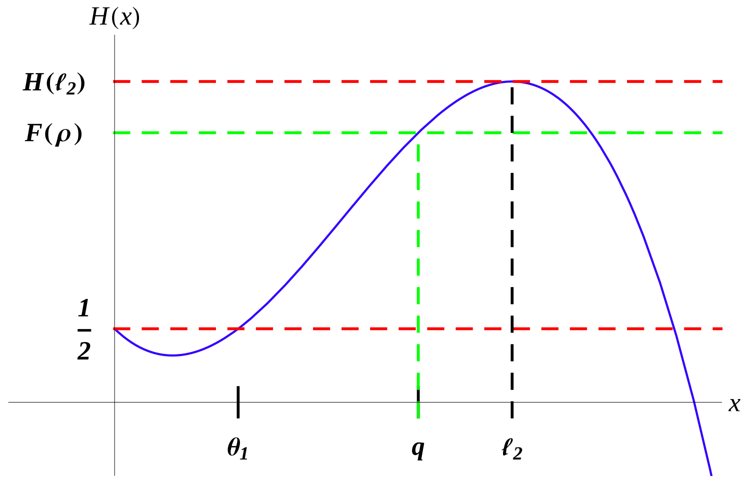

Figure 6.4

Graph of $H(x)$.

$$F(\ell_2) > \frac{1}{2} \tag{6.21}$$

then clearly for $\rho \in (\theta_1, \ell_2)$ with $\rho \approx \ell_2$ we have $F(\rho) > \frac{1}{2}$. It is easy to see that (6.21) will be satisfied if (solving using Mathematica)

$$c < c_3 := \frac{9a^2}{144b} - \frac{9(a^4 - 96ab^2)}{144b\left(-a^6 - 240a^3b^2 + 16\left(72b^4 + \sqrt{3}\sqrt{b^2(a^3+12b^2)^3}\right)\right)^{\frac{1}{3}}} \\ - \frac{9}{144b}\left(-a^6 - 240a^3b^2 + 16\left(72b^4 + \sqrt{3}\sqrt{b^2(a^3+12b^2)^3}\right)\right)$$

and for c_3 to be positive (again using Mathematica)

$$a > a_0 := \sqrt[3]{3\,b^2}$$

both hold. Hence, if $a \leq a_0$ or $c \geq c^* = c^*(a,b) := \min\{c_1, c_2, c_3\}$ then (1.43) - (1.45) has no positive solution.

■

6.4 Computational results

We present the structure of positive solutions of (1.40) - (1.42) in Section 6.4.1. The structure of positive solutions of (1.43) - (1.45) and (1.46) - (1.48) are given in Section 6.4.2 and (1.49) - (1.51) in Section 6.4.3. Finally, the complete evolution of the bifurcation curve of (1.37) - (1.39) is presented in Section 6.4.4. For what follows in this section, we are particularly interested in the case when $b = 1$.

6.4.1 Positive solutions of (1.40) - (1.42)

For the case when $b = 1$ and from Theorem 15 the bifurcation diagrams of (1.40) - (1.42) are illustrated in Figures 6.5 and 6.6. Note that in this case $\lambda_1 = \pi^2$.

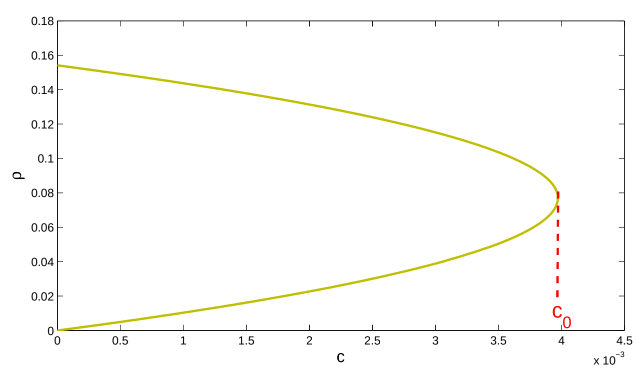

Figure 6.5

$||u||_\infty$ vs c for $a = 10$ and $b = 1$.

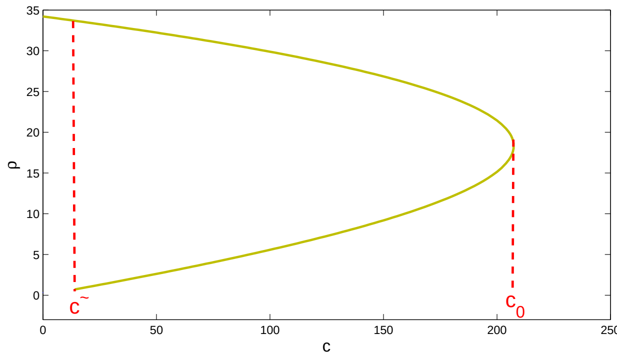

Figure 6.6

$||u||_\infty$ vs c for $a = 40$ and $b = 1$.

6.4.2 Positive solutions of (1.43) - (1.45) and (1.46) - (1.48)

From Theorem 19, we plot the level sets when $b = 1$ of

$$G_1(\rho) - \sqrt{2} = 0 \tag{6.22}$$

for $a > \sqrt[3]{3}$ and $\rho \in [r, \ell_2)$. By implementing a numerical root-finding algorithm in Mathematica we were able to solve equation (6.22). Explicit formulas were used to calculate the unique $r = r(a, b, c)$ and $q = q(\rho)$ values. Note that these computations are expensive due to the natural of the improper integral equations involved. Figures 6.7 - 6.11 depict several level sets plotted within $[r, \ell_2) \times [0, c^*)$. In what follows, the green curve represents ρ vs c while the upper and lower branches of the dotted black curve represent ℓ_2 and r, respectively. We note that the green curve's lower branch begins to shrink for $a \geq 10.1388$. This is due to the fact that solutions of (6.22) are outside of $[r, \ell_2)$.

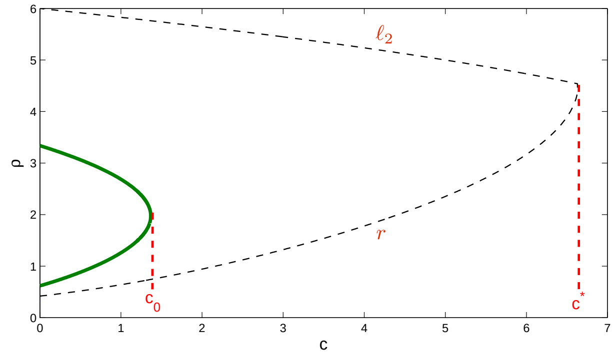

Figure 6.7

$||u||_\infty$ vs c for $a = 6$ and $b = 1$.

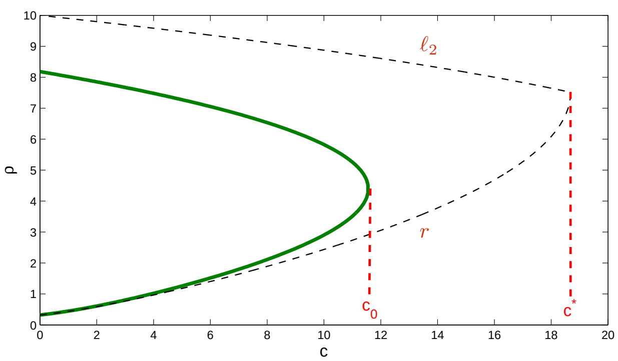

Figure 6.8

$||u||_\infty$ vs c for $a = 10$ and $b = 1$.

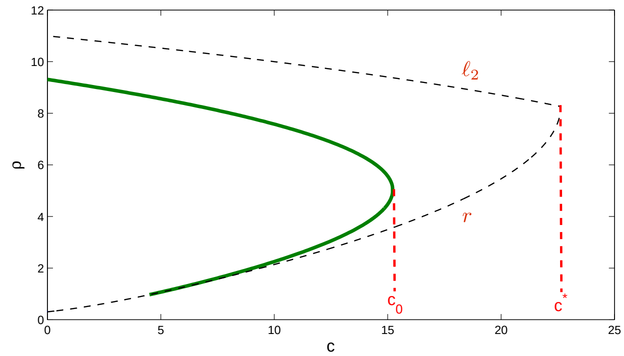

Figure 6.9

$||u||_\infty$ vs c for $a = 11$ and $b = 1$.

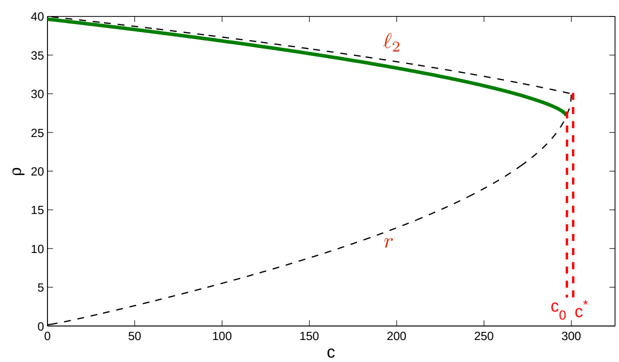

Figure 6.10

$||u||_\infty$ vs c for $a = 40$ and $b = 1$.

6.4.3 Positive solutions of (1.49) - (1.51)

Again, we are particularly interested in the case when $b = 1$. Recalling Theorem 24, we plot the level sets of

$$G_2(\rho) - \sqrt{2} = 0 \tag{6.23}$$

Using our numerical root-finding algorithm in Mathematica to solve equation (6.23) and explicit formulas to calculate the unique $r = r(a, b, c)$ and $q = q(\rho)$ values, level sets

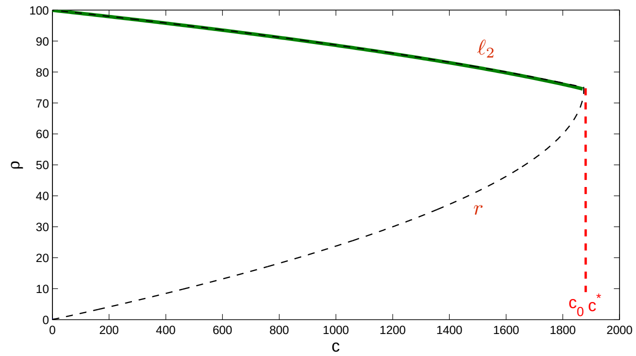

Figure 6.11

$||u||_\infty$ vs c for $a = 100$ and $b = 1$.

were plotted within $[r, \ell_2) \times [0, c^*)$. Notice that the blue curve breaks into two components somewhere around $a = 4.39$, with the lower component vanishing for $a > 10.1387$. This is due to the fact that the ρ-values, which are solutions of (6.23), are outside of $[r, \ell_2)$.

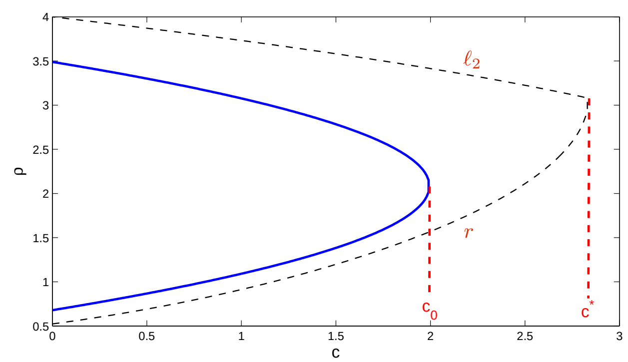

Figure 6.12

$||u||_\infty$ vs c for $a = 4$ and $b = 1$.

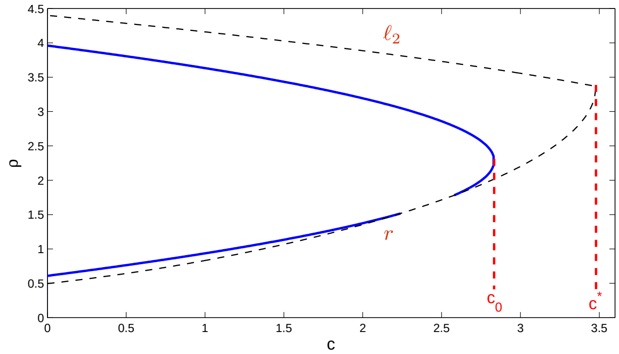

Figure 6.13

$||u||_\infty$ vs c for $a = 4.4$ and $b = 1$.

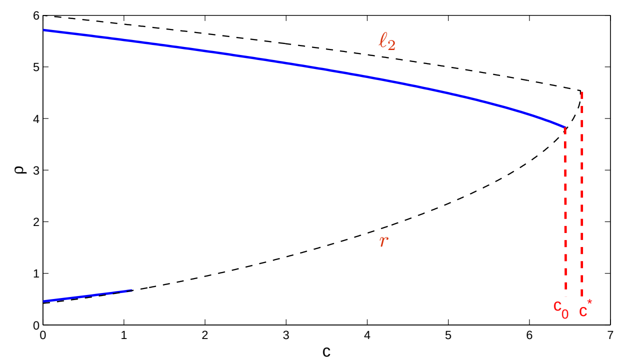

Figure 6.14

$||u||_\infty$ vs c for $a = 6$ and $b = 1$.

6.4.4 Complete structure of positive solutions for (1.37) - (1.39)

Combining results from the three cases, (1.40) - (1.42), (1.43) - (1.45), and (1.49) - (1.51) while recalling that the (1.43) - (1.45) case represents two symmetric solutions,

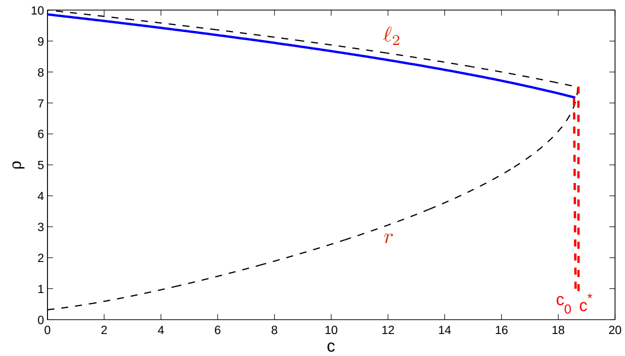

Figure 6.15

$||u||_\infty$ vs c for $a = 10$ and $b = 1$.

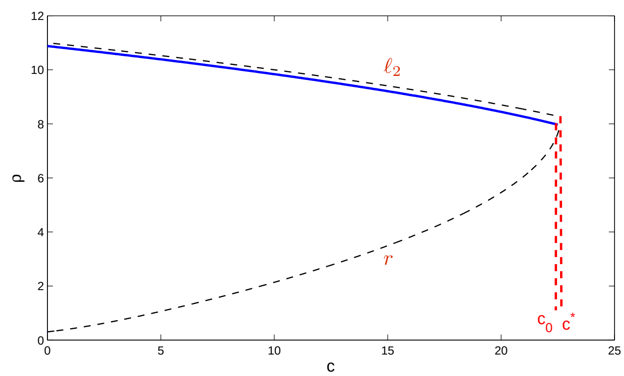

Figure 6.16

$||u||_\infty$ vs c for $a = 11$ and $b = 1$.

we are able to completely determine the structure of positive solutions to (1.37) - (1.39). As before, we are primarily interested in the case when $b = 1$. Also, our computations indicate the following existence results. For what follows, (1.40) - (1.42) is depicted in yellow, (1.43) - (1.45) and (1.46) - (1.48) both in green, and (1.49) - (1.51) in blue.

Case 1 For $b = 1$, if $a < a_1$ (for $a_1 \approx 2.8324$) then (1.37) - (1.39) has no positive solution for any $c \geq 0$.

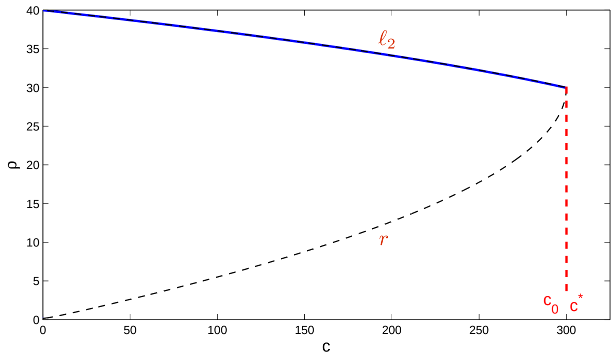

Figure 6.17

$||u||_\infty$ vs c for $a = 40$ and $b = 1$.

Case 2 For $b = 1$, if $a \in [a_1, a_2)$ (for some $a_2 > a_1$) (for $a_2 \approx 4.39$) then there exists a $c_0 > 0$ such that if

(1) $0 \leq c < c_0$ then (1.37) - (1.39) has exactly 2 positive solutions.

(2) $c = c_0$ then (1.37) - (1.39) has a unique positive solution.

(3) $c > c_0$ then (1.37) - (1.39) has no positive solution.

A bifurcation diagram of the case when $b = 1$ and $a = 4$ is shown in Figure 6.18.

Case 3 For $b = 1$, if $a \in [a_2, a_3)$ (some $a_3 \in (4.4, 5)$) then there exist $c_i > 0$, $i = 0, 1, 2$, such that if

(1) $0 \leq c \leq c_2$ or $c_1 \leq c < c_0$ then (1.37) - (1.39) has exactly 2 positive solutions.

(2) $c_2 < c < c_1$ or $c = c_0$ then (1.37) - (1.39) has a unique positive solution.

(3) $c > c_0$ then (1.37) - (1.39) has no positive solution.

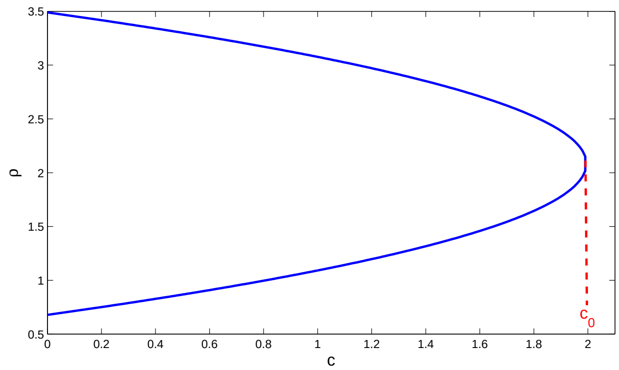

Figure 6.18

$||u||_\infty$ vs c for $a = 4$ and $b = 1$.

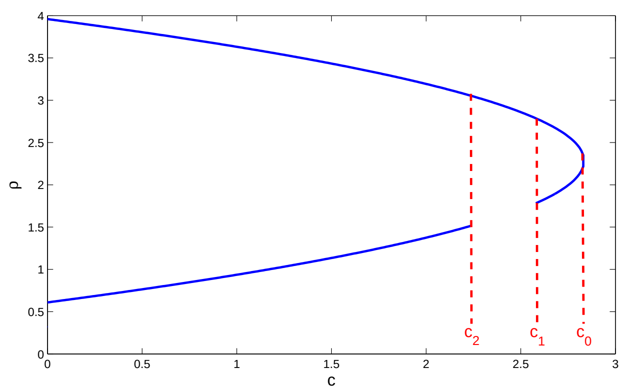

Figure 6.19

$||u||_\infty$ vs c for $a = 4.4$ and $b = 1$.

Figure 6.19 exemplifies Case 3.

Case 4 For $b = 1$, if $a \in [a_3, a_4)$ (for $a_4 \approx 5.0407$) then there exist $c_i > 0$, $i = 0, 1$, such that if

(1) $0 \leq c \leq c_1$ then (1.37) - (1.39) has exactly 2 positive solutions.

(2) $c_1 < c \leq c_0$ then (1.37) - (1.39) has a unique positive solution.

(3) $c > c_0$ then (1.37) - (1.39) has no positive solution.

Case 4 is illustrated in Figure 6.20.

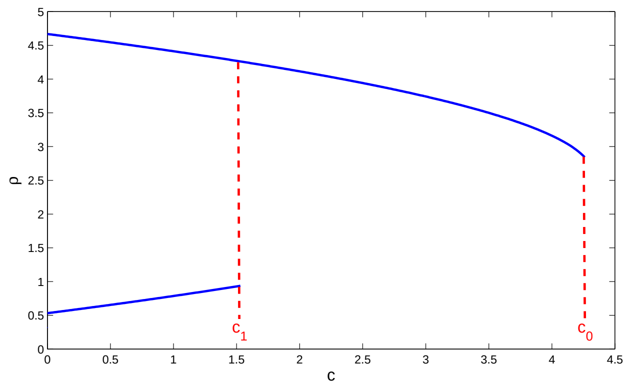

Figure 6.20

$||u||_\infty$ vs c for $a = 5.03$ and $b = 1$.

Case 5 For $b = 1$, if $a \in [a_4, a_5)$ (for $a_5 = \pi^2$) then there exist $c_i > 0$, $i = 0, 1, 2$, such that if

(1) $0 \leq c \leq c_2$ then (1.37) - (1.39) has exactly 6 positive solutions.

(2) $c_2 < c < c_1$ then (1.37) - (1.39) has exactly 5 positive solutions.

(3) $c = c_1$ then (1.37) - (1.39) has exactly 3 positive solutions.

(4) $c_1 < c \leq c_0$ then (1.37) - (1.39) has a unique positive solution.

(5) $c > c_0$ then (1.37) - (1.39) has no positive solution.

Case 5 is depicted in Figure 6.21.

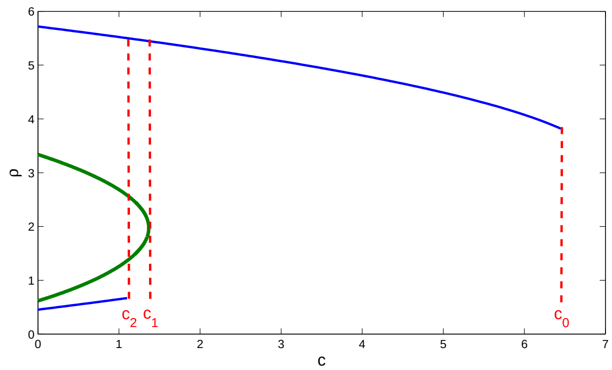

Figure 6.21

$||u||_\infty$ vs c for $a = 6$ and $b = 1$.

Case 6 For $b = 1$, if $a \in [a_5, a_6)$ (some $a_6 \in (10, 10.1388)$) then there exist $c_i > 0$, $i = 0, 1, 2, 3$, such that if

(1) $0 \leq c < c_3$ then (1.37) - (1.39) has exactly 8 positive solutions.

(2) $c = c_3$ then (1.37) - (1.39) has exactly 7 positive solutions.

(3) $c_3 < c \leq c_2$ then (1.37) - (1.39) has exactly 6 positive solutions.

(4) $c_2 < c < c_1$ then (1.37) - (1.39) has exactly 5 positive solutions.

(5) $c = c_1$ then (1.37) - (1.39) has exactly 3 positive solutions.

(6) $c_1 < c \leq c_0$ then (1.37) - (1.39) has a unique positive solution.

(7) $c > c_0$ then (1.37) - (1.39) has no positive solution.

Figure 6.22 shows the bifurcation diagram for $a = 10, b = 1$ along with Figures 6.23 and 6.24, which give two small cross sections of the diagram.

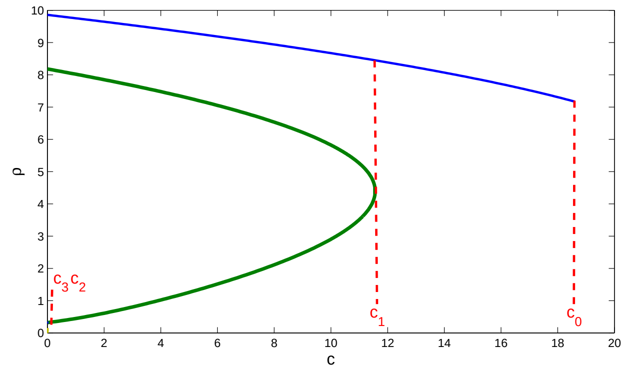

Figure 6.22

$||u||_\infty$ vs c for $a = 10$ and $b = 1$.

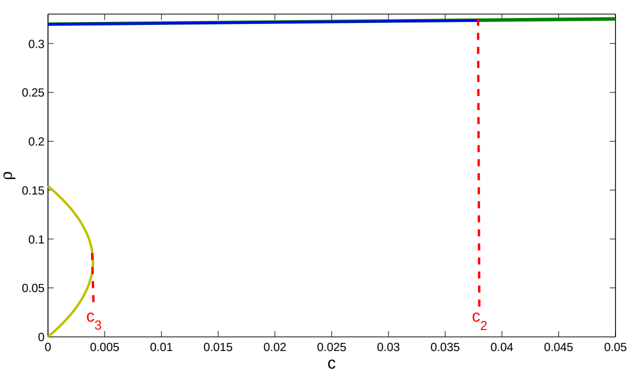

Figure 6.23

$||u||_\infty$ vs c cross-section 1 for $a = 10$ and $b = 1$.

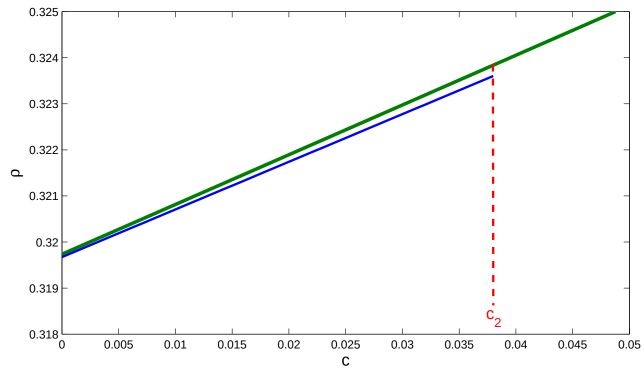

Figure 6.24

$||u||_\infty$ vs c cross-section 2 for $a = 10$ and $b = 1$.

Case 7 For $b = 1$, if $a \in [a_6, a_7)$ (for $a_7 \approx 10.1388$) then there exist $c_i > 0$, $i = 0, 1, 2, 3$, such that if

(1) $0 \leq c \leq c_3$ then (1.37) - (1.39) has exactly 8 positive solutions.

(2) $c_3 < c < c_2$ then (1.37) - (1.39) has exactly 7 positive solutions.

(3) $c_2 \leq c < c_1$ or $c = c_2$ then (1.37) - (1.39) has exactly 6 positive solutions.

(4) $c_2 < c < c_1$ then (1.37) - (1.39) has exactly 5 positive solutions.

(5) $c = c_1$ then (1.37) - (1.39) has exactly 3 positive solutions.

(6) $c_1 < c \leq c_0$ then (1.37) - (1.39) has a unique positive solution.

(7) $c > c_0$ then (1.37) - (1.39) has no positive solution.

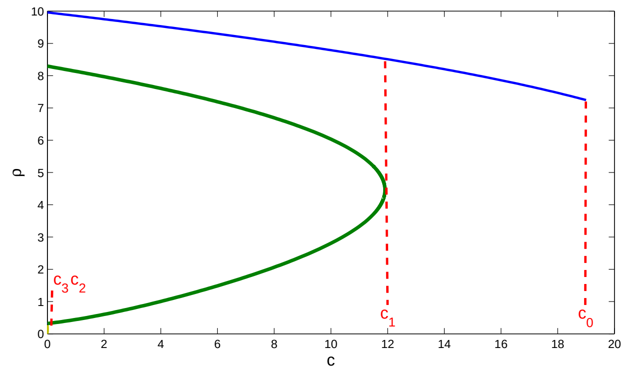

Figure 6.25

$||u||_\infty$ vs c for $a = 10.1$ and $b = 1$.

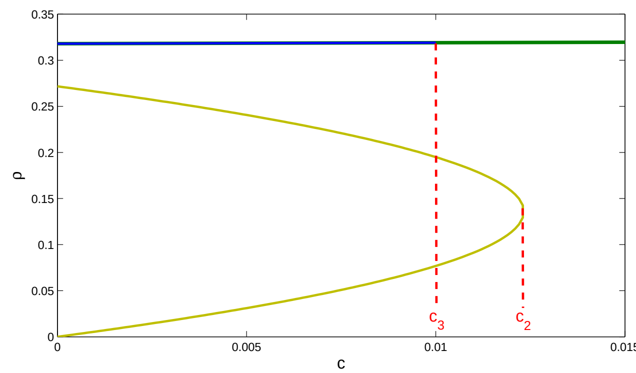

Figure 6.26

$||u||_\infty$ vs c cross-section 1 for $a = 10.1$ and $b = 1$.

The bifurcation diagram for $a = 10.1, b = 1$ is depicted in Figures 6.25 - 6.27.

Case 8 For $b = 1$, if $a \in [a_7, a_8]$ (for $a_8 = 4\pi^2$) then there exist $c_i > 0$, $i = 0, 1, 2, 3$, such that if

(1) $0 \le c < c_3$ or $c_2 \le c < c_1$ then (1.37) - (1.39) has exactly 5 positive solutions.

(2) $c = c_3$ then (1.37) - (1.39) has exactly 4 positive solutions.

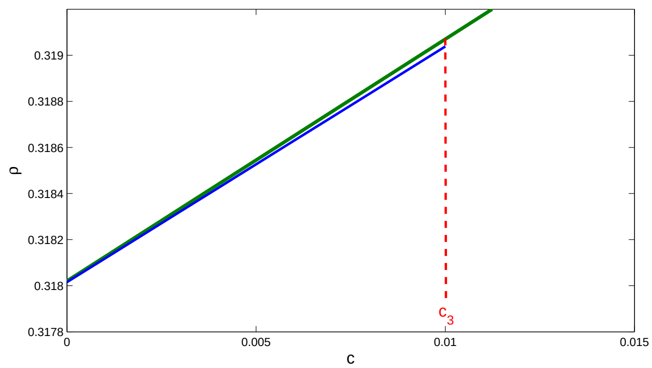

Figure 6.27

$||u||_\infty$ vs c cross-section 2 for $a = 10.1$ and $b = 1$.

(3) $c_3 < c < c_2$ or $c = c_1$ then (1.37) - (1.39) has exactly 3 positive solutions.

(4) $c_1 < c \leq c_0$ then (1.37) - (1.39) has a unique positive solution.

(5) $c > c_0$ then (1.37) - (1.39) has no positive solution.

Figure 6.28 shows the bifurcation diagram for $a = 11, b = 1$.

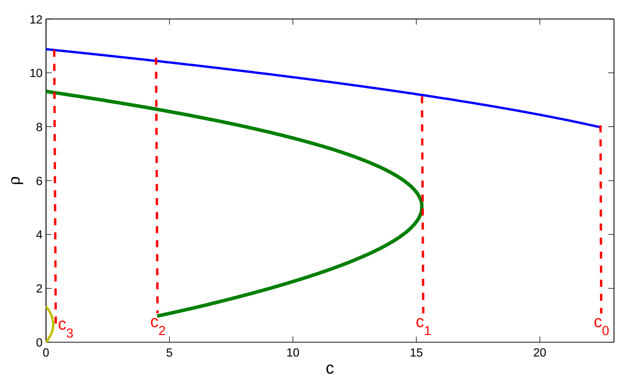

Figure 6.28

$||u||_\infty$ vs c for $a = 11$ and $b = 1$.

Case 9 For $b = 1$, if $a \in (a_8, \infty)$ then there exist $c_i > 0$, $i = 0, 1, 2, 3$, such that if

(1) $c_3 \leq c < c_2$ then (1.37) - (1.39) has exactly 5 positive solutions.

(2) $0 \leq c < c_3$ or $c = c_2$ then (1.37) - (1.39) has exactly 4 positive solutions.

(3) $c_2 < c \leq c_1$ then (1.37) - (1.39) has exactly 3 positive solutions.

(4) $c_1 < c \leq c_0$ then (1.37) - (1.39) has a unique positive solution.

(5) $c > c_0$ then (1.37) - (1.39) has no positive solution.

The bifurcation diagram for $a = 40, b = 1$ is shown in Figure 6.29.

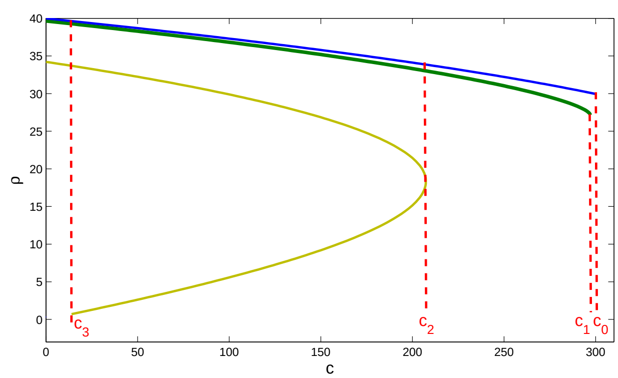

Figure 6.29

$||u||_\infty$ vs c for $a = 40$ and $b = 1$.

CHAPTER 7

COMBUSTION MODEL IN ONE-DIMENSION

In this chapter we present the proofs of Theorems 29 - 33. We prove Theorem 29 in Section 7.1, Theorems 30 and 31 in Section 7.2, Theorem 32 in Section 7.3, and Theorem 33 in Section 7.4. To conclude the chapter, we provide the complete evolution of the bifurcation curve of (1.52) - (1.54) in Section 7.5.

7.1 Proof of Theorem 29

Define $F(u) = \int_0^u f(s)ds$, the primitive of $f(u)$. Figures 7.1 and 7.2 show $f(u)$ plotted for $\beta = 2$ and $\beta = 5$, respectively. Notice that $f(u)$ is concave on $[0, \infty)$ for $\beta \in (0, 2]$. When $\beta \in (2, \infty)$, there exists a $\mu_0 \in [0, \infty)$ such that $f(u)$ is convex on $[0, \mu_0)$ and concave on (μ_0, ∞). For all $\beta > 0$, $f(u)$ is increasing on $[0, \infty]$ and bounded above by the horizonal asymptote, $y = e^{\beta}$. Also, $F(u)$ is shown in Figure 7.3. Fix $\beta \in (0, \infty)$.

($\Rightarrow$:) Suppose $u(x)$ is a positive solution to (1.55) - (1.57) with $||u||_\infty = \rho$. First note that (1.55) is an autonomous differential equation. Thus, if there exists a $x_0 \in (0, 1)$ such that

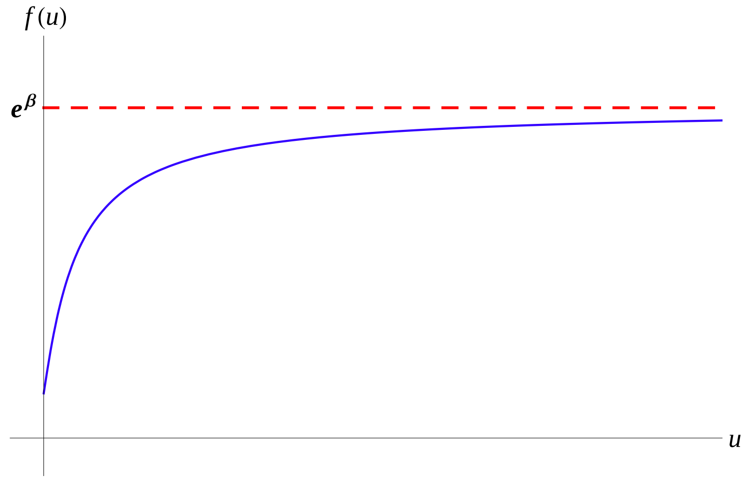

Figure 7.1

Graph of $f(u)$ when $\beta = 2$.

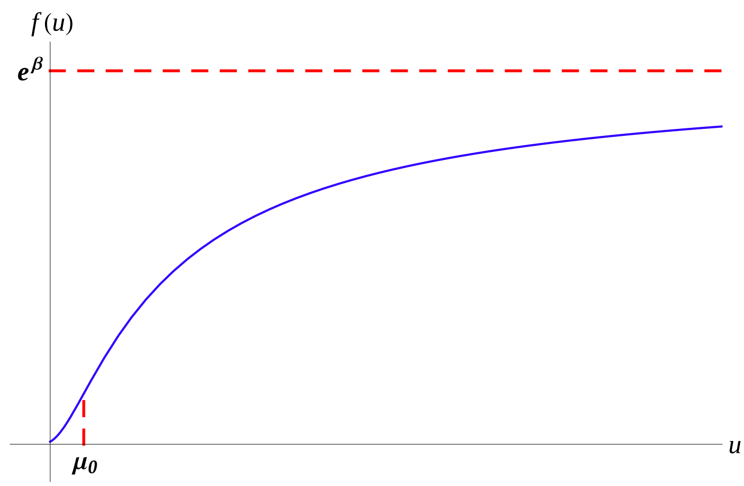

Figure 7.2

Graph of $f(u)$ when $\beta = 5$.

$u'(x_0) = 0$ then both $v(x) := u(x_0 + x)$ and $w(x) := u(x_0 - x)$ satisfy the initial value problem,

$$\begin{aligned} -z'' &= \lambda f(z) \\ z(0) &= u(x_0) \\ z'(0) &= 0 \end{aligned} \tag{7.1}$$

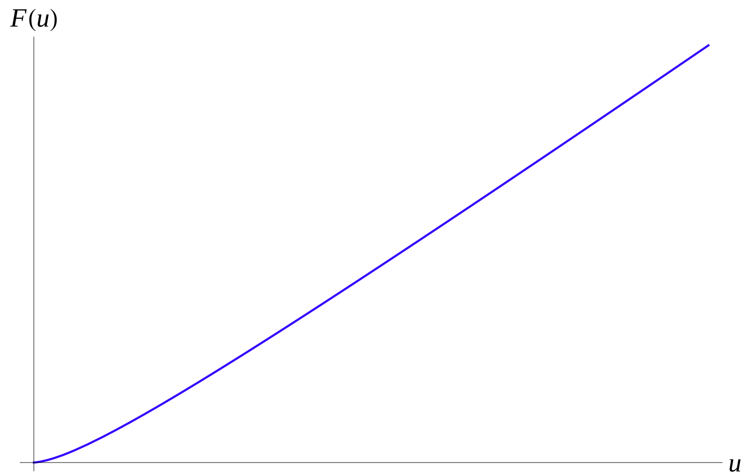

Figure 7.3

Graph of $F(u)$ when $\beta = 5$.

for all $x \in [0, d)$ where $d = \min\{x_0, 1 - x_0\}$. By Picard's Existence and Uniqueness Theorem, $u(x_0 + x) \equiv u(x_0 - x)$. Hence, $u(x)$ must be symmetric about $x_0 = \frac{1}{2}$ and $u'(x) \geq 0; x \in [0, x_0]$ while $u'(x) \leq 0; x \in [x_0, 1]$. Now, multiplying (1.55) by $u'(x)$ yields,

$$-\left[\frac{[u'(x)]^2}{2}\right]' = \lambda[F(u(x))]'. \tag{7.2}$$

Integrating throughout (7.2) from x to $\frac{1}{2}$ we have,

$$\frac{u'(x)}{\sqrt{F(\rho) - F(u(x))}} = \sqrt{2\lambda}; \quad x \in [0, \frac{1}{2}). \tag{7.3}$$

Integration of (7.3) from 0 to x gives,

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x; \quad x \in [0, \frac{1}{2}]. \tag{7.4}$$

Using the fact that $u(\frac{1}{2}) = \rho$, (7.4) becomes,

$$G_3(\rho) := \sqrt{2}\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{\lambda}. \tag{7.5}$$

($\Leftarrow$:) Suppose: there exist $\lambda, \rho \in (0, \infty)$ such that $G_3(\rho) = \sqrt{\lambda}$. Now, define $u : [0, \frac{1}{2}) \to \mathbb{R}$ by

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x. \tag{7.6}$$

We will show that $u(x)$ is a positive solution of (1.55). It follows that the left-hand side of (7.6) is a differentiable function of u which is strictly increasing from 0 to $\frac{1}{2}$ as u increases from 0 to ρ. Hence, for each $x \in [0, \frac{1}{2})$ there exists a unique $u(x)$ that satisfies

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x. \tag{7.7}$$

By the Implicit Function Theorem, $u(x)$ is differentiable as a function of x. Differentiating (7.7), we have

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]}; \quad x \in [0, \frac{1}{2}]. \tag{7.8}$$

Simplifying (7.8) gives,

$$-\frac{[u'(x)]^2}{2} = \lambda[F(u(x)) - F(\rho)]; \quad x \in [0, \frac{1}{2}]. \tag{7.9}$$

Differentiating (7.9), we have

$$-u''(x) = f(u(x)).$$

Thus, $u(x)$ satisfies the differential equation in (1.55). Also, it is clear that $u(0) = 0$. Finally, defining $u(x)$ as a symmetric function on (0, 1), gives a positive solution to (1.55) - (1.57) with $||u||_\infty = \rho$ and $u(0) = 0 = u(1)$.

Remark 3 (see [7]) $G_3(\rho)$ is well defined and the included improper integral is convergent since $f(\rho) > 0$ and $F(u)$ is strictly increasing. Moreover, $G_3(\rho)$ is a continuous and differentiable function.

■

7.2 Proof of Theorems 30 and 31

Define $F(u) = \int_0^u f(s)ds$, the primitive of $f(u)$. Using a similar argument to the one in Section 7.1, if there exists a $x_0 \in (0,1)$ such that $u'(x_0) = 0$ then $u(x)$ is symmetric about x_0. Now, assume that $u(x)$ is a positive solution of (1.58) - (1.60) with $\rho := ||u||_\infty = u(x_0)$ for some $x_0 \in (0,1)$ such that $u'(x_0) = 0$. Define $q := u(1)$. Clearly, $u'(x) \geq 0$ on $[0, x_0]$ and $u'(x) \leq 0$ on $[x_0, 1]$. Hence, $u(x)$ must resemble Figure 7.4. Fix $\beta \in (0, \infty)$. ($\Rightarrow$:) Multiplying (1.58) by u', we have $-u'u'' = \lambda f(u)u'$. Integrating with respect to x gives,

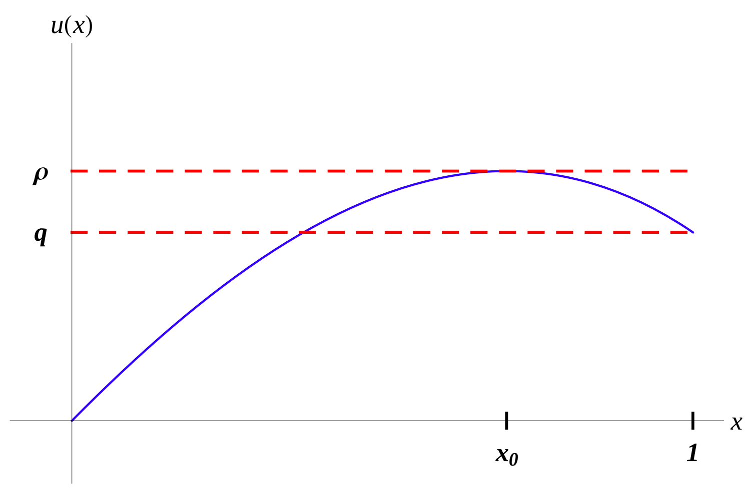

Figure 7.4

Typical solution of (1.58) - (1.60).

$$\frac{-(u')^2}{2} = \lambda F(u) + K. \tag{7.10}$$

Substituting $x = x_0$ and $x = 1$ into (7.10) while using $u'(x_0) = 0$, $u(x_0) = \rho$, $u(1) = q$, and $u'(1) = -1$ yields,

$$F(\rho) = -\frac{K}{\lambda} \tag{7.11}$$

and

$$F(q) + \frac{1}{2\lambda} = -\frac{K}{\lambda}. \tag{7.12}$$

Combining (7.11) and (7.12) gives,

$$F(\rho) = F(q) + \frac{1}{2\lambda}. \tag{7.13}$$

Substitution of (7.11) into (7.10) yields,

$$\frac{-(u')^2}{2} = \lambda(F(u) - F(\rho)). \tag{7.14}$$

Now, solving for u' in (7.14), we have

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]}; \quad x \in [0, x_0] \tag{7.15}$$

$$u'(x) = -\sqrt{2\lambda[F(\rho) - F(u(x))]}; \quad x \in [x_0, 1]. \tag{7.16}$$

Integration of (7.15) and (7.16) combined with $u(0) = 0$ and $u(x_0) = \rho$ gives,

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x; \quad x \in [0, x_0] \tag{7.17}$$

$$\int_\rho^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2\lambda}(x - x_0); \quad x \in [x_0, 1]. \tag{7.18}$$

We substitute $x = x_0$ into (7.17) and $x = 1$ into (7.18) giving

$$\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda}x_0 \tag{7.19}$$

$$\int_\rho^{q} \frac{ds}{\sqrt{F(\rho) - F(s)}} = -\sqrt{2\lambda}(1 - x_0). \tag{7.20}$$

Subtract (7.20) from (7.19) yielding,

$$2\int_0^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} - \int_0^{q} \frac{ds}{\sqrt{F(\rho)-F(s)}} \quad = \quad \sqrt{2\lambda} \tag{7.21}$$

Solving (7.14) for $\sqrt{2\lambda}$ and using $u'(1) = -1$ and $u(1) = q$, we have

$$\sqrt{2\lambda} \quad = \quad \frac{1}{\sqrt{F(\rho)-F(q)}} \tag{7.22}$$

Combining (7.22) with (7.21) we define,

$$\widetilde{G_4}(\rho, q) \quad := \quad 2\int_0^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} - \int_0^{q} \frac{ds}{\sqrt{F(\rho)-F(s)}} - \frac{1}{\sqrt{F(\rho)-F(q)}}$$

Now, for each $\rho \in (0, \infty)$, we need to find a $q = q(\rho) \in [0, \rho)$ such that $\widetilde{G_4}(\rho, q(\rho)) = 0$.

If for fixed $\rho \in (0, \infty)$ there is a unique $q(\rho) \in [0, \rho)$ with $\widetilde{G_4}(\rho, q(\rho)) = 0$ then

$$2\int_0^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} - \int_0^{q(\rho)} \frac{ds}{\sqrt{F(\rho)-F(s)}} \quad = \quad \frac{1}{\sqrt{F(\rho)-F(q(\rho))}} = \sqrt{2\lambda}$$

will be satisfied for a unique $\lambda \in (0, \infty)$. As a result, we need to analyze the existence and uniqueness of such a $q = q(\rho)$. The following lemma lists several properties of $\widetilde{G_4}(\rho, q)$.

Lemma 10

1. For every $\rho > 0$, $\widetilde{G_4}(\rho, q) \to -\infty$ as $q \to \rho$
2. For all $\rho > 0$ and $q \in [0, \rho)$ we have that $\widetilde{G_{4q}}(\rho, q) < 0$.
3. $\widetilde{G_4}(\rho, 0) \to \infty$ as $\rho \to \infty$
4. $\widetilde{G_4}(\rho, 0) \to -\infty$ as $\rho \to 0$

Proof: **(1)** Follows from the fact that $F(u)$ is increasing and the Mean Value Theorem.

(2) Fix $\rho > 0$. Then for all $q \in [0, \rho)$,

$$\widetilde{G_{4q}}(\rho, q) = -\frac{1}{\sqrt{F(\rho)-F(q)}} - \frac{f(q)}{2[F(\rho)-F(q)]^{\frac{3}{2}}} < 0.$$

(3) For all $\rho > 0$,

$$\widetilde{G_4}(\rho, 0) = 2\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} - \frac{1}{\sqrt{F(\rho)}}$$

$\Rightarrow \widetilde{G_4}(\rho, 0) \to \infty$ as $\rho \to \infty$.

(4) Again, this follows from the Mean Value Theorem and monotonicity of $F(u)$. Hence the lemma is proved. Notice that if $\widetilde{G_4}(\rho, 0) > 0$ then there will be a unique $q(\rho) \in [0, \rho)$ such that $\widetilde{G_4}(\rho, q(\rho)) = 0$. From Lemma 10, $\widetilde{G_4}(\rho, q)$ must resemble Figure 7.5. Figures 7.6 and 7.7 show what $\widetilde{G_4}(\rho, 0)$ resembles for $\beta \in (0, 4]$ and $\beta \in (4, \infty)$ respectively.

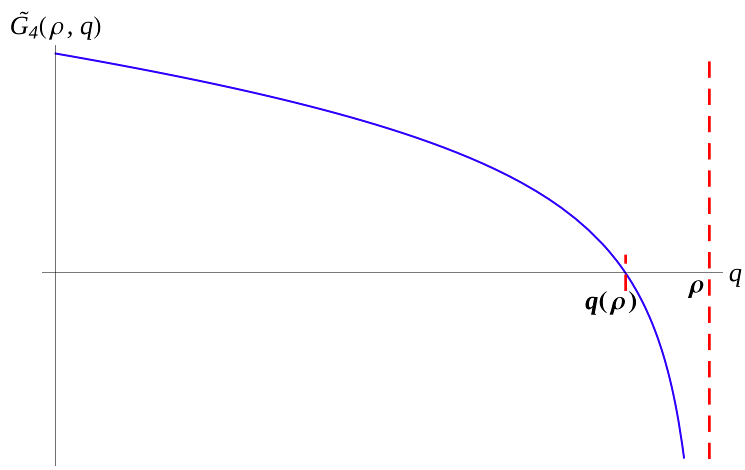

Figure 7.5

Graph of $\widetilde{G_4}(\rho, q)$.

For $\beta \in (0, 4]$ there exists a unique $\rho_0 > 0$ such that if $\rho \geq \rho_0$ then $\widetilde{G_4}(\rho, 0) \geq 0$ and if $\rho < \rho_0$ then $\widetilde{G_4}(\rho, 0) < 0$. In the second case, $\beta \in (4, \infty)$, the shape of $\widetilde{G_4}(\rho, 0)$ changes from that of the first case with the addition of both a local maximum and a local minimum. However, based on our computations, we conjecture that there exists a unique

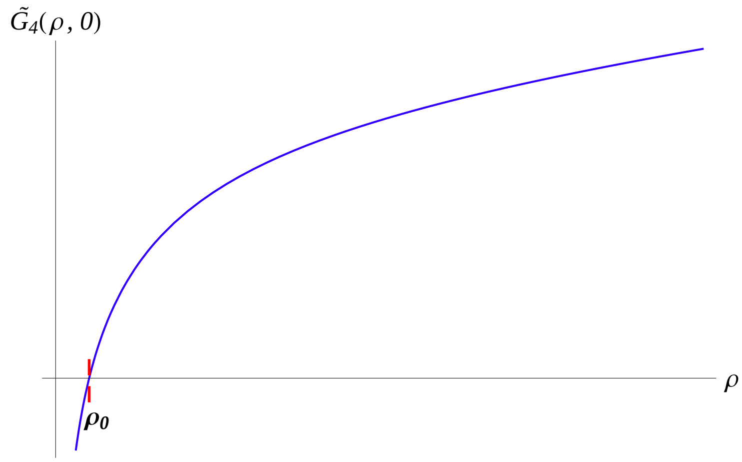

Figure 7.6

Graph of $\widetilde{G_4}(\rho, 0)$ for $\beta \in (0, 4]$.

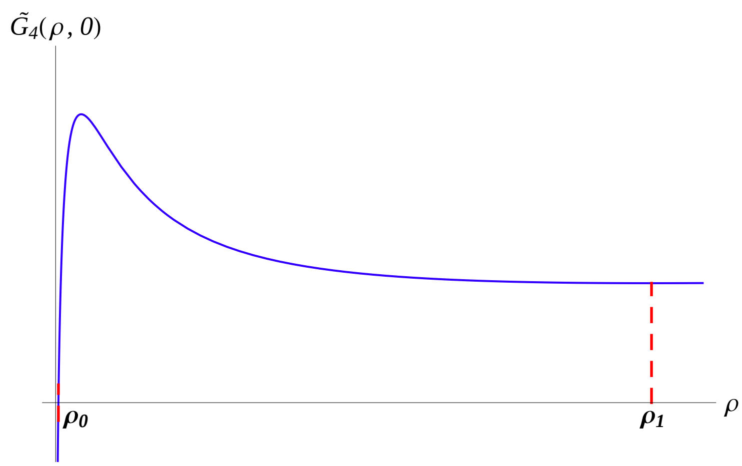

Figure 7.7

$\widetilde{G_4}(\rho, 0)$ for $\beta \in (4, \infty)$.

$\rho_0 > 0$ such that if $\rho \geq \rho_0$ then $\widetilde{G_4}(\rho, 0) \geq 0$ and if $\rho < \rho_0$ then $\widetilde{G_4}(\rho, 0) < 0$. Hence, for each $\rho \in [\rho_0, \infty)$ there is a unique $q(\rho) \in [0, \rho)$ such that $\widetilde{G_4}(\rho, q(\rho)) = 0$. Now we define

$$G_4(\rho, q(\rho)) := \frac{1}{2[F(\rho) - F(q(\rho))]}$$

for all $\rho \in [\rho_0(\beta), \infty)$ and $q(\rho) \in [0, \rho)$.

($\Leftarrow$:) Suppose: there exist $\lambda \in (0, \infty), \rho \in S(\beta)$ such that $G_4(\rho, q(\rho)) = \lambda$ where $q(\rho) \in [0, \rho)$ is the unique solution of $\widetilde{G_4}(\rho, q(\rho)) = 0$. Define $u(x) : (0, 1) \to \mathbb{R}$ by

$$\begin{aligned} \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} &= \sqrt{2\lambda} x; \quad x \in [0, x_0] \\ \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} &= -\sqrt{2\lambda}(x - x_0); \quad x \in [x_0, 1]. \end{aligned} \tag{7.23}$$

We will show that $u(x)$ is a positive solution to (1.58) - (1.60). Notice that the turning point of $u(x), x_0$, is given by

$$x_0 = \frac{1}{\sqrt{2\lambda}} \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}. \tag{7.24}$$

Clearly, for fixed λ,

$$\frac{1}{\sqrt{2\lambda}} \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}}$$

is a differentiable function of u and is strictly increasing from 0 to x_0 as u increases from 0 to ρ. Hence, for each $x \in [0, x_0]$ there exists a unique $u(x)$ such that

$$\int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} = \sqrt{2\lambda} x.$$

By the Implicit Function Theorem, $u(x)$ is differentiable with respect to x. This implies that,

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u(x))]}; \quad x \in [0, x_0]. \tag{7.25}$$

A similar argument can be made to show that

$$u'(x) = -\sqrt{2\lambda[F(\rho) - F(u(x))]}; \quad x \in [x_0, 1]. \tag{7.26}$$

From (7.25) and (7.26) we have,

$$\frac{[u'(x)]^2}{2} = \lambda[F(\rho) - F(u(x))]; \quad x \in [0, 1]. \tag{7.27}$$

Differentiating (7.27) gives,

$$-u''u' = \lambda f(u)u'; \quad x \in (0, 1)$$

$$\Rightarrow -u'' = \lambda f(u); \quad x \in (0, 1).$$

Hence, $u(x)$ satisfies (1.58). It only remains to show that $u(x)$ satisfies (1.59) and (1.60). But, it is clear that $u(0) = 0$. Also, since $G_4(\rho, q(\rho)) = \lambda$, we have

$$2\lambda = \frac{1}{F(\rho) - F(q(\rho))},$$

or equivalently,

$$F(\rho) - F(q(\rho)) = \frac{1}{2\lambda} \tag{7.28}$$

Substituting $x = 1$ into (7.26) gives,

$$u'(1) = -\sqrt{2\lambda}\sqrt{F(\rho) - F(q(\rho))}. \tag{7.29}$$

Combining (7.28) and (7.29),

$$u'(1) = -1.$$

Hence, $u(x)$ satisfies both (1.59) and (1.60).

■

7.3 Proof of Theorem 32

The proof follows easily from observing that

$$\begin{aligned} G_4(\rho, q(\rho)) &= \frac{1}{2[F(\rho) - F(q(\rho))]} \\ &= \left[\sqrt{2}\int_0^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} - \frac{1}{\sqrt{2}}\int_0^{q(\rho)} \frac{ds}{\sqrt{F(\rho)-F(s)}}\right]^2 \\ &\leq \left[\sqrt{2}\int_0^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}}\right]^2 = [G_3(\rho)]^2. \end{aligned}$$

■

7.4 Proof of Theorem 33

Fix $\beta > 0$ and let $\rho \geq \rho_0(\beta)$, $q(\rho)$, and λ be as in Theorem 31 with $G_4(\rho, q(\rho)) = \lambda$.

(a) Claim: $\rho - q(\rho) \leq \frac{e^{\beta}}{4\rho}$

With this claim, it is clear that for fixed β, $[\rho - q(\rho)] \to 0$ as $\rho \to \infty$. Now to prove the claim. Since $G_4(\rho, q(\rho)) = \lambda$, we have that

$$\int_0^{\rho} \frac{ds}{\sqrt{F(\rho)-F(s)}} + \int_{q(\rho)}^{\rho} \frac{dt}{\sqrt{F(\rho)-F(t)}} = \frac{1}{\sqrt{F(\rho)-F(q(\rho))}}.$$

By the Mean Value Theorem, there exist $\theta_1 \in (s, \rho), \theta_2 \in (t, \rho)$, and $\theta_3 \in (q(\rho), \rho)$ with $s \in (0, \rho)$ and $t \in (q(\rho), \rho)$ such that

$$\int_0^{\rho} \frac{ds}{\sqrt{f(\theta_1)}\sqrt{\rho - s}} + \int_{q(\rho)}^{\rho} \frac{dt}{\sqrt{f(\theta_2)}\sqrt{\rho - t}} = \frac{1}{\sqrt{f(\theta_3)}\sqrt{\rho - q(\rho)}}.$$

Now, since $f(u)$ is monotone increasing, we have that $f(\theta_1), f(\theta_2) \leq f(\rho)$ and

$$\frac{1}{\sqrt{f(\rho)}}\int_0^{\rho} \frac{ds}{\sqrt{\rho - s}} + \frac{1}{\sqrt{f(\rho)}}\int_{q(\rho)}^{\rho} \frac{dt}{\sqrt{\rho - t}} \leq \frac{1}{\sqrt{f(\theta_3)}\sqrt{\rho - q(\rho)}}. \tag{7.30}$$

A change of variables in the integrals of (7.30) yields,

$$\frac{\sqrt{\rho}}{\sqrt{f(\rho)}}\int_0^1 \frac{dv}{\sqrt{1-v}} + \frac{\sqrt{\rho}}{\sqrt{f(\rho)}}\int_{\frac{q}{\rho}}^1 \frac{dw}{\sqrt{1-w}} \leq \frac{1}{\sqrt{f(\theta_3)}\sqrt{\rho-q(\rho)}}. \tag{7.31}$$

This implies that,

$$\rho - q(\rho) \leq \frac{f(\rho)}{4f(\theta_3)\rho} \leq \frac{e^\beta}{4\rho} \tag{7.32}$$

since $f(\rho) \leq e^\beta$ and $f(\theta_3) \geq 1$.

(b) By the Mean Value Theorem there exists a $\theta \in (q(\rho), \rho)$ such that

$$\lambda = \frac{1}{2f(\theta)[\rho - q(\rho)]}.$$

But, $1 \leq f(\theta) \leq e^\beta$, which implies

$$\frac{1}{2e^\beta[\rho - q(\rho)]} \leq \lambda \leq \frac{1}{2[\rho - q(\rho)]}. \tag{7.33}$$

Part (a) combined with (7.33) completes (b).

(c) Finally, (7.33) combined with (7.32) yields (c).

■

7.5 Computational results

We present the structure of positive solutions of (1.55) - (1.57) in Section 7.5.1, (1.58) - (1.60) in Section 7.5.2, and (1.52) - (1.54) in Section 7.5.3.

7.5.1 Positive solutions of (1.55) - (1.57)

In this subsection, we present the evolution of the bifurcation curve for (1.55) - (1.57) suggested by our computational results. Mathematica was employed to plot $G_3(\rho)$ from

Theorem 29 for various values of β. Our results agree with those of previous authors such as [7], who was first to present them.

Case 1 (See [7]) If $\beta \in (0, \beta_0)$ (some $\beta_0 \approx 4.25$) then (1.55) - (1.57) has a unique positive solution for all $\lambda > 0$.

Figure 7.8 gives a typical bifurcation diagram for $\beta \in (0, \beta_0)$. Note that the following figures are log plots.

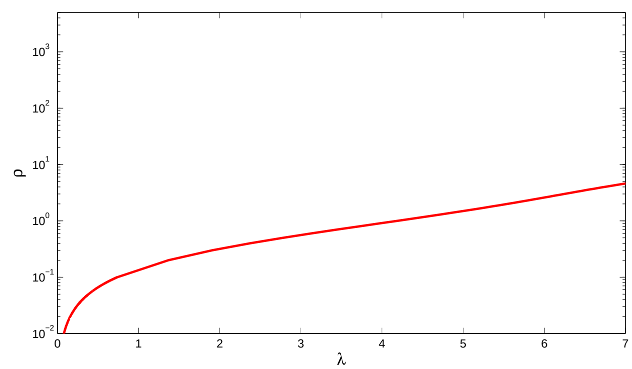

Figure 7.8

$||u||_\infty$ vs λ for $\beta = 3$.

Case 2 (See [7]) If $\beta \in (\beta_0, \infty)$ then there exist $\lambda_0, \lambda_1 > 0$ such that if

(1) $\lambda_0 < \lambda < \lambda_1$ then (1.55) - (1.57) has exactly 3 positive solutions;

(2) $\lambda = \lambda_0$ or $\lambda = \lambda_1$ then (1.55) - (1.57) has exactly 2 positive solutions;

(3) $0 < \lambda < \lambda_0$ or $\lambda > \lambda_1$ then (1.55) - (1.57) has a unique positive solution.

Figure 7.9 gives a typical bifurcation diagram for $\beta \in (\beta_0, \infty)$.

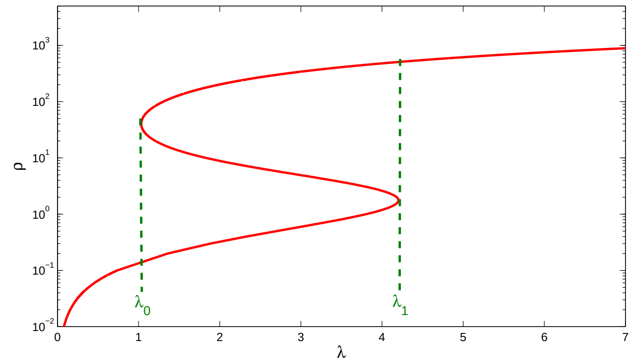

Figure 7.9

$||u||_\infty$ vs λ for $\beta = 7$.

7.5.2 Positive solutions of (1.58) - (1.60)

This subsection will present computational results for (1.58) - (1.60). In order to produce bifurcation diagrams, Mathematica was employed in a two-step process. Recalling Theorems 30 and 31 from Chapter 1, for fixed $\beta > 0$ the corresponding unique $\rho_0(\beta)$ is first found using a standard root-finding algorithm. Then for each $\rho \geq \rho_0(\beta)$, the same root-finding algorithm is employed to solve $\widetilde{G_4}(\rho, q(\rho)) = 0$ for the unique q-value. Finally, $G_4(\rho, q(\rho))$ is evaluated for the given ρ and its unique $q(\rho)$ to obtain the corresponding unique λ. The result is a bifurcation diagram portraying λ vs ρ. Due to the improper integrals in $\widetilde{G_4}(\rho, q(\rho))$, this procedure is computationally expensive. The numerical investigations suggest the following evolution of the bifurcation curve:

Case 1 For $\beta \in (0, \beta_1)$ (for some $\beta_1 < 4$), there exists a $\lambda_0 > 0$ such that if

(1) $\lambda \geq \lambda_0$ then (1.58) - (1.60) has a unique solution;

(2) $\lambda < \lambda_0$ then (1.58) - (1.60) has NO positive solution.

The following bifurcation diagram illustrates Case 1.

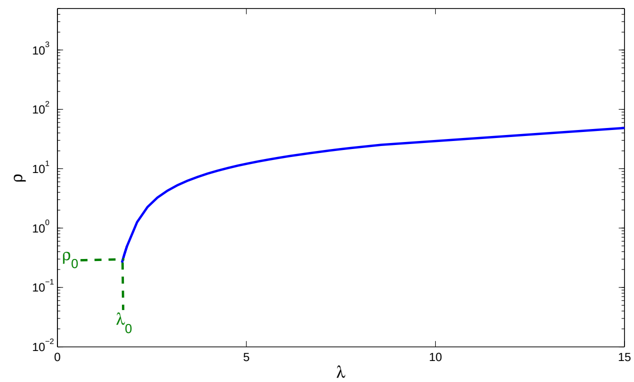

Figure 7.10

$||u||_\infty$ vs λ for $\beta = 2$.

Case 2 For $\beta \in [\beta_1, \infty)$, there exists $\lambda_0, \lambda_1, \lambda_2 > 0$ such that if

(1) $\lambda_0 \leq \lambda < \lambda_2$ then (1.58) - (1.60) has exactly 3 positive solutions;

(2) $\lambda_1 < \lambda < \lambda_0$ or $\lambda = \lambda_2$ then (1.58) - (1.60) has exactly 2 positive solutions;

(3) $\lambda > \lambda_2$ or $\lambda = \lambda_1$ then (1.58) - (1.60) has a unique positive solution.

Figure 7.11 shows a typical bifurcation curve for Case 2. Notice that (λ_0, ρ_0) corresponds to the case when $q(\rho) = u(1) = 0$ and thus satisfies both (1.55) - (1.57) and (1.58) - (1.60). We would then expect this to be the point at which the branch of solutions of (1.52) - (1.54) bifurcates into the separate cases.

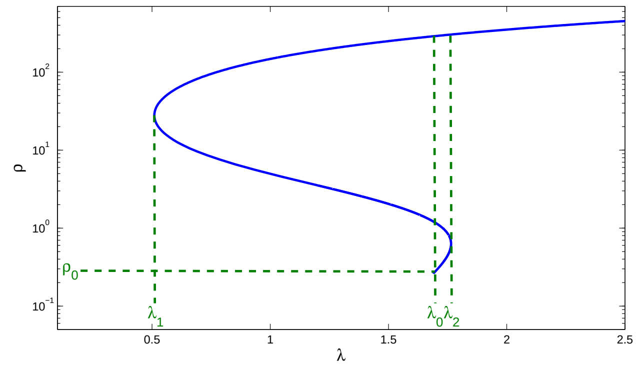

Figure 7.11

$||u||_\infty$ vs λ for $\beta = 6$.

7.5.3 Complete structure of positive solutions for (1.52) - (1.54)

To conclude the chapter, we present the computational results for (1.52) - (1.54) by combining the solutions of (1.55) - (1.57) in Section 7.5.2 and (1.58) - (1.60) in Section 7.5.3.

Theorem 35

For $\beta > 0$, (1.52) - (1.54) has at least one positive solution for every $\lambda > 0$.

Case 1 For $\beta \in (0, \beta_1)$, there exists a $\lambda_0 > 0$ such that if

(1) $\lambda > \lambda_0$ then (1.52) - (1.54) has 2 positive solutions

(2) $\lambda \leq \lambda_0$ then (1.52) - (1.54) has a unique positive solution.

Figure 7.12 shows the complete bifurcation diagram of (1.52) - (1.54) for $\beta = 2$.

Case 2 For $\beta \in [\beta_1, \beta_0)$, there exists $\lambda_0, \lambda_1, \lambda_2 > 0$ such that if

(1) $\lambda_0 < \lambda < \lambda_2$ then (1.52) - (1.54) has exactly 4 positive solutions;

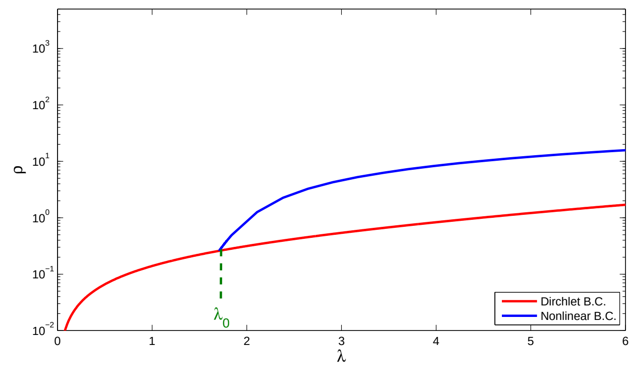

Figure 7.12

$||u||_\infty$ vs λ for $\beta = 2$.

(2) $\lambda_1 < \lambda \leq \lambda_0$ or $\lambda = \lambda_0, \lambda_2$ then (1.52) - (1.54) has exactly 3 positive solutions;

(3) $\lambda > \lambda_2$ or $\lambda = \lambda_1$ then (1.52) - (1.54) has exactly 2 positive solutions;

(4) $\lambda < \lambda_1$ then (1.52) - (1.54) has a unique positive solution.

Case 2 is illustrated in Figure 7.13.

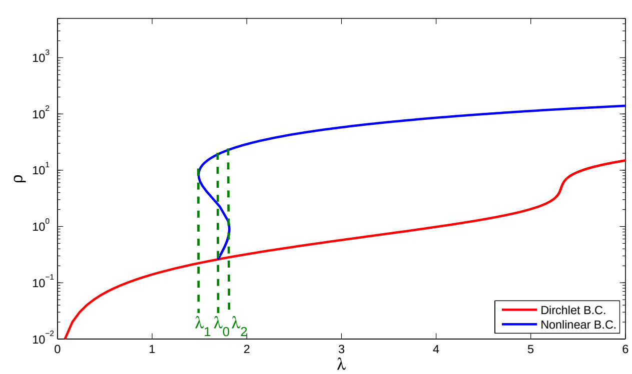

Figure 7.13

$||u||_\infty$ vs λ for $\beta = 4$.

Case 3 For $\beta \in [\beta_0, \beta_2)$ (some $\beta_2 \in (6, 6.5)$), there exists $\lambda_i > 0$ for $i = 0, 1, 2, 3, 4$ such that if

(1) $\lambda_0 < \lambda < \lambda_2$ or $\lambda_3 < \lambda < \lambda_4$ then (1.52) - (1.54) has exactly 4 positive solutions;

(2) $\lambda_1 < \lambda \leq \lambda_0$ or $\lambda = \lambda_2, \lambda_3, \lambda_4$ then (1.52) - (1.54) has exactly 3 positive solutions;

(3) $\lambda_2 < \lambda < \lambda_3$ or $\lambda > \lambda_4$ or $\lambda = \lambda_1$ then (1.52) - (1.54) has exactly 2 positive solutions;

(4) $\lambda < \lambda_1$ then (1.52) - (1.54) has a unique positive solution.

Case 3 is shown in Figure 7.14. Notice that the bifurcation diagram contains two S-shaped curves that do not overlap.

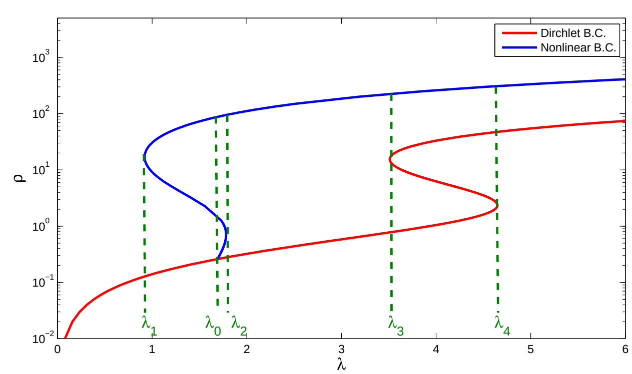

Figure 7.14

$||u||_\infty$ vs λ for $\beta = 5$.

Case 4 For $\beta \in [\beta_2, \infty)$, there exists $\lambda_i > 0$ for $i = 0, 1, 2, 3, 4$ such that if

(1) $\lambda_0 < \lambda < \lambda_3$ then (1.52) - (1.54) has exactly 6 positive solutions;

(2) $\lambda_2 < \lambda \leq \lambda_0$ or $\lambda = \lambda_3$ then (1.52) - (1.54) has exactly 5 positive solutions;

(3) $\lambda_3 < \lambda < \lambda_4$ or $\lambda = \lambda_2$ then (1.52) - (1.54) has exactly 4 positive solutions;

(4) $\lambda_1 < \lambda < \lambda_2$ or $\lambda = \lambda_4$ then (1.52) - (1.54) has exactly 3 positive solutions;

(5) $\lambda > \lambda_4$ or $\lambda = \lambda_1$ then (1.52) - (1.54) has exactly 2 positive solutions;

(6) $\lambda < \lambda_1$ then (1.52) - (1.54) has a unique positive solution.

Figure 7.15 exemplifies Case 4 with the complete bifurcation curve for $\beta = 8$. Again the double S-shape appears but in this case the S's overlap, yielding exactly 6 positive solutions for a certain range of λ.

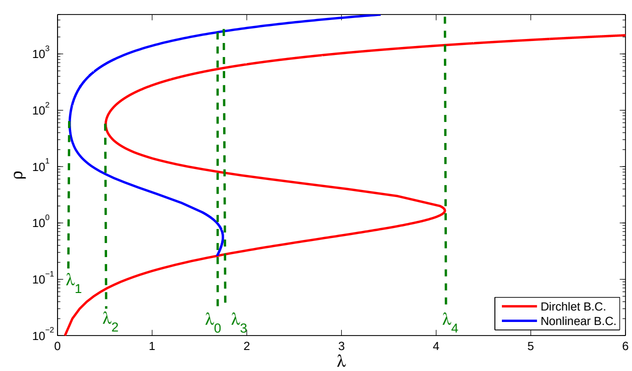

Figure 7.15

$||u||_\infty$ vs λ for $\beta = 8$.

CHAPTER 8

CONCLUSIONS AND FUTURE DIRECTIONS

8.1 Conclusions

An extensive study of the existence of positive solutions to classes of elliptic boundary value problems with nonlinear boundary conditions is initiated by this thesis. For the population dynamics case, it focused on determining the evolution of the bifurcation curve of positive solutions for one-dimensional boundary value problems and the challenging problem of constructing positive sub-solutions to higher dimensional semipositone problems when constant yield harvesting is employed. For boundary value problems arising from combustion theory, this thesis is focused on determining the structure of positive solutions for one-dimensional boundary value problems and in higher dimensions it employed degree theory to prove existence of positive solutions to boundary value problems.

8.2 Future directions

We plan to

(A) Investigate stability, uniqueness, and exact multiplicity results,

(B) Consider other nonlinear reaction terms and nonlinear boundary conditions,

(C) Expand our study to include reaction convection Biological models,

(D) Extend this study to systems.

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