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Classes of Singular Nonlinear Eigenvalue Problems with Semipositone Structure

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By

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The investigation of positive steady states to reaction diffusion models in bounded domains with Dirichlet boundary conditions has been of great interest since the 1960's. We study reaction diffusion models where the reaction term is negative at the origin. In the literature, such problems are referred to as semipositone problems and have been studied for the last 30 years. In this dissertation, we extend the theory of semipositone problems to classes of singular semipositone problems where the reaction term has singularities at certain locations in the domain. In particular, we consider problems where the reaction term approaches negative infinity at these locations. We establish several existence results when the domain is a smooth bounded region or an exterior domain. Some uniqueness results are also obtained. Our existence results are achieved by the method of sub and super solutions, while our uniqueness results are proved by establishing a priori estimates and analyzing structural properties of the solution. We also extend many of our results to systems.

Key words: Boundary value problems, Semipositone, Exterior Domains, Existence results, Uniqueness results, Method of sub and super solutions, A priori estimates

DEDICATION

To my parents, brother and Sarath

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CHAPTER 1

INTRODUCTION

We consider boundary value problems of the form:

$$\begin{cases} -\Delta u & = \lambda g(u) \text{ in } \Omega \\ \\ u & = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter, $\Delta z = \operatorname{div}\big(\nabla z\big)$ is the Laplacian of z, Ω is a smooth bounded domain in $\mathbb{R}^n$, and $g : (0, \infty) \to \mathbb{R}$ is a C^1 function. Such problems arise naturally in applications to nonlinear heat generation, combustion theory, chemical reactor theory, and population dynamics (see [6], [32], and [38]). In the case when $g(0) > 0$ (positone problems) there is a very rich history in the study of positive solutions (see [2], [8], [17], [20], [22], [25], [26], [27], [33]). In this dissertation, we will investigate positive solutions to problems of the form (1.1) when $g(0) < 0$ (semipositone case) or $\lim_{s \to 0^+} g(s) = -\infty$ (infinite semipositone case). The study of positive solutions to semipositone problems has been of great interest in the recent past (see [1], [3], [4], [5], [9], [10], [11], [12], [13], [16], [19], and [31]) and has been well documented to be mathematically challenging (see [7], [30]). Our focus will be to analyze classes of semipositone problems with singularities in the reaction term (To date, only a few results exist in this direction. See [14], [21], [24], [28], [29], [34], [36], and [40]). We will discuss existence results for (1.1) in the case $\lim_{s\to 0^+} g(s) = -\infty$, and also existence and uniqueness results for positive radial solutions to exterior domain problems of the form

$$\begin{cases} -\Delta v = \lambda K(|x|)g(v), & x \in \Omega_e \\ \quad v = 0 & \text{if } |x| = r_0 \\ \quad v \to 0 & \text{as } |x| \to \infty, \end{cases} \tag{1.2}$$

where $\lambda, \Delta v$ are as before, $|x|$ is the Euclidean norm of x, $\Omega_e = \{x \in \mathbb{R}^n \,|\, |x| > r_0\}$, $n > 2$, K belongs to a class of functions such that $\lim_{r\to\infty} K(r) = 0$, and $g : (0, \infty) \to \mathbb{R}$ is a C^1 function such that $g(0) < 0$ or $\lim_{s\to 0^+} g(s) = -\infty$. Using certain transformations (discussed in Section 2.4), equation (1.2) can be reduced to the two point boundary value problem

$$\begin{cases} -u''(t) = \lambda h(t)g(u), & 0 < t < 1 \\ \quad u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $h(t) = \frac{r_0^2}{(2-n)^2} t^{\frac{-2(n-1)}{n-2}} K(r_0 t^{\frac{1}{2-n}})$. We note here that $h(t)$ may be singular at $t = 0$ (namely, $\lim_{t\to 0} h(t) = +\infty$), depending on the function K, which will cause an added singularity.

We will extend many of our existence results to systems, and to problems involving the $p-$Laplacian operator ($\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$).

We will obtain our existence results by the method of sub and super solutions. By a subsolution of (1.1) we mean a function $\psi \in C^2(\Omega) \bigcap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} -\Delta\psi \leq \lambda g(\psi), & \text{in } \Omega \\ \psi > 0, & \text{in } \Omega \\ \psi = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

and by a supersolution of (1.1) we mean a function $Z \in C^2(\Omega) \cap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} -\Delta Z \geq \lambda g(Z), & \text{in } \Omega \\ Z > 0, & \text{in } \Omega \\ Z = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Then by the following lemma there exists a positive solution (see [2, 35, 18]).

Lemma 1

Let ψ be a subsolution of (1.1) and Z be a supersolution of (1.1) such that $\psi \leq Z$. Then (1.1) has a solution u such that $\psi \leq u \leq Z$.

The construction of a subsolution is challenging in the semipositone case (see [7] and [30]). Here our test functions for a positive subsolution must come from positive functions ψ such that $-\Delta\psi < 0$ near the boundary and $-\Delta\psi > 0$ in a large part of the interior. Infinite semipositone problems are even more challenging because in this case the subsolution must also satisfy $\lim_{x\to\partial\Omega} -\Delta\psi = -\infty$, since $\lim_{s\to 0^+} g(s) = -\infty$. We will prove our uniqueness results by establishing a priori estimates and analyzing structural properties of solutions.

In the following sections, we provide details of our results and examples of reaction terms that satisfy our hypotheses.

1.1 Existence of positive solutions for classes of infinite semipositone problems on exterior domains (Theorems 1-6)

Consider the boundary value problem of the form

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda h(s)\frac{g(u(s))}{u^{\rho}}, \quad 0 < s < 1 \\ \\ u(0) = u(1) = 0, \end{cases} \tag{1.6}$$

where λ is a positive parameter, $p > 1$, $0 \leq \rho < 1$, $g \in C([0, \infty), \mathbb{R})$ with $g(0) < 0$, and $h \in C((0, 1], (0, \infty))$ satisfies: $\exists\, \epsilon_1 > 0$, $d > 0$, and $\beta \in (0, 1 - \rho)$ such that

$$h(t) \leq \frac{d}{t^{\beta}} \text{ for all } t \in (0, \epsilon_1),$$

h may be singular at 0, and $\hat{h} = \inf_{t \in (0,1)} h(t) > 0$. A motivation for studying this boundary value problem is discussed in Section 2.4.

For the case $\rho = 0$, we assume :

(A_1) $\lim_{s \to \infty} g(s) = \infty$,

(A_2) $\lim_{s \to \infty} \frac{g(s)}{s^{p-1}} = 0$,

and prove :

Theorem 1

Let $\rho = 0$ and assume (A_1) and (A_2) are satisfied. Then (1.6) has a positive solution for $\lambda \gg 1$.

An example of a function satisfying (A_1) and (A_2) is $g(s) = s^{\gamma} - k$, where $0 < \gamma < p - 1$, and $k > 0$.

For the case $0 < \rho < 1$, we assume:

(A_3) there exist $\delta > 0, A > 0$ such that $g(s) \geq As^{\delta}$ for $s \gg 1$,

(A_4) there exist $\gamma > 0, B > 0$ such that $\gamma < \rho + p - 1$, and $g(s) \leq Bs^{\gamma}$ for all $s \geq 0$,

and prove :

Theorem 2

Let $0 < \rho < 1$ and assume (A_3) and (A_4) are satisfied. Then (1.6) has a positive solution for $\lambda \gg 1$.

An example of a function satisfying (A_3) and (A_4) is $g(s) = s^{\gamma} - k$, where $0 < \gamma < \rho + p - 1$, and $k > 0$.

Next we consider problems of the form

$$\begin{cases} -(|u'|^{p-2}u')' = h(t)[au^{p-1} - bu^{\gamma-1} - \frac{c}{u^{\rho}}], \quad 0 < t < 1 \\ u(0) = u(1) = 0. \end{cases} \tag{1.7}$$

Here a, b, c are positive constants, $p > 1, 0 \leq \rho < 1, \gamma > p$, and h is as before. Let λ_1 be the first eigenvalue of the problem $-(|\phi'|^{p-2}\phi')' = \lambda|\phi|^{p-2}\phi,\ t \in (0,1), \phi(0) = \phi(1) = 0$.

We prove :

Theorem 3

Let $a > \frac{\lambda_1}{\hat{h}}$. Then $\exists\, c^* = c^*(a, b, p, \rho)$ such that for $c < c^*$, (1.7) has a positive solution.

We also extend these results to corresponding systems. Consider

$$\begin{cases} -(|u'|^{p-2}u')' = \lambda h_1(t)\frac{g_1(v(t))}{u^{\rho}}, \quad 0 < t < 1 \\ -(|v'|^{p-2}v')' = \lambda h_2(t)\frac{g_2(u(t))}{v^{\rho}}, \quad 0 < t < 1 \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{1.8}$$

where λ is a positive parameter, $p > 1$, $0 \leq \rho < 1$, $h_1, h_2 \in C((0,1],(0,\infty))$ satisfy: $\exists\, \epsilon_1 > 0$, $d > 0$, and $\beta \in (0, 1-\rho)$ such that

$$h_i(t) \leq \frac{d}{t^\beta} \text{ for all } t \in (0,\epsilon_1) \text{ for } i = 1,2,$$

the h_i's may be singular at 0, and $\hat{h} = \min\{\inf_{t\in(0,1)} h_1(t), \inf_{t\in(0,1)} h_2(t)\} > 0$. Under the assumptions that the g_i's $i = 1,2$ are continuous and satisfy

(A_5) $\lim_{s\to\infty} g_i(s) = \infty$, $i = 1,2$,

(A_6) $\lim_{s\to\infty} \frac{g_1(M(g_2(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0$ for every $M > 0$,

(A_7) There exist $\delta > 0, A > 0$ such that $g_i(s) \geq As^\delta$ for $s \gg 1$, $i = 1,2$,

(A_8) There exist $\gamma > 0, B > 0$ such that $\gamma < \rho + p - 1$ and $g_i(s) \leq Bs^\gamma$for all $s \geq 0$, $i = 1,2$,

we establish :

Theorem 4

Let $\rho = 0$ and assume (A_5) and (A_6) are satisfied. Then (1.8) has a positive solution for $\lambda \gg 1$.

Examples of functions satisfying (A_5) and (A_6) are $g_1(s) = s^{\gamma_1} - k$, and $g_2(s) = s^{\gamma_2}$, where $k > 0$, and $\gamma_1 > 0, \gamma_2 > 0$ are such that $\gamma_1\gamma_2 < (p-1)^2$.

Theorem 5

Let $0 < \rho < 1$, and assume (A_7) and (A_8) are satisfied. Then (1.8) has a positive solution for $\lambda \gg 1$.

Examples of functions satisfying (A_7) and (A_8) are $g_1(s) = s^{\gamma_1} - k_1$, and $g_2(s) = s^{\gamma_2} - k_2$, where $k_1, k_2 > 0$, and $\gamma_i, i = 1,2$ are such that $0 < \gamma_i < p + \rho - 1$.

Finally we consider the system:

$$\begin{cases} -(|u'|^{p-2}u')' = h_1(t)[a_1u^{p-1} - b_1u^{\gamma-1} - \frac{c_1}{v^\rho}], \quad 0 < t < 1, \quad 0 < \rho < 1 \\ -(|v'|^{p-2}v')' = h_2(t)[a_2v^{p-1} - b_2v^{\gamma-1} - \frac{c_2}{u^\rho}], \quad 0 < t < 1, \quad 0 < \rho < 1 \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{1.9}$$

where a_i, b_i, c_i are positive constants, $p > 1, \gamma > p$ and the h_i's are as before. In this setting, we establish:

Theorem 6

Let $\min\{a_1, a_2\} > \frac{\lambda_1}{\hat{h}}$. *Then* $\exists\ c^* = c^*(a_i, b_i, p, \rho) > 0$ *such that* (1.9) *has a positive solution when* $\max\{c_1, c_2\} < c^*$.

1.2 Uniqueness of nonnegative solutions for semipositone problems on exterior domains (Theorem 7)

We consider the boundary value problem

$$\begin{cases} -u''(s) = \lambda h(s) f(u(s)), \quad 0 < s < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{1.10}$$

where λ is a positive parameter, and $h \in C^1((0,1], (0,\infty))$ satisfies: $\exists\ \epsilon_1 > 0$, $d > 0$, and $\beta \in (0,1)$ such that

$$h(t) \leq \frac{d}{t^\beta} \text{ for all } t \in (0, \epsilon_1),$$

h may be singular at 0, $\hat{h} = \inf_{t\in(0,1)} h(t) > 0$, and $h(s)$ is decreasing for $s > 0$. When $f \in C^1([0,\infty), \mathbb{R})$, and satisfies:

(B_1) f is increasing, $f(0) < 0$, and $\lim_{s\to\infty} f(s) = \infty$,

(B_2) $\lim_{s\to\infty} \frac{f(s)}{s} = 0$,

(B_3) f is concave,

we establish :

Theorem 7

Assume $(B_1) - (B_3)$ *are satisfied. Then* (1.10) *has a unique nonnegative solution for* $\lambda \gg 1$.

An example of a function satisfying $(B_1) - (B_3)$ is $f(s) = (s+1)^\gamma - k$, where $k > 1$, and $0 < \gamma < 1$.

1.3 Existence and uniqueness results for semipositone problems with falling zeros on exterior domains (Theorems 8-9)

We consider the boundary value problem

$$\begin{cases} -u''(s) = \lambda h(s) f(u(s)), \quad 0 < s < 1 \\ \\ \quad u(0) = u(1) = 0, \end{cases} \tag{1.11}$$

where λ is a positive parameter, $h \in C^1((0,1],(0,\infty))$ satisfies : there exist $\epsilon_1 > 0, c > 0,$ and $\beta \in (0,1)$ such that $h(t) \leq \frac{c}{t^\beta}$ for all $t \in (0,\epsilon_1)$, h may be singular at 0, h is decreasing, and $\hat{h} = \inf_{t \in (0,1)} h(t) > 0$. When $f \in C^1$ satisfies :

(C_1) there exists ρ_1, ρ_2 such that $0 < \rho_1 < \rho_2$, $f(\rho_1) = f(\rho_2) = 0$ and $f > 0$ in (ρ_1, ρ_2),

(C_2) $\int_t^{\rho_2} f(s)ds > 0$ for every $t \in [0, \rho_2)$,

we prove :

Theorem 8

Assume $(C_1) - (C_2)$ *are satisfied. Then* (1.11) *has a nonnegative solution for* $\lambda \gg 1$.

Under the additional assumption

(C_3) f is concave and $f'(s) < 0$ in $(\rho_2 - \tau, \rho_2]$ for some $\tau > 0$,

we establish :

Theorem 9

Assume (C_1)–(C_3) *are satisfied. Then* (1.11) *has a unique nonnegative solution for* $\lambda \gg 1$.

An example of a function satisfying $(C_1) - (C_3)$ is $f(s) = -s^2 + 5s - 4$.

1.4 Existence of positive solutions for classes of infinite semipositone problems with asymptotically linear growth forcing terms (Theorems 10-13)

We study the problem

$$\begin{cases} -\Delta_p u = g(\lambda, u) & \text{in } \Omega \\ \quad u \ = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.12}$$

where $g(\lambda, u) = \lambda f(u) - \frac{1}{u^\alpha}$, λ is a positive parameter, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $p > 1$, Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$, $0 < \alpha < 1$, and $f : [0, \infty) \to \mathbb{R}$ is a continuous function. Under the assumptions

(D_1) there exist $\sigma_1 > 0, k > 0$, and $s_0 > 0$ such that $f(s) \geq \sigma_1 s^{p-1} - k$ for every $0 \leq s \leq s_0$,

(D_2) $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \sigma$ for some $\sigma > 0$,

we establish :

Theorem 10

Assume $(D_1) - (D_2)$ *are satisfied. Then there exist positive constants* $s_0^*(\sigma, \Omega)$, $J(\Omega)$, $\underline{\lambda}$, *and* $\hat{\lambda}(> \underline{\lambda})$ *such that if* $s_0 \geq s_0^*$, *and* $\frac{\sigma_1}{\sigma} \geq J$, (1.12) *has a positive solution for* $\lambda \in [\underline{\lambda}, \hat{\lambda}]$.

We also extend our results to systems of the form:

$$\begin{cases} -\Delta_p u = \lambda f_1(v) - \frac{1}{u^\alpha} & \text{in } \Omega \\ -\Delta_p v = \lambda f_2(u) - \frac{1}{v^\alpha} & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where λ is a positive parameter, $\alpha \in (0,1)$, and the nonlinearities $f_i's, i = 1, 2$ are continuous, nondecreasing, and satisfy:

(D_3) There exist $\sigma_i > 0, k_i > 0$, and $s_i > 0$ such that $f_i(s) \geq \sigma_i s^{p-1} - k_i$ for every $0 \leq s \leq s_i, i = 1, 2$.

(D_4) $\lim_{s\to\infty} \frac{f_1([f_2(s)]^{p-1})}{s^{p-1}} = \sigma$ for some $\sigma > 0$.

(D_5) There exists $\tau \in \mathbb{R}$ such that for each $M > 0, f_1(Ms) \leq M^\tau f_1(s)$ for $s \gg 1$.

We prove :

Theorem 11

Assume $(D_3) - (D_5)$ are satisfied. Then there exist positive constants $s_0^*(\sigma, \Omega)$, $J^*(\Omega)$, λ_*, and $\lambda_{**}(> \lambda_*)$ such that if $\min\{s_1, s_2\} \geq s_0^*$, and $\frac{\min(\sigma_1,\sigma_2)}{\sigma^{\frac{p-1}{p-1+\tau}}} \geq J^*$, (1.13) has a positive solution for $\lambda \in [\lambda_*, \lambda_{**}]$.

We also study corresponding problems on exterior domains, which reduce to the two point boundary value problem:

$$\begin{cases} -(|u'|^{p-2}u')' = h(s)g(\lambda, u), & 0 < s < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{1.14}$$

where $g(\lambda, u)$ is as before, and $h \in C((0,1], (0,\infty))$ may be singular at 0, and satisfies:

$$\text{there exist } \epsilon_1 > 0, d > 0, \beta \in (0, 1-\alpha) \text{ such that } h(s) \leq \frac{d}{s^\beta} \text{ for all } s \in (0, \epsilon_1].$$

We establish :

Theorem 12

Assume $(D_1) - (D_2)$ *are satisfied. Then there exist positive constants* $s^*(\sigma, \Omega)$, $\bar{J}(\Omega)$, $\tilde{\underline{\lambda}}$, *and* $\hat{\lambda}(> \tilde{\underline{\lambda}})$ *such that if* $s_0 \geq s^*$, *and* $\frac{\sigma_1}{\sigma} \geq \bar{J}$, (1.14) *has a positive solution for* $\lambda \in [\tilde{\underline{\lambda}}, \hat{\lambda}]$.

Finally, we also extend these results to the systems:

$$\begin{cases} -(|u'|^{p-2}u')' = h_1(t)\big(\lambda f_1(v) - \frac{1}{u^\alpha}\big), \quad 0 < t < 1 \\ -(|v'|^{p-2}v')' = h_2(t)\big(\lambda f_2(u) - \frac{1}{v^\alpha}\big), \quad 0 < t < 1 \\ u(0) = u(1) = 0, \quad v(0) = v(1) = 0, \end{cases} \tag{1.15}$$

where λ, α, f_i's are as before, and h_i's $\in C((0,1], (0,\infty))$ may be singular at 0, and satisfy:

$$\text{there exist } \epsilon > 0, d > 0, \text{ and } \beta \in (0, 1-\alpha) \text{ such that } h_i(s) \leq \frac{d}{s^\beta} \text{ for all } s \in (0, \epsilon],$$

i=1, 2. We prove :

Theorem 13

Assume $(D_3) - (D_5)$ *are satisfied. Then there exist positive constants* $s^*(\sigma, \Omega)$, $\bar{J}^*(\Omega)$, $\tilde{\lambda_*}$, *and* $\lambda_{**}(> \tilde{\lambda_*})$ *such that if* $\min\{s_1, s_2\} \geq s^*$, *and* $\frac{\min(\sigma_1, \sigma_2)}{\sigma^{\frac{p-1}{p-1+\tau}}} \geq \bar{J}^*$, (1.15) *has a positive solution for* $\lambda \in [\tilde{\lambda_*}, \lambda_{**}]$.

Here we give an example of a function satisfying our hypotheses for Theorem 10. Note that the same example satisfies the hypotheses of Theorem 12. Consider the function $f(s, m_0) = \sigma s^{p-1} + m_0 s^\gamma - k$ where $\sigma > 0, m_0 > 0, p > 1$, $\gamma \in (0, p-1)$ and k is a real number. Now let $s_0 = (\frac{m_0}{m_0^\nu - \sigma})^{\frac{1}{p-1-\gamma}}$ for some $\nu \in (0,1)$. Then for every $0 \leq s \leq s_0$, $m_0 \geq m_0^\nu s^{p-1-\gamma} - \sigma s^{p-1-\gamma}$. Multiplying by s^γ we see that

$$\sigma s^{p-1} + m_0 s^\gamma \geq m_0^\nu s^{p-1}.$$

This implies $f(s) \geq \sigma_1 s^{p-1} - k$ for every $0 \leq s \leq s_0$ where $\sigma_1 = m_0^{\nu}$. Hence (D_1) is satisfied. Also f satisfies (D_2) since $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \sigma$. Clearly, when m_0 is large s_0 and $\frac{\sigma_1}{\sigma}$ are also large and hence Theorem 10 holds. In particular if $\lambda \in [\frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{m_0^{\nu}}, \frac{1}{2\sigma||e_p||_{\infty}^{p-1}}]$, (1.12) has a positive solution. Note that $\frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{m_0^{\nu}} \to 0$ as $m_0 \to \infty$ and hence this interval is nonempty when the constant m_0 in f is large enough. In fact given a $\lambda \in (0, \frac{1}{2\sigma||e_p||_{\infty}^{p-1}}]$, there exists $m^*(\lambda)$ such that if $m_0 > m^*(\lambda)$, (1.12) has a positive solution.

We now give examples of functions satisfying our hypotheses for Theorem 11. Here again we note that the same examples satisfy the hypotheses for Theorem 13. Consider $f_1(s) = s^{p-1}$ and $f_2(s, a, b) = as^{\frac{1}{p-1}} + bs^{\gamma} - k$ where $p > 1, a, b > 0, 0 < \gamma < \frac{1}{p-1}$, and k is a real number. Clearly f_1 satisfies (D_3) and (D_5) with $\sigma_1 = 1, s_1 = \infty$ and $\tau = p - 1$. Now, set $s_2 = (b^{1-\nu})^{\frac{1}{p-1-\gamma}}$, for some $\nu \in (0, 1)$. This implies for $s \leq s_2$, $bs^{\gamma} \geq b^{\nu} s^{p-1}$. Thus, f_2 satisfies (D_3) with $\sigma_2 = b^{\nu}$. Also, $\lim_{s\to\infty} \frac{f_1(f_2(s)^{p-1})}{s^{p-1}} = a^{(p-1)^2}$. Next when $b \gg 1, \min\{s_1, s_2\} = s_2$ is large and $\frac{\min\{\sigma_1,\sigma_2\}}{\sigma^{\frac{p-1}{p-1+\tau}}} = \frac{1}{a^{\frac{(p-1)^3}{p-1+\tau}}}$. Hence when b is large and a is small the hypotheses of Theorem 11 hold and we obtain a nonempty interval of λ where a positive solution exists.

1.5 Existence results for classes of infinite semipositone problems with falling zeros (Theorems 14-16)

We study positive solutions to the boundary value problem

$$\begin{cases} -\Delta_p u = \dfrac{au^{p-1} - bu^{\gamma-1} - c}{u^{\alpha}}, & x \in \Omega \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$, $a > 0, b > 0, c \geq 0$, and $\alpha \in (0, 1), p > 1$, and $\gamma > p$. For (1.16), we prove:

Theorem 14

Given $a, b > 0, \gamma > p$, and $\alpha \in (0,1)$, there exists a $c_1 = c_1(a, b, \alpha, p, \gamma, \Omega) > 0$ such that for $c < c_1$, (1.16) has a positive solution.

Next we study this problem on an exterior domain. Namely, we consider

$$\begin{cases} -(|u'|^{p-2}u')' = h(s)(\frac{au^{p-1}-bu^{\gamma-1}-c}{u^\alpha}), \quad 0 < s < 1 \\ \\ u(0) = u(1) = 0, \end{cases} \tag{1.17}$$

where $a, b, c, \alpha, p, \gamma$ are as before and $h \in C((0,1],(0,\infty))$ may be singular at 0, $\hat{h} = \inf_{t\in(0,1)} h(t) > 0$, and satisfies: there exists $\epsilon_1 > 0$, $d > 0$, and $\beta \in (0, 1-\alpha)$ such that

$$h(t) \leq \frac{d}{t^\beta} \text{ for all } t \in (0, \epsilon_1).$$

Then we prove:

Theorem 15

Given $a, b > 0, \gamma > p$, and $\alpha \in (0,1)$, there exists a $c_2 = c_2(a, b, \alpha, p, \gamma)$ such that for $c < c_3$, (1.17) has a positive solution.

We also discuss a bifurcation result for the problem

$$\begin{cases} -\Delta_p u = \dfrac{au^{p-1} - bu^{\gamma-1}}{u^\alpha}, & x \in \Omega \\ \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.18}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, a is a positive parameter, $b, \alpha > 0$, $p > 1 + \alpha$ and $\gamma > p$. We prove:

Theorem 16

The boundary value problem (1.18) has a branch of positive solutions bifurcating from the trivial branch of solutions $(a, 0)$ at $(0, 0)$.

Now we provide an outline of this thesis. In Chapter 2, we introduce some preliminary results, which are needed for establishing our theorems. Proofs of the results stated in Section 1.1 are provided in Chapter 3. In Chapter 4, we present the proof of the uniqueness result discussed in Section 1.2. Proofs of the results in Section 1.3 are provided in Chapter 5. Chapter 6 contains proofs of the results in Section 1.4. In Chapter 7, the results in Section 1.5 are proved. We provide some computational results for (1.12), (1.16), and (1.18) in the one dimensional case in Chapter 8. Conclusions and future directions are discussed in Chapter 9.

CHAPTER 2

PRELIMINARIES

In this chapter we provide some preliminary results which will be used to establish our main theorems. In particular, we will discuss maximum principles, anti maximum principles, the method of sub and super solutions, a sweeping principle, and the reduction of an exterior domain problem to a two point boundary value problem.

2.1 Maximum and anti maximum principles

For the following, we assume that Ω is a smooth bounded domain in $\mathbb{R}^n$ and $u \in C^2(\Omega) \cap C(\overline{\Omega})$.

Lemma 2 (Maximum principle)

Let $\Delta u \geq 0$ in Ω. If u attains its maximum M at some interior point in Ω, then $u \equiv M$ in Ω.

Lemma 3 (Hopf's maximum principle)

Let $\Delta u \geq 0$ in Ω. Suppose that $u \leq M$ in Ω and $u = M$ at some $p \in \partial\Omega$. Then $\dfrac{\partial u}{\partial \nu} > 0$ at p unless $u \equiv M$ where $\dfrac{\partial}{\partial \nu}$ denotes the outward normal derivative.

Lemma 4 (Anti-maximum principle, Clement and Peletier [15])

Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. Then there exists a $\delta = \delta(\Omega) > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta)$, the problem

$$\begin{cases} -\Delta z - \lambda z = -1, \quad x \in \Omega \\ \qquad\qquad z = 0, \quad x \in \partial\Omega, \end{cases} \tag{2.1}$$

has a solution z_λ such that $z_\lambda > 0$ in Ω and $\dfrac{\partial z_\lambda}{\partial \nu} < 0$ on $\partial\Omega$, where ν is the outer unit normal to Ω.

Maximum and anti maximum principles also hold when the Laplacian is replaced by a more general operator, the $p-$Laplacian, $\Delta_p z = div(|\nabla z|^{p-2}\nabla z)$ (see [37], [39]).

2.2 The method of sub and super solutions

Consider

$$\begin{cases} -\Delta_p u = \lambda g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where λ is a positive parameter, $\Delta_p u = div(|\nabla u|^{p-2}\nabla u), p > 1$, Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with smooth boundary $\partial\Omega$. We use the following definition of sub and super solutions. Let $W^{1,p}(\Omega)$ denote the set of all functions $u \in L^p(\Omega)$ such that the weak derivative Du is in $L^p(\Omega)$. By a subsolution of (2.2) we mean a function $\psi \in W^{1,p}(\Omega) \cap C(\bar{\Omega})$ that satisfies

$$\begin{cases} \int_\Omega |\nabla\psi|^{p-2}\nabla\psi.\nabla w \leq \lambda \int_\Omega g(\psi)w, & \text{for every } w \in W \\ \psi > 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

and by a supersolution we mean a function $Z \in W^{1,p}(\Omega) \bigcap C(\bar{\Omega})$ that satisfies:

$$\begin{cases} \int_\Omega |\nabla Z|^{p-2}\nabla Z.\nabla w \geq \lambda \int_\Omega g(Z)w, & \text{for every } w \in W \\ Z > 0 & \text{in } \Omega \\ Z = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4}$$

where $W = \{\xi \in C_0^\infty(\Omega) : \xi \geq 0 \text{ in } \Omega\}$. Then the following lemma holds.

Lemma 5

(see [2, 28, 35, 18]) *Let ψ be a subsolution of (2.2) and Z be a supersolution of (2.2) such that $\psi \leq Z$ in Ω. Then (2.2) has a solution u such that $\psi \leq u \leq Z$ in Ω.*

For problems of the form

$$\begin{cases} -u''(s) = \lambda h(s)g(u(s)), & 0 < s < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{2.5}$$

where λ is a positive parameter, $g \in C^1([0,\infty),\mathbb{R})$, $h \in C^1((0,1],(0,\infty))$, and h may be singular at 0, we also use the following definition of sub and super solutions. Here we do not require the sub and super solutions to be strictly positive in the interior. By a subsolution of (2.5) we mean a function $\psi \in W^{1,2}(0,1) \bigcap C[0,1]$ that satisfies:

$$\begin{cases} \int_0^1 -\psi\phi'' \leq \lambda \int_0^1 h(t)g(\psi)\phi, & \text{for every } \phi \in V \\ \\ \quad \psi(0) \leq 0, \psi(1) \leq 0, \end{cases} \tag{2.6}$$

and by a supersolution we mean a function $Z \in W^{1,2}(0,1) \bigcap C[0,1]$ that satisfies:

$$\begin{cases} \int_0^1 -Z\phi'' \geq \lambda \int_0^1 h(t)g(Z)\phi, & \text{for every } \phi \in V \\ \\ \quad Z(0) \geq 0, Z(1) \geq 0, \end{cases} \tag{2.7}$$

where $V = \{\zeta \in C_0^\infty(0,1) : \zeta \geq 0 \text{ in } (0,1)\}$. Then we have the following lemma (see [23]).

Lemma 6

Let ψ be a subsolution and Z be a supersolution such that $\psi \leq Z$ in $(0,1)$. Then (2.5) has a solution $u \in C^2((0,1)) \cap C^1([0,1])$ such that $\psi \leq u \leq Z$ in $(0,1)$.

2.3 A sweeping principle

Here we state and prove a version of a sweeping principle for the problem

$$\begin{cases} -u''(s) = \lambda h(s) f(u(s)), & 0 < s < 1 \\ \\ \quad u(0) = u(1) = 0, \end{cases} \tag{2.8}$$

where λ is a positive parameter, $g \in C^1([0,\infty),\mathbb{R})$, $h \in C^1((0,1],(0,\infty))$, and h may be singular at 0.

Lemma 7

Let u be a solution of (2.8), B be a connected topological space and let $A = \{w_t : t \in B\}$ be a family of subsolutions satisfying $w_t(x) < 0$ at $x = 0, 1$ for all $t \in B$. If

- *$t \to w_t$ is continuous with respect to $||.||_\infty$ and*
- *$w_{t_0} \le u$ in $[0,1]$ for some $t_0 \in B$,*

then $w_t \le u$ for all $t \in B$.

Proof: Set $I = \{t \in B : w_t \le u \text{ in } [0,1]\}$. I is nonempty as $w_{t_0} \le u$ in $[0,1]$. We will show that I is both closed and open. Then the connectedness of B would imply that $I = B$. Clearly I is closed since $t \to w_t$ is continuous with respect to $||.||_\infty$. In order to show that I is open we will prove that every point in I is an interior point. Let $t \in I$ be given. Then $\int_0^1 -(w_t - u)\phi'' \le \lambda \int_0^1 h(x)[f(w_t) - f(u)]\phi,$ for every $\phi \in V$ and $w_t(x) - u(x) < -\xi_t$ for some $\xi_t > 0$ at $x = 0, 1$. Define

$$g(x) = \begin{cases} \dfrac{f(w_t(x)) - f(u(x))}{w_t(x) - u(x)}; & w_t(x) \neq u(x) \\ \frac{\partial}{\partial x} f(w_t(x)) \qquad ; & w_t(x) = u(x). \end{cases}$$

Then

$$\begin{aligned} \int_0^1 -(w_t + \xi - u)\phi'' \;\; \le \;\; & \lambda \int_0^1 h(x)[g^+ - g^-][w_t + \xi - u]\phi - \xi \int_0^1 \phi'' \\ & -\xi\lambda \int_0^1 h(x)[g^+ - g^-]\phi, \forall \phi \in V, x \in [0,1], \end{aligned}$$

and $w_t(x) + \xi - u(x) < 0$ at $x = 0, 1$ for all $\xi < \xi_t$. Rearranging the terms we have

$$\begin{aligned} \int_0^1 -(w_t + \xi - u)\phi'' - \int_0^1 h(x)g^+(w_t + \xi - u)\phi \;\; \le \;\; & -\lambda \int_0^1 h(x)g^-(w_t + \xi - u) - \xi \\ & \int_0^1 \phi'' - \xi\lambda \int_0^1 h(x)[g^+ - g^-]\phi, \end{aligned}$$

for all $\phi \in V, x \in [0,1]$. Now for ξ small enough we have

$$\int_0^1 -(w_t + \xi - u)\phi'' - \int_0^1 h(x)g^+(w_t + \xi - u)\phi \le 0,$$

$\forall\phi \in V, x \in [0,1]$ and $w_t(x) + \xi - u(x) < 0$ at $x = 0, 1$. By the weak maximum principle we obtain $w_t(x) + \xi - u(x) \le 0$ on $[0,1]$. Hence $w_t(x) < u(x)$ in $[0,1]$. This implies that t is in the interior of I. Thus I is both closed and open and therefore $I = B$ i.e., $w_t \le u$ for all $t \in B$.

2.4 The reduction of an exterior domain problem to a two point boundary value problem

Consider the problem

$$\begin{cases} -\Delta u &= \lambda K(|x|) f(u), \quad x \in \Omega \\ u &= 0, \qquad \text{if} \quad |x| = r_0 \\ u &\to 0 \qquad \text{as} \quad |x| \to \infty, \end{cases} \tag{2.9}$$

where $K : [r_0, \infty) \to (0, \infty)$ is continuous, $\Omega = \{x \in \mathbb{R}^n | \, |x| > r_0\}$, $n > 2$, and $f : (0, \infty) \to \mathbb{R}$ is continuous. We set $r = |x| = \sqrt{x_1^2 + x_2^2 + ... + x_n^2}$ and $v(r) = u(x)$. Then

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r)$$

which reduces (2.9) to the following:

$$\begin{cases} -v''(r) - \frac{n-1}{r} v'(r) &= \lambda K(r) f(v(r)), \quad r_0 < r < \infty \\ v(r_0) = 0, \quad v(r) &\to 0, \qquad \text{as} \quad r \to \infty. \end{cases} \tag{2.10}$$

Now set $s = (\frac{r}{r_0})^{2-n}$ and $z(s) = v(r)$, then

$$-v''(r) - \frac{n-1}{r} v'(r) = -\frac{(2-n)^2}{r_0^2} s^{\frac{2(1-n)}{2-n}} z''(s).$$

This reduces the problem (2.10) to the following boundary value problem,

$$\begin{cases} -z'' &= \lambda h(s) f(z(s)), \quad 0 < s < 1 \\ z(0) &= z(1) = 0, \end{cases} \tag{2.11}$$

where $h(s) = \frac{r_0^2}{(2-n)^2} s^{\frac{-2(n-1)}{n-2}} K(r_0 s^{\frac{1}{2-n}})$. Thus studying positive radial solutions to the problem (2.9) is equivalent to studying positive solutions to (2.11).

In a very similar way, by using the transformations $r = |x|$, $s = (\frac{r}{r_0})^{\frac{-n+p}{p-1}}$, we can reduce the problem

$$\begin{cases} -\Delta_p u &= \lambda K(|x|) f(u), \quad x \in \Omega \\ u &= 0, \qquad \text{if} \quad |x| = r_0 \\ u &\to 0 \qquad \text{as} \quad |x| \to \infty \end{cases} \tag{2.12}$$

to the two point boundary value problem

$$\begin{cases} -(|u'|^{p-2}u')' = h(s) f(u), \quad 0 < s < 1 \\ u(0) = u(1) = 0, \end{cases} \tag{2.13}$$

where $h(s) = (\frac{p-1}{n-p})^p r_0^p s^{\frac{-p(n-1)}{n-p}} K(r_0 s^{\frac{-(p-1)}{n-p}})$.

CHAPTER 3

PROOFS OF THEOREMS 1-6

3.1 Proof of Theorem 1

Consider

$$-(|\phi'|^{p-2}\phi')' = \lambda|\phi|^{p-2}\phi,\ t \in (0,1), \phi(0) = \phi(1) = 0. \tag{3.1}$$

Let $\phi_1 \in C^2[0,1]$ be an eigenfunction corresponding to the first eigenvalue λ_1 of (3.1) such that $\phi_1 > 0$ and $||\phi_1||_\infty = 1$. Then there exist $d_1 > 0$ such that

$$0 < \phi_1(t) \leq d_1 t(1-t) \text{ for } t \in (0,1).$$

Let $\alpha \in (1, \frac{p-\beta}{p-1})$, $\epsilon < \epsilon_1$, $m > 0$ and $\mu > 0$ be such that

$$-m > [\lambda_1 \alpha^{p-1}\phi_1^p - \alpha^{p-1}(\alpha-1)(p-1)\,|\phi_1'|^p] \ \text{ in } (0,\epsilon] \cup [1-\epsilon, 1)$$

and $\phi_1 > \mu$ in $(\epsilon, 1-\epsilon)$. This is possible since $\phi_1 = 0$ and $|\phi_1'| > 0$ at $t = 0, 1$. Define $\psi = \lambda k_0 \phi_1^\alpha$ where $-k_0 < \frac{d_1^{p-\alpha(p-1)}d}{m} \min_{t\in[0,\infty)} g(t)$. Then

$$\begin{aligned}
\psi' &= \lambda k_0 \alpha(\phi_1)^{\alpha-1}\phi_1', \\
-(|\psi'|^{p-2}\psi')' &= -\lambda k_0^{p-1}\alpha^{p-1}(\alpha-1)(p-1)\phi_1^{(\alpha-1)(p-1)-1}\,|\phi_1'|^p \\
&\quad -\lambda k_0^{p-1}\alpha^{p-1}\phi_1^{(\alpha-1)(p-1)}(|\phi_1'|^{p-2}\phi_1')' \\
&= \lambda[\lambda_1 k_0^{p-1}\alpha^{p-1}\phi_1^{\alpha(p-1)} - k_0^{p-1}\alpha^{p-1}(\alpha-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\alpha(p-1)}}].
\end{aligned}$$

For $t \in (0, \epsilon]$,

$$
\begin{aligned}
-(|\psi'|^{p-2}\psi')' &= \lambda \frac{k_0^{p-1}}{\phi_1^{p-\alpha(p-1)}}[\lambda_1 \alpha^{p-1}\phi_1^p - \alpha^{p-1}(\alpha-1)(p-1)\,|\phi_1'|^p] \\
&\leq -\lambda \frac{k_0}{d_1^{p-\alpha(p-1)} t^{p-\alpha(p-1)}} m \leq -\lambda \frac{k_0}{d_1^{p-\alpha(p-1)} t^{\beta}} m \leq -\lambda \frac{k_0 h(t)}{d_1^{p-\alpha(p-1)} d} m \\
&\leq \lambda h(t) \min_{t\in[0,\infty)} g(t) \leq \lambda h(t) g(\psi).
\end{aligned}
$$

Since h does not have a singularity in $[1-\epsilon, 1]$, it is easier to prove $-(|\psi'|^{p-2}\psi')' \leq \lambda h(t) g(\psi)$ for $t \in [1-\epsilon, 1)$. Now for $t \in (\epsilon, 1-\epsilon)$, since $\phi_1(t) \geq \mu$ and $\lim_{s\to\infty} g(s) = \infty$, $g(\lambda k_0 \phi_1^\alpha(t)) \geq \frac{1}{\hat{h}} \lambda_1 k_0^{p-1} \alpha^{p-1} \phi_1^{\alpha(p-1)}(t)$ for $\lambda \gg 1$. Thus for $\lambda \gg 1$,

$$
-(|\psi'|^{p-2}\psi')' \leq \lambda\lambda_1 k_0^{p-1}\alpha^{p-1}\phi_1^{\alpha(p-1)}(t) \leq \lambda \hat{h} g(\lambda k_0 \phi_1^\alpha(t)) \leq \lambda h(t) g(\psi).
$$

Hence for $\lambda \gg 1$, ψ is a positive subsolution of (1.6). Next we construct a positive supersolution. Let $Z = M(\lambda)e$ where e is the solution of

$$
-(|e'|^{p-2}e')' = h(t), \;\; 0 < t < 1, \quad e(0) = e(1) = 0.
$$

Define $\hat{g}(x) = \max_{u\in[0,x]} g(u)$, then $\hat{g}$ satisfies (A_1) and (A_2) and is nondecreasing. Choose $M(\lambda) \gg 1$ such that

$$
\frac{1}{||e||_\infty{}^{p-1}\lambda} \geq \frac{\hat{g}(M(\lambda)||e||_\infty)}{(M(\lambda)||e||_\infty)^{p-1}}.
$$

Then

$$
-(|Z'|^{p-2}Z')' = (M(\lambda))^{p-1}h(t) \geq \lambda \hat{g}(M(\lambda)||e||_\infty)h(t) \geq \lambda \hat{g}(M(\lambda)e)h(t) \geq \lambda h(t) g(Z).
$$

Hence Z is a positive supersolution of (1.6) . Choose $M(\lambda) \gg 1$ such that $\psi \leq Z$. Thus we know that (1.6) has a positive solution $u \in \; [\psi, Z]$.

3.2 Proof of Theorem 2

Let ϕ_1 be as defined before, $\alpha \in (1, \frac{p-\beta}{p-1+\rho})$ and $r \in (\frac{1}{1+\rho}, \frac{1}{1+\rho-\delta})$. Define $\psi = \lambda^r \phi_1^\alpha$. Then

$$\psi' = \lambda^r \alpha \phi_1^{\alpha-1}\phi_1',$$
$$-(|\psi'|^{p-2}\psi')' = \lambda^r[\lambda_1 \alpha^{p-1}\phi_1^{\alpha(p-1)} - \alpha^{p-1}(\alpha-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\alpha(p-1)}}].$$

Let $m > 0$, $\epsilon > 0$ be such that $\alpha^{p-1}(\alpha-1)(p-1)|\phi_1'|^p - \lambda_1\alpha^{p-1}\phi_1^p \geq m$ in $(0,\epsilon]\cup[1-\epsilon,1)$ where $\epsilon < \epsilon_1$ as in the previous section. Let $k > 0$ be such that $g(s) \geq -k$ for all $s \geq 0$. Then in $(0,\epsilon]\cup[1-\epsilon,1)$, for $\lambda \gg 1$

$$\lambda_1\alpha^{p-1}\phi_1^p - \alpha^{p-1}(\alpha-1)(p-1)|\phi_1'|^p \leq -m \leq \frac{\lambda d d_1^\beta(-k)}{\lambda^r\lambda^{r\rho}},$$

since $1-r-r\rho < 0$. Hence in $(0,\epsilon]$, for $\lambda \gg 1$

$$\begin{aligned}
-(|\psi'|^{p-2}\psi')' &= \lambda^r[\lambda_1\alpha^{p-1}\phi_1^{\alpha(p-1)} - \alpha^{p-1}(\alpha-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\alpha(p-1)}}] \leq \frac{\lambda d d_1^\beta(-k)}{\lambda^{r\rho}\phi_1^{p-\alpha(p-1)}} \\
&\leq \frac{\lambda d d_1^\beta(-k)}{\lambda^{r\rho}\phi_1^{\beta+\alpha\rho}} \leq \frac{\lambda d(-k)}{(\lambda^r\phi_1^\alpha)^\rho t^\beta} \leq \frac{\lambda(-k)h(t)}{(\lambda^r\phi_1^\alpha)^\rho} \leq \frac{\lambda g(\lambda^r\phi_1^\alpha)h(t)}{(\lambda^r\phi_1^\alpha)^\rho}. \qquad (3.2)
\end{aligned}$$

Here again we note that since h does not have any singularity near $t = 1$, an easier proof will show that $-(|\psi'|^{p-2}\psi')' \leq \lambda\frac{h(t)g(\psi)}{\psi^\rho}$ in $[1-\epsilon, 1)$.

Next in $(\epsilon, 1-\epsilon)$, since there exist $\mu > 0$ such that $\phi_1 \geq \mu$, from (A_3)

$$g(\lambda^r\phi_1^\alpha) \geq A(\lambda^r\phi_1^\alpha)^\delta, \text{ for } \lambda \gg 1.$$

Since $1 + r(\delta-\rho) - r > 0$, in $(\epsilon, 1-\epsilon)$,

$$-(|\psi'|^{p-2}\psi')' \leq \lambda^r\lambda_1\alpha^{p-1}\phi_1^{\alpha(p-1)} \leq \lambda\hat{h}A(\lambda^r\phi_1^\alpha)^{\delta-\rho}, \text{ for } \lambda \gg 1.$$

Hence, for $\lambda \gg 1$ we have,

$$-(|\psi'|^{p-2}\psi')' \leq \frac{\lambda \hat{h} A(\lambda^r \phi_1^\alpha)^\delta}{(\lambda^r \phi_1^\alpha)^\rho} \leq \frac{\lambda h(t) g(\lambda^r \phi_1^\alpha)}{(\lambda^r \phi_1^\alpha)^\rho}. \tag{3.3}$$

Combining (3.2) and (3.3) we see that

$$-(|\psi'|^{p-2}\psi')' \leq \lambda h(t) \frac{g(\psi)}{\psi^\rho} \text{ in } (0,1) \text{ for } \lambda \gg 1.$$

Thus ψ is a positive subsolution. Now we construct a supersolution $Z \geq \psi$. Note that in (A_4), without loss of generality we can choose $\rho \leq \gamma < \rho + p - 1$. Hence for $m(\lambda) \gg 1$,

$$(m(\lambda))^{p-1+\rho-\gamma} \geq \lambda B e^{\gamma-\rho},$$

where e is as before. Hence for $m(\lambda) \gg 1$

$$m(\lambda)^{p-1} \geq \lambda \frac{B(m(\lambda)e)^\gamma}{(m(\lambda)e)^\rho}.$$

Define $Z = m(\lambda)e$. Then

$$-(|Z'|^{p-2}Z')' = m(\lambda)^{p-1}h(t) \geq \lambda \frac{B(m(\lambda)e)^\gamma}{(m(\lambda)e)^\rho} h(t) \geq \lambda h(t) \frac{g(m(\lambda)e)}{(m(\lambda)e)^\rho}.$$

Thus Z is a supersolution. Further $m(\lambda)$ can be chosen large such that $Z \geq \psi$. Hence (1.6) has a positive solution for $\lambda \gg 1$ when $0 < \rho < 1$.

3.3 Proof of Theorem 3

Consider the boundary value problem

$$-(|z'|^{p-2}z')' - \lambda |z|^{p-2} z = -1, \;\; 0 < t < 1, \qquad z(0) = z(1) = 0. \tag{3.4}$$

From an anti-maximum principle (see [37]) there exist $\delta_1 > 0$ such that for $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ the solution, z_λ of (3.4) is positive in $(0,1)$ and $|z'_\lambda| > 0$ at $t = 0, 1$. Also there exists $d_2 > 0$ such that

$$0 < z_\lambda \leq d_2 t(1-t) \text{ for } t \in (0,1).$$

Let $\alpha \in (1, \min\{(\frac{a\hat{h}}{\lambda_1})^{\frac{1}{p-1}}, \frac{p-\beta}{p-1+\rho}\})$, and fix $\lambda^* \in (\lambda_1, \min\{\frac{a\hat{h}}{\alpha^{p-1}}, \lambda_1 + \delta_1\})$. Define $\psi = k_0 z^{\alpha}_{\lambda^*}$ where z_{λ^*} is the solution of (3.4) for $\lambda = \lambda^*$ and

$$k_0 = \min\{(\frac{\alpha^{p-1}}{b||z_{\lambda^*}||_\infty^{(\gamma-1)\alpha-1} d_2^{p-\alpha(p-1)} d})^{\frac{1}{\gamma-p}}, (\frac{(a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}})}{2b||z_{\lambda^*}||_\infty^{\alpha(\gamma-p)}})^{\frac{1}{\gamma-p}}\}.$$

Then

$$\begin{aligned}
\psi' &= k_0 \alpha z_{\lambda^*}^{\alpha-1} z'_{\lambda^*}, \\
-(|\psi'|^{p-2}\psi')' &= -k_0^{p-1}\alpha^{p-1}(\alpha-1)(p-1) z_{\lambda^*}^{(\alpha-1)(p-1)-1} |z'_{\lambda^*}|^p \\
&\quad -k_0^{p-1}\alpha^{p-1} z_{\lambda^*}^{(\alpha-1)(p-1)} (|z'_{\lambda^*}|^{p-2} z'_{\lambda^*})'
\end{aligned}$$

$$= k_0^{p-1}\alpha^{p-1} z_{\lambda^*}^{\alpha(p-1)} \lambda^* - k_0^{p-1}\alpha^{p-1} z_{\lambda^*}^{(\alpha-1)(p-1)} - k_0^{p-1}\alpha^{p-1}(\alpha-1)(p-1)\frac{|z'_{\lambda^*}|^p}{z_{\lambda^*}^{p-\alpha(p-1)}} \quad (3.5)$$

and

$$h(t)(a\psi^{p-1} - b\psi^{\gamma-1} - \frac{c}{\psi^\rho}) = h(t)(ak_0^{p-1} z_{\lambda^*}^{\alpha(p-1)} - bk_0^{\gamma-1} z_{\lambda^*}^{\alpha(\gamma-1)} - \frac{c}{(k_0 z_{\lambda^*}^{\alpha})^\rho}). \quad (3.6)$$

Let $\mu > 0, m > 0$ be such that $|z_{\lambda^*}| \leq 1$, and $|z'_{\lambda^*}| \geq m$ in $(0, \epsilon] \cup [1-\epsilon, 1)$ and $z_{\lambda^*} \geq \mu$ in $(\epsilon, 1-\epsilon)$ where $\epsilon < \epsilon_1$. Also let

$$c^* = \min\{\frac{k_0^{p-1+\rho}\alpha^{p-1}(\alpha-1)(p-1)m^p}{d_2^\beta d}, \frac{1}{2} k_0^{p-1+\rho} \mu^{\alpha(p-1+\rho)} (a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}})\}.$$

In $(0, \epsilon]$ we compare (3.5) and (3.6) term by term to see that for $c < c^*$

$$-(|\psi'|^{p-2}\psi')' \leq h(t)(a\psi^{p-1} - b\psi^{\gamma-1} - \frac{c}{\psi^\rho}).$$

Since $\lambda^*\alpha^{p-1} < a\hat{h}$,

$$k_0^{p-1}\alpha^{p-1}z_{\lambda^*}^{\alpha(p-1)}\lambda^* \leq k_0^{p-1}z_{\lambda^*}^{\alpha(p-1)}h(t)a. \tag{3.7}$$

Next, we see that

$$\begin{aligned}
-k_0^{p-1}\alpha^{p-1}z_{\lambda^*}^{(\alpha-1)(p-1)} &= \frac{-k_0^{p-1}\alpha^{p-1}z_{\lambda^*}}{z_{\lambda^*}^{p-\alpha(p-1)}} \leq \frac{-k_0^{p-1}\alpha^{p-1}z_{\lambda^*}}{d_2^{p-\alpha(p-1)}t^{p-\alpha(p-1)}} \leq \frac{-k_0^{p-1}\alpha^{p-1}z_{\lambda^*}}{d_2^{p-\alpha(p-1)}t^{\beta}} \\
&\leq -k_0^{p-1}\alpha^{p-1}z_{\lambda^*}\frac{h(t)}{d_2^{p-\alpha(p-1)}d} = \frac{-k_0^{\gamma-1}\alpha^{p-1}z_{\lambda^*}h(t)}{k_0^{\gamma-p}d_2^{p-\alpha(p-1)}d}.
\end{aligned}$$

Now from the choice of k_0, $\frac{-1}{k_0{}^{\gamma-p}} \leq \frac{-b||z_{\lambda^*}||_\infty^{(\gamma-1)\alpha-1}d_2^{p-\alpha(p-1)}d}{\alpha^{p-1}}$. Hence,

$$\begin{aligned}
-k_0^{p-1}\alpha^{p-1}z_{\lambda^*}^{(\alpha-1)(p-1)} &\leq -bk_0^{\gamma-1}||z_{\lambda^*}||_\infty^{(\gamma-1)\alpha-1}z_{\lambda^*}h(t) \leq -bk_0^{\gamma-1}z_{\lambda^*}^{(\gamma-1)\alpha-1}z_{\lambda^*}h(t) \\
&= -bk_0^{\gamma-1}z_{\lambda^*}^{(\gamma-1)\alpha}h(t).
\end{aligned}$$

Since $p-\alpha(p-1) > \beta+\alpha\rho$ and $c < \frac{k_0^{\rho+p-1}\alpha^{p-1}(\alpha-1)(p-1)m^p}{d_2^{\beta}d}$,

$$\begin{aligned}
\frac{-k_0^{p-1}\alpha^{p-1}(\alpha-1)(p-1)|z'_{\lambda^*}|^p}{z_{\lambda^*}^{p-\alpha(p-1)}} &\leq \frac{-k_0^{p-1}\alpha^{p-1}(\alpha-1)(p-1)m^p}{z_{\lambda^*}{}^{\beta}z_{\lambda^*}{}^{\alpha\rho}} \\
&\leq \frac{-k_0^{p-1}\alpha^{p-1}(\alpha-1)(p-1)m^ph(t)}{d_2^{\beta}dz_{\lambda^*}^{\alpha\rho}} \\
&= \frac{-k_0^{p-1+\rho}\alpha^{p-1}(\alpha-1)(p-1)m^ph(t)}{d_2^{\beta}d(k_0z_{\lambda^*}^{\alpha})^{\rho}} \\
&\leq -\frac{ch(t)}{(k_0z_{\lambda^*}^{\alpha})^{\rho}}.
\end{aligned}$$

Hence we get $-(|\psi'|^{p-2}\psi')' \leq h(t)(a\psi^{p-1} - b\psi^{\gamma-1} - \frac{c}{\psi^\rho})$ in $(0,\epsilon]$. It is easier to prove this in $[1-\epsilon,1)$, as h is not singular. Now in $(\epsilon, 1-\epsilon)$ since $z_\lambda^* \geq \mu$, we have

$c \leq \frac{1}{2}k_0^{p-1+\rho}(z_{\lambda^*}^{\alpha})^{p-1+\rho}(a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}})$ and by our choice of k_0, $bk_0^{\gamma-p}z_{\lambda^*}^{\alpha(\gamma-p)} \leq \frac{1}{2}(a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}})$. Hence, for t $\in (\epsilon, 1-\epsilon)$,

$$\begin{aligned}
-(|\psi'|^{p-2}\psi')' &\leq k_0^{p-1}\alpha^{p-1}z_{\lambda^*}^{\alpha(p-1)}\lambda^* = \frac{\hat{h}k_0^{p-1}\alpha^{p-1}z_{\lambda^*}^{\alpha(p-1)}\lambda^*}{\hat{h}} \\
&\leq h(t)[\frac{1}{2}\frac{\alpha^{p-1}}{\hat{h}}\lambda^* k_0^{p-1}z_{\lambda^*}^{\alpha(p-1)} + \frac{1}{2}\frac{\alpha^{p-1}}{\hat{h}}\lambda^* k_0^{p-1}z_{\lambda^*}^{\alpha(p-1)}] \\
&\leq h(t)\big[\frac{1}{2}ak_0^{p-1}z_{\lambda^*}^{\alpha(p-1)} - \frac{1}{2}k_0^{p-1}z_{\lambda^*}^{\alpha(p-1)}(a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}}) \\
&\quad +\frac{1}{2}ak_0^{p-1}z_{\lambda^*}^{\alpha(p-1)} - \frac{1}{2}k_0^{p-1}z_{\lambda^*}^{\alpha(p-1)}(a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}})\big] \\
&\leq h(t)[\frac{1}{2}k_0^{p-1}z_{\lambda^*}^{\alpha(p-1)}a - \frac{c}{(k_0 z_{\lambda^*}^{\alpha})^{\rho}} + \frac{1}{2}k_0^{p-1}z_{\lambda^*}^{\alpha(p-1)}a - bk_0^{\gamma-1}z_{\lambda^*}^{\alpha(\gamma-1)}] \\
&= h(t)[ak_0^{p-1}z_{\lambda^*}^{\alpha(p-1)} - bk_0^{\gamma-1}z_{\lambda^*}^{\alpha(\gamma-1)} - \frac{c}{(k_0 z_{\lambda^*}^{\alpha})^{\rho}}].
\end{aligned}$$

Hence ψ is a positive subsolution of (1.7). Next we construct a supersolution. We know that there exist a large $\bar{M} > 0$ such that $au^{p-1} - bu^{\gamma-1} - \frac{c}{u^{\rho}} \leq \bar{M}^{p-1}$ for all $u > 0$ and $\bar{M}e \geq \psi$ in (0,1) where e is as defined before. Let $Z = \bar{M}e$. Then Z is a positive supersolution of (1.7). Thus Theorem 3 is proven.

3.4 Proof of Theorem 4

Let ϕ_1 be as defined before, $\alpha \in (1, \frac{p-\beta}{p-1})$ and $-k_0 < \frac{d_1^{p-\alpha(p-1)}d}{m}\min\{\bar{g}_1, \bar{g}_2\}$, where $\bar{g}_i = \min_{x\in[0,\infty)} g_i(x)$, $i = 1, 2$ and d_1, m are as in the proof of Theorem 1. Define $\psi_1 = \psi_2 = \lambda k_0 \phi_1^{\alpha}$. Following the steps in the proof of Theorem 1, it is now easy to show that for $\lambda \gg 1$, (ψ_1, ψ_2) is a subsolution of (1.8). Now we define

$$Z_1 = M(\lambda)e_1,$$

$$Z_2 = \big(\lambda g_2(M(\lambda)||e_1||_{\infty})\big)^{\frac{1}{p-1}}e_2,$$

where e_i is solution of $-(|e_i'|^{p-2}e_i')' = h_i(t), \quad 0 < t < 1,\, e_i(0) = e_i(1) = 0,\, i = 1, 2.$ Choose $M(\lambda) \gg 1$ such that

$$\frac{1}{||e_1||_\infty{}^{p-1}\lambda} \geq \frac{g_1\big(\lambda^{\frac{1}{p-1}}||e_2||_\infty(g_2(M(\lambda)||e_1||_\infty))^{\frac{1}{p-1}}\big)}{M(\lambda)^{p-1}||e_1||_\infty{}^{p-1}}.$$

Now

$$\begin{aligned}-(|Z_1'|^{p-2}Z_1')' = M(\lambda)^{p-1}h_1(t) &\geq \lambda g_1\big(\lambda^{\frac{1}{p-1}}||e_2||_\infty(g_2(M(\lambda)||e_1||_\infty))^{\frac{1}{p-1}}\big)h_1(t)\\ &\geq \lambda g_1\big(\lambda^{\frac{1}{p-1}}e_2(g_2(M(\lambda)||e_1||_\infty))^{\frac{1}{p-1}}\big)h_1(t) = \lambda g_1(Z_2)h_1(t),\\ -(|Z_2'|^{p-2}Z_2')' = \lambda g_2(M(\lambda)||e_1||_\infty)h_2(t) &\geq \lambda g_2(M(\lambda)e_1)h_2(t) = \lambda g_2(Z_1)h_2(t).\end{aligned}$$

Hence (Z_1, Z_2) is a positive supersolution of (1.8). Choose $M(\lambda) \gg 1$ such that $\psi_1 \leq Z_1$ and $\psi_2 \leq Z_2$. Thus Theorem 4 is proven.

3.5 Proof of Theorem 5

Let ϕ_1 be as defined before, $\alpha \in (1, \frac{p-\beta}{p-1+\rho})$ and $r \in (\frac{1}{1+\rho}, \frac{1}{1+\rho-\delta})$. Define $\psi_1 = \psi_2 = \lambda^r\phi_1^\alpha$. A similar proof as in Theorem 2 will show that (ψ_1, ψ_2) is a subsolution of (1.8) for $\lambda \gg 1$. Now we construct a supersolution $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. There exist $\tau_1 > 0$ and $\tau_2 > 0$ such that

$$e_2 \leq \tau_1 e_1, \quad \text{and} \quad e_1 \leq \tau_2 e_2,$$

where $e_i's$ are as in the proof of Theorem 4. As in Theorem 2 we can choose $\rho \leq \gamma < \rho + p - 1$, hence for $m(\lambda) \gg 1$,

$$(m(\lambda))^{p-1+\rho-\gamma} \geq \lambda B\tau_i^\gamma e_i^{\gamma-\rho}, \quad i = 1, 2.$$

Hence, for $m(\lambda) \gg 1$,

$$m(\lambda)^{p-1} \geq \lambda \tau_1^{\gamma} \frac{B(m(\lambda)e_1)^{\gamma}}{(m(\lambda)e_1)^{\rho}} \geq \lambda \frac{B(m(\lambda)e_2)^{\gamma}}{(m(\lambda)e_1)^{\rho}}.$$

Similarly

$$m(\lambda)^{p-1} \geq \lambda \frac{B(m(\lambda)e_1)^{\gamma}}{(m(\lambda)e_2)^{\rho}}.$$

Define $(Z_1, Z_2) = (m(\lambda)e_1, m(\lambda)e_2)$. Then

$$\begin{aligned} -(|Z_1'|^{p-2}Z_1')' &= m(\lambda)^{p-1}h_1(t) \geq \lambda \frac{B(m(\lambda)e_2)^{\gamma}}{(m(\lambda)e_1)^{\rho}} h_1(t) \\ &\geq \lambda h_1(t) \frac{g_1(m(\lambda)e_2)}{(m(\lambda)e_1)^{\rho}} = \lambda h_1(t) \frac{g_1(Z_2)}{(Z_1)^{\rho}} \end{aligned}$$

and similarly

$$-(|Z_2'|^{p-2}Z_2')' \geq \lambda h_2(t) \frac{g_2(Z_1)}{(Z_2)^{\rho}}$$

Thus (Z_1, Z_2) is a supersolution. Further $m(\lambda)$ can be chosen large such that $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. Hence (1.8) has a positive solution for $\lambda \gg 1$ when $0 < \rho < 1$.

3.6 Proof of Theorem 6

Let $a = \min(a_1, a_2)$ and $b = \max(b_1, b_2)$. Define $\psi_1 = \psi_2 = k_0 z_{\lambda^*}^{\alpha}$ where z_{λ^*} is the solution of (3.4) for $\lambda = \lambda^* \in (\lambda_1, \min(\frac{a\hat{h}}{\alpha^{p-1}}, \lambda_1 + \delta_1))$,

$$k_0 = \min\{(\frac{\alpha^{p-1}}{b||z_{\lambda^*}||_{\infty}^{(\gamma-1)\alpha-1} d_2^{p-\alpha(p-1)} d})^{\frac{1}{\gamma-p}}, (\frac{(a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}})}{2b||z_{\lambda^*}||_{\infty}^{\alpha(\gamma-p)}})^{\frac{1}{\gamma-p}}\},$$

and $\alpha \in (1, \min((\frac{a\hat{h}}{\lambda_1})^{\frac{1}{p-1}}, \frac{p-\beta}{p-1+\rho}))$. By following the proof of Theorem 3 we can easily show that there exists

$$c^* = \min\{\frac{k_0^{p-1+\rho}\alpha^{p-1}(\alpha-1)(p-1)m^p}{d_2^{\beta}d}, \frac{1}{2}k_0^{p-1+\rho}\mu^{\alpha(p-1+\rho)}(a - \frac{\alpha^{p-1}\lambda^*}{\hat{h}})\}$$

such that for $\max\{c_1, c_2\} < c^*$, (ψ_1, ψ_2) is a positive subsolution of (1.9). Define $Z_1 = \bar{M}e_1$ and $Z_2 = \bar{M}e_2$ where $\bar{M} > 0$ is such that $a_i u^{p-1} - b_i u^{\gamma-1} - \frac{c_i}{u^\rho} \leq \bar{M}$ for $i = 1, 2$ and $\bar{M}e_1 > \psi_1$, $\bar{M}e_2 > \psi_2$. It is easy to see that (Z_1, Z_2) is a supersolution of (1.9). Hence Theorem 6 is proven.

CHAPTER 4

PROOF OF THEOREM 7

We first establish some a priori estimates which are needed to prove Theorem 7.

4.1 A priori estimates

Let $F(s) = \int_0^s f(t)dt$. Note that there exist positive real numbers β, θ such that $f(\beta) = 0$ and $F(\theta) = 0$ and $\beta < \theta$. (See Figure 4.1).

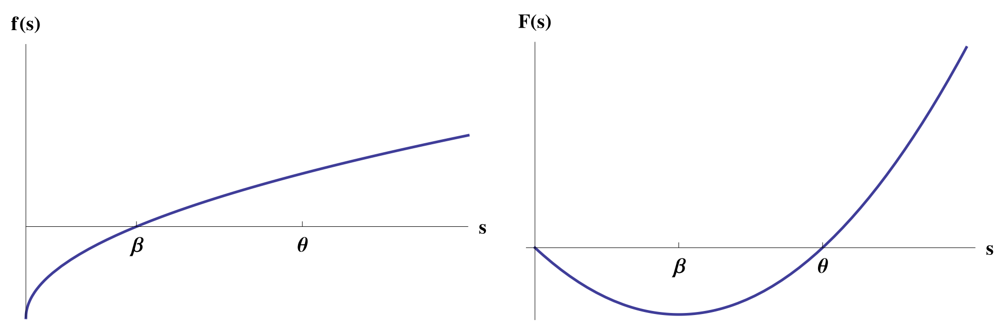

Figure 4.1

Graphs of $f(s)$ and $F(s)$

Lemma 8

Let u be a nonnegative solution of (1.10). Then u has only one interior maximum, say at t_0, and $u(t_0) > \theta$.

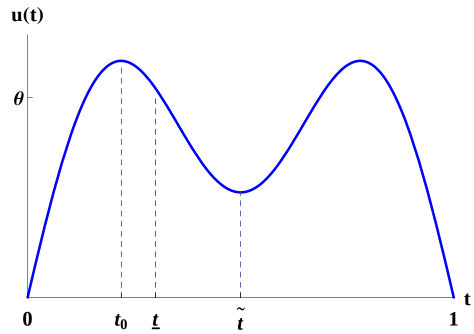

Figure 4.2

A solution with more than one maximum

Proof. Let $E(t) := \lambda F(u(t))h(t) + \frac{[u'(t)]^2}{2}, t \in (0,1)$. Hence $E'(t) = \lambda F(u(t))h'(t)$. Since, $h(s)$ decreases for $s > 0$, $E(t)$ increases when $u(t) < \theta$ and decreases when $u(t) > \theta$. Let $t_0 \in (0,1)$ be the first point at which u has a local maximum, and assume $u(t) \leq \theta, \forall t \leq t_0$. Integrating (1.10) from t to $t_0, t < t_0$, and using properties of h,

$$u'(t) \;=\; \lambda \int_t^{t_0} h(s)f(u(s))ds \leq \lambda \frac{df(\theta)}{1-\alpha}(t_0^{1-\alpha} - t^{1-\alpha}) \leq \lambda \frac{df(\theta)}{1-\alpha} \tag{4.1}$$

where $d \geq c$ is such that $h(t) \leq \frac{d}{t^\alpha}$ for all $t \in (0,1)$ and $\alpha \in (0,1)$. Integrating (4.1) again from 0 to $t, t < t_0$, $u(t) \leq \lambda M_0 t$ where $M_0 = \frac{df(\theta)}{1-\alpha}$. Since f is continuous, there exists $k_0 > 0$ such that $|F(u)| \leq k_0 u$ for $u \in [0,\theta]$. Hence

$$\lim_{t\to 0+} \lambda |F(u(t))| h(t) \leq \lim_{t \to 0+} k_0 \lambda M_0 d t^{1-\alpha} = 0,$$

which implies $\lim_{t\to 0+} E(t) \geq 0$. Since $E(t)$ increases on $[0, t_0]$, $E(t_0) = \lambda F(u(t_0))h(t_0) > 0$ which is a contradiction if $u(t_0) \leq \theta$. Hence $u(t_0) > \theta$.

Now suppose u has two interior maxima. Let $\tilde{t} \in (t_0, 1)$ be such that $u'(\tilde{t}) = 0$ and $u''(\tilde{t}) \geq 0$ (as in Figure 4.2). Since $u''(\tilde{t}) = -\lambda h(\tilde{t}) f(u(\tilde{t})) \geq 0$ we see that $u(\tilde{t}) \leq \beta$ and thus $E(\tilde{t}) < 0$. Let $\underline{t} \in (t_0, \tilde{t})$ be such that $u(\underline{t}) = \theta$. Since $E(\underline{t}) \geq 0$ and E increases in $(\underline{t}, \tilde{t})$, $E(\tilde{t}) > 0$ which is contradiction. Hence u can have only one interior maximum and that maximum value is bigger than θ.

Lemma 9

If $t_1, \hat{t_1}$ are such that $t_1 < \hat{t_1}$ and $u(t_1) = u(\hat{t_1}) = \beta$, then $t_1, 1 - \hat{t_1} \leq O(\lambda^{-\frac{1}{2}})$.

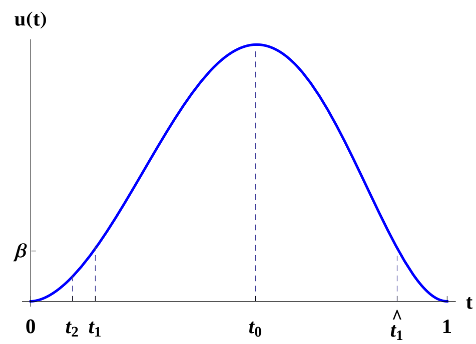

Figure 4.3

Graph of a solution

Proof. Let t_2 be the first point in (0, 1) such that $u(t_2) = \frac{\beta}{2}$. Integrating (1.10) from 0 to $t, t < t_2$,

$$\begin{aligned} u'(t) &= u'(0) - \lambda \int_0^t h(s) f(u(s)) ds \\ &\geq \lambda \hat{h} t \big(- f(\frac{\beta}{2}) \big). \end{aligned}$$

Integrating again from 0 to t_2, we obtain

$$t_2 \leq \tilde{c} \lambda^{-\frac{1}{2}}, \text{ where } \tilde{c} = \Big(\frac{-\beta}{\hat{h} f(\frac{\beta}{2})} \Big)^{\frac{1}{2}} > 0. \tag{4.2}$$

By the Mean Value Theorem, there exists a $\bar{t} \in [0, t_2]$ such that $u(t_2) - u(0) = u'(\bar{t})(t_2)$ and by (4.2), $u'(\bar{t}) \geq \frac{\beta}{2\tilde{c}}\lambda^{\frac{1}{2}}$. Since u' increases in $[0, t_1]$, this implies

$$u'(t) \geq \frac{\beta}{2\tilde{c}}\lambda^{\frac{1}{2}}, \ \forall t \in [t_2, t_1]. \tag{4.3}$$

Integrating (4.3) from t_2 to t_1 we see that $(t_1 - t_2) \leq \tilde{c}\lambda^{-\frac{1}{2}}$. This and (4.2) implies $t_1 \leq O(\lambda^{-\frac{1}{2}})$. Similarly we can also prove $1 - \hat{t_1} \leq O(\lambda^{-\frac{1}{2}})$.

Lemma 10

Given $M > 0$, there exists $\lambda(M)$ such that if $\lambda > \lambda(M)$ then $u(\hat{t}) \geq M$ for some $\hat{t} \in (t_1, \hat{t_1})$.

Proof. Let $v := u - \beta$, then $v > 0$ in $(t_1, \hat{t_1})$ and satisfies:

$$\begin{cases} -v'' = \lambda h(t)\dfrac{f(u)}{u-\beta}v, \quad 0 < t < 1 \\ \\ v(t_1) = v(\hat{t_1}) = 0. \end{cases} \tag{4.4}$$

Also

$$-\Big(sin\Big(\frac{\pi(t-t_1)}{(\hat{t_1}-t_1)}\Big)\Big)'' = \frac{\pi^2}{(\hat{t_1}-t_1)^2}sin\Big(\frac{\pi(t-t_1)}{(\hat{t_1}-t_1)}\Big). \tag{4.5}$$

Multiplying (4.4) by $sin\big(\frac{\pi(t-t_1)}{(\hat{t_1}-t_1)}\big)$ and integrating from t_1 to $\hat{t_1}$, we have

$$\int_{t_1}^{\hat{t_1}} cos\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big)\frac{\pi}{(\hat{t_1}-t_1)}v'ds = \int_{t_1}^{\hat{t_1}} \lambda h(s)\frac{f(u)}{u-\beta}vsin\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big)ds \tag{4.6}$$

and multiplying (4.5) by v and integrating from t_1 to $\hat{t_1}$, we have

$$\int_{t_1}^{\hat{t_1}} cos\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big)\frac{\pi}{(\hat{t_1}-t_1)}v'ds = \int_{t_1}^{\hat{t_1}} \frac{\pi^2}{(\hat{t_1}-t_1)^2}vsin\Big(\frac{\pi(s-t_1)}{(\hat{t_1}-t_1)}\Big)ds. \tag{4.7}$$

Now subtracting (4.7) from (4.6) we see easily that

$$\lambda\frac{f(u)}{u-\beta}h(t) = \frac{\pi^2}{(\hat{t_1}-t_1)^2} \text{ for some } t \in (t_1, \hat{t_1}). \tag{4.8}$$

Note that $\inf_{t\in(0,1)} h(t) > 0$ and from Lemma 9 without loss of generality we can assume $(\hat{t_1} - t_1) > \frac{1}{2}$. Thus for $\lambda \gg 1$, (4.8) is true only if $\frac{f(u)}{u-\beta} \to 0$. Since f satisfies (B_2), this implies $||u||_\infty \to \infty$ when $\lambda \to \infty$.

Lemma 11

There exists $k > 0$ such that $u(t) > \lambda k$ for $t \in [\frac{1}{4}, \frac{3}{4}]$ if $\lambda \gg 1$.

Proof. We first claim $u(t) > \frac{\beta+\theta}{2}$ for $t \in [\frac{1}{4}, \frac{3}{4}]$. Recall $t_0 \in (t_1, \hat{t_1})$ is the point at which u has it's maximum. By Lemma 10 given $M > 0, \exists\ \lambda(M)$ such that if $\lambda > \lambda(M)$ then $u(t_0) \geq M$. Since $u'' < 0$ in (t_1, t_0), for $t \in [t_1, t_0]$, we have

$$u(t) \geq \frac{(u(t_0) - \beta)}{t_0 - t_1}(t - t_1) + \beta. \tag{4.9}$$

Similarly for $t \in [t_0, \hat{t_1}]$, we can get

$$u(t) \geq \frac{(u(t_0) - \beta)}{\hat{t_1} - t_0}(\hat{t_1} - t) + \beta. \tag{4.10}$$

Now by Lemma 9, for $\lambda \gg 1$ we can assume $t_1 < 0.2$ and $\hat{t_1} > 0.8$. Hence from (4.9), (4.10) and Lemma 10, the claim $u(t) > \frac{\beta+\theta}{2}$ holds when λ is large. Now let $G(t,s)$ be the Green's function associated with problem (1.10). Then

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)h(s)f(u(s))ds \\ &\geq \lambda \Big[\int_0^{t_1} G(t,s)h(s)f(u(s))ds + \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)h(s)f(u(s))ds \\ &\quad + \int_{\hat{t_1}}^1 G(t,s)h(s)f(u(s))ds \Big]. \end{aligned}$$

But by Lemma 9, $t_1 \to 0$ and $\hat{t_1} \to 1$ as $\lambda \to \infty$. Hence for $\lambda \gg 1$, $u(t) \geq \lambda k$ for $t \in [\frac{1}{4}, \frac{3}{4}]$, where $k = \frac{1}{2}\hat{h} f(\frac{\beta+\theta}{2}) \inf_{[0,1]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)ds$, which proves the lemma.

Lemma 12

There exists $\bar{\lambda}$ *such that if* $\lambda \geq \bar{\lambda}$, $u(t) \geq \lambda d(t, \partial\Omega)$, *where* $\Omega = (0, 1)$.

Proof. Let σ be the unique solution of

$$\begin{cases} -\sigma''(t) = \chi_{[\frac{1}{4},\frac{3}{4}]} h(t), \quad 0 < t < 1 \\ \\ \sigma(0) = \sigma(1) = 0, \end{cases} \tag{4.11}$$

where χ is the characteristic function. By Hopf's maximum principle there exists $\bar{c} > 0$ such that $\sigma(t) \geq \bar{c}e(t)$ $\forall t \in [0, 1]$, where e is the solution of $-e''(t) = h(t)$ in $(0, 1)$ and $e(0) = e(1) = 0$. Let $\underline{M} > 0$ be such that $P = \bar{c}f(\underline{M}) + f(0) > 0$ and let u_1, u_2 satisfy $-u_1'' = \lambda f(\underline{M})\chi_{[\frac{1}{4},\frac{3}{4}]} h(t)$ in $(0, 1)$, $u_1(0) = u_1(1) = 0$ and $-u_2'' = -\lambda f(0)h(t)$ in $(0, 1)$, $u_2(0) = u_2(1) = 0$. Then by Lemma 11, if $\lambda > \frac{\underline{M}}{k}$, we have

$$\begin{aligned} -u'' &= \lambda f(u)h(t) \\ &\geq \lambda f(\underline{M})\chi_{[\frac{1}{4},\frac{3}{4}]} h(t) + \lambda f(0)h(t) \end{aligned}$$

and thus, by the maximum principle, $u(t) \geq u_1(t) - u_2(t) = \lambda f(\underline{M})\sigma(t) + \lambda f(0)e(t)$. Hence

$$u(t) \;\; \geq \;\; \lambda f(\underline{M})\bar{c}e(t) + \lambda f(0)e(t) = \lambda P e(t), \;\; \forall t \in (0, 1).$$

Let $L > 0$ be such that $e(t) \geq Ld(t, \partial\Omega)$ for all $t \in [0, 1]$. Hence $u(t) \geq \lambda\tilde{K}d(t, \partial\Omega)$ for all $t \in (0, 1)$ where $\tilde{K} = PL$. Now let $D := [\epsilon, 1 - \epsilon]$, for some $\epsilon > 0$. Then $u(t) \geq \lambda\tilde{K}\epsilon$ for all $t \in D$. Let u_3 be the unique solution to $-u_3''(t) = \chi_D h(t)$ in $(0, 1)$, $u_3(0) = u_3(1) = 0$. Since f satisfies (B_1), for $\lambda \gg 1$, $f(\lambda\tilde{K}\epsilon)u_3(t) + f(0)e(t) \geq d(t, \partial\Omega)$ in $[0, 1]$. Hence for $\lambda \gg 1$, $-u'' = \lambda h(t)f(u(t)) \geq \lambda\Big(f(\lambda\tilde{K}\epsilon)\chi_D h(t) + f(0)h(t)\Big)$, and thus by the maximum principle $u(t) \geq \lambda\Big(f(\lambda\tilde{K}\epsilon)u_3(t) + f(0)e(t)\Big) \geq \lambda d(t, \partial\Omega)$ for all $t \in [0, 1]$, if λ is large, which proves the lemma.

Lemma 13

For each $\lambda > 0$, *there exists* $\bar{M}(\lambda)$ *such that* $||u||_\infty \leq \bar{M}(\lambda)$.

Proof. Due to our assumptions on h, $\int_0^1 h(s)ds \equiv A < \infty$. By (B_2), there exists $\bar{K}$ such that $f(z) \leq \lambda^{-1}(A+1)^{-1}z + \bar{K}$, for all $z \geq 0$. Since $G(s,t) \leq 1/4$ for all $s, t, \in [0, 1]$, we have

$$\begin{aligned}
\|u\|_\infty &= u(t_0) \\
&= \lambda \int_0^1 G(s, t_0)h(s)f(u(s))ds \\
&\leq \lambda \int_0^1 G(s, t_0)h(s)(\lambda^{-1}(A+1)^{-1}u(t_0) + \bar{K})ds \\
&\leq \frac{1}{2}u(t_0) + \lambda\bar{K}A.
\end{aligned} \tag{4.12}$$

Therefore $\|u\|_\infty \leq 2\lambda\bar{K}A$, which proves the lemma.

4.2 Proof of Theorem 7

We first claim that (1.10) has a maximal positive solution $\bar{u}$ for $\lambda \gg 1$. Given $\lambda > 0$, choose $J = J(\lambda) > \lambda f(\bar{M}(\lambda))$ where $\bar{M}(\lambda)$ is as in the previous section. Further choose $J \gg 1$ so that $J \geq \lambda f(J||e||_\infty)$, where e is as before (see Lemma 12). This is possible since f satisfies (B_2). Now if v is any solution of (1.10), then $-(Je - v)''(t) = Jh(t) - \lambda f(v)h(t) \geq h(t)(J - \lambda f(\bar{M}(\lambda))) > 0$ in $(0, 1)$. By the maximum principle we obtain $Je \geq v$ in [0, 1]. Also, $-(Je)''(t) = Jh(t) \geq \lambda f(Je(t))h(t)$ in $(0, 1)$. Hence Je is a supersolution of (1.10) larger than any solution of (1.10). However, by Theorem 1, we know that (1.10) has a positive solution for $\lambda \gg 1$. Hence (1.10) must have a maximal positive solution $\bar{u}$ for $\lambda \gg 1$.

Now let u be any other positive solution of (1.10). To establish our theorem, we will now show that $\bar{u} \equiv u$ for $\lambda \gg 1$. Since $\bar{u}$ and u are solutions of (1.10), we obtain

$$\begin{aligned} -(\bar{u}-u)''(t) &= \lambda h(t)\Big(f(\bar{u}(t)) - f(u(t))\Big), \quad 0 < t < 1 \\ (\bar{u}-u)(0) &= (\bar{u}-u)(1) = 0. \end{aligned} \tag{4.13}$$

By the Mean Value Theorem there exists ξ such that $u \leq \xi \leq \bar{u}$ in $[0,1]$ and

$$\begin{aligned} -(\bar{u}-u)''(t) &= \lambda h(t) f'(\xi)(\bar{u}(t) - u(t)), \quad 0 < t < 1 \\ (\bar{u}-u)(0) &= (\bar{u}-u)(1) = 0. \end{aligned} \tag{4.14}$$

Multiplying (1.10) by $(\bar{u}-u)$, (4.14) by u, integrating and using the fact that f is concave, we obtain

$$\lambda \int_0^1 \Big(f(u) - f'(u)u\Big) h(s)(\bar{u}-u)ds \leq 0. \tag{4.15}$$

Now by (B_2), there exists $a > 0, b > 0$ such that $f(z) - f'(z)z \geq b$ whenever $z \geq a$ and from Lemma 12, $u(t) \geq a$ if $d(t, \partial\Omega) \geq \frac{a}{\lambda}$ when $\lambda \gg 1$. Let $\Omega_+ = [\frac{a}{\lambda}, 1 - \frac{a}{\lambda}]$ and $\Omega_- = (0, \frac{a}{\lambda}) \cup (1 - \frac{a}{\lambda}, 1)$. Then from (4.15) we obtain

$$I = \int_{\Omega_+} b(\bar{u}-u)h(s)ds + \int_{\Omega_-} f(0)(\bar{u}-u)h(s)ds \leq 0. \tag{4.16}$$

Here we have used $f(z) - zf'(z) \geq f(0)\ \forall z \geq 0$, which follows from the fact that f is concave.

Next let m_1, m_2 satisfy $-m_1''(t) = \chi_{\Omega_+} h(t)$ in $(0,1), m_1(0) = m_1(1) = 0$ and $-m_2''(t) = \chi_{\Omega_-} h(t)$ in $(0,1), m_2(0) = m_2(1) = 0$ respectively. Multiplying (4.14) by $bm_1(t) + f(0)m_2(t)$ and integrating by parts we obtain

$$\begin{aligned} I &= \int_{\Omega_+} b(\bar{u} - u)h(s)ds + \int_{\Omega_-} f(0)(\bar{u} - u)h(s)ds \\ &= \lambda \int_0^1 f'(\xi)(\bar{u} - u)h(s)[bm_1(s) + f(0)m_2(s)]ds. \end{aligned} \tag{4.17}$$

As λ tends to $+\infty$, m_1 tends to e and m_2 tends to 0 in $C^1[0,1]$. Hence for $\lambda \gg 1$ $bm_1(t) + f(0)m_2(t) > 0$ in $(0,1)$. Thus from (4.16) and (4.17) we see that $I = 0$ for $\lambda \gg 1$, and from (4.17), we see that this is possible only if $\bar{u} \equiv u$ in $[0,1]$, which proves Theorem 7.

CHAPTER 5

PROOFS OF THEOREMS 8-9

5.1 Proof of Theorem 8

We first establish two useful results for such nonlinear eigenvalue problems when the nonlinearities are zero at the origin. Namely, we consider $\tilde{f} \in C^1((0,\infty), R)$ such that $\tilde{f}(0) = 0$ and satisfies:

$(\tilde{C}_1)$ there exists $\tilde{\rho}_1, \tilde{\rho}_2$ such that $0 < \tilde{\rho}_1 < \tilde{\rho}_2$, $\tilde{f}(\tilde{\rho}_1) = \tilde{f}(\tilde{\rho}_2) = 0$ and $\tilde{f} > 0$ in $(\tilde{\rho}_1, \tilde{\rho}_2)$,

$(\tilde{C}_2)$ $\int_t^{\tilde{\rho}_2} \tilde{f}(s)ds > 0$ for every $t \in [0, \tilde{\rho}_2)$,

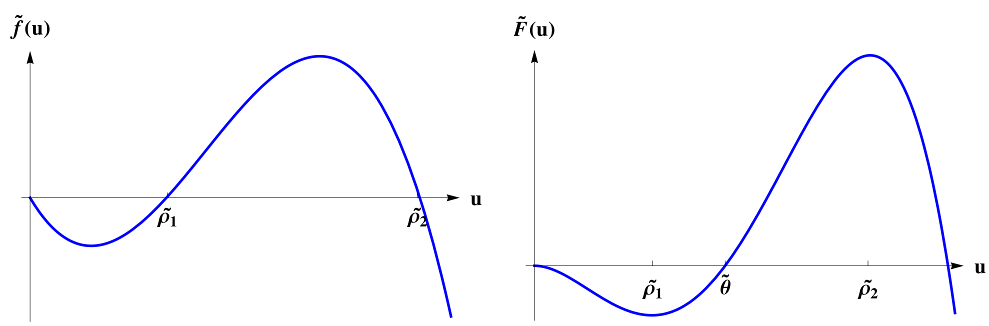

Figure 5.1

Graphs of $\tilde{f}(u)$ and $\tilde{F}(u)$

(see Figure 5.1) and study the boundary value problem:

$$\begin{cases} -u''(s) = \lambda h(s)\tilde{f}(u(s)), \quad 0 < s < 1 \\ \\ \quad u(0) = u(1) = 0 \end{cases} \tag{5.1}$$

where $h(s)$ is as before. First we establish:

Lemma 14

Assume $(\tilde{C}_1)$, *and* $(\tilde{C}_2)$ *hold and* $\tilde{f}(0) = 0$. *Then* (5.1) *has a positive solution* u_λ *for* $\lambda \gg 1$ *with* $\max u_\lambda \in (\tilde{\rho_1}, \tilde{\rho_2}]$.

Proof. First modify $\tilde{f}$ outside $[0, \tilde{\rho_2}]$ as $\tilde{f}(s) = 0$ if $s > \tilde{\rho_2}$ and $\tilde{f}(s) = -\tilde{f}(-s)$ for $s < 0$. Define $I_\lambda(u) = \frac{1}{2}\int_0^1 (u'(x))^2 dx - \lambda \int_0^1 h(x)\tilde{F}(u(x))dx$ in $W_0^{1,2}(0,1)$, where $\tilde{F}(u) = \int_0^u \tilde{f}(s)ds$. Since h is integrable and $\tilde{F}$ is bounded, it is easy to see that $I_\lambda(u)$ is bounded below, weakly lower semi continuous and coercive for $\lambda > 0$. Also since $\tilde{F}$ is an even function and $h(s) > 0$, $I_\lambda(|u|) \leq I_\lambda(u)$, for all $\lambda > 0$. Hence $I_\lambda(u)$ has a nonnegative minimizer, say u_λ.

We now prove $||u_\lambda||_\infty \leq \tilde{\rho_2}$. Suppose $||u_\lambda||_\infty > \tilde{\rho_2}$ and let ${t_1}^*$ be such that $u_\lambda({t_1}^*) = ||u_\lambda||_\infty$. Then there exists a ${t_0}^* < {t_1}^*$ such that $u_\lambda({t_0}^*) = \tilde{\rho_2}$ and u_λ is nondecreasing in $[{t_0}^*, {t_1}^*]$. Integrating (5.1) from t to ${t_1}^*$ where ${t_0}^* < t < {t_1}^*$ we see that

$$u'(t) = \lambda \int_t^{{t_1}^*} h(s)\tilde{f}(u_\lambda(s))ds = 0 \text{ (since } \tilde{f}(s) = 0 \text{ for } s > \tilde{\rho_2}),$$

which is a contradiction.

Next we prove $||u_\lambda||_\infty > \tilde{\rho_1}$ for $\lambda \gg 1$. Suppose $||u_\lambda||_\infty \le \tilde{\rho_1}$ for all positive λ. We choose a $w \in C_0^\infty(0,1)$ such that $0 \le w \le \tilde{\rho_2}$ in $[0,\delta] \cup [1-\delta, 1]$ and $w = \tilde{\rho_2}$ in $(\delta, 1-\delta)$ where δ will be chosen later. Then

$$\begin{aligned}
I_\lambda(w) - I_\lambda(u_\lambda) &= \frac{1}{2}\int_0^1 ((w')^2 - (u_\lambda')^2)dx - \lambda \int_0^1 h(x)(\tilde{F}(w) - \tilde{F}(u_\lambda))dx \\
&\le \frac{1}{2}\int_0^1 (w')^2 dx - \lambda\Big[\int_0^1 h(x)\tilde{F}(\tilde{\rho_2})dx + \int_0^\delta h(x)(\tilde{F}(w) - \tilde{F}(\tilde{\rho_2}))dx \\
&\quad + \int_{1-\delta}^1 h(x)(\tilde{F}(w) - \tilde{F}(\tilde{\rho_2}))dx - \int_0^1 h(x)\tilde{F}(u_\lambda)dx\Big] \\
&\le \frac{1}{2}\int_0^1 (w')^2 dx - \lambda\int_0^\delta h(x)(\tilde{F}(w) - \tilde{F}(\tilde{\rho_2}))dx \\
&\quad -\lambda\int_{1-\delta}^1 h(x)(\tilde{F}(w) - \tilde{F}(\tilde{\rho_2}))dx - \lambda\int_0^1 h(x)\int_{u_\lambda}^{\tilde{\rho_2}} \tilde{f}(s)dsdx.
\end{aligned}$$

Let $\beta = \min\{\int_\rho^{\tilde{\rho_2}} \tilde{f}(s)ds; 0 \le \rho \le \tilde{\rho_1}\}$. By our assumption $\beta > 0$. Also $\tilde{F}(u_\lambda) \le m$ for some $m > 0$ and $h(s) \le \frac{d}{t^\alpha}$ for all $t \in (0,1)$, thus

$$I_\lambda(w) - I_\lambda(u_\lambda) \le \frac{1}{2}\int_0^1 (w')^2 dx + \frac{2\lambda m d\delta^{1-\alpha}}{1-\alpha} + \frac{2\lambda m d(1-(1-\delta)^{1-\alpha})}{1-\alpha} - \frac{\lambda\beta d}{1-\alpha}.$$

We now choose $\delta \approx 0$. Then it follows that $I_\lambda(w) < I_\lambda(u_\lambda)$ for $\lambda \gg 1$, a contradiction. Thus $||u_\lambda||_\infty > \tilde{\rho_1}$ for $\lambda \gg 1$.

Next we prove that $u_\lambda > 0$ in $(0,1)$. Suppose $u_\lambda(\hat{t}) = 0$ for some $\hat{t} \in (0,1)$. Then u_λ satisfies the initial value problem

$$\begin{cases} -u_\lambda''(s) = \lambda h(s)\tilde{f}(u_\lambda(s)), \\ \quad u_\lambda'(\hat{t}) = u_\lambda(\hat{t}) = 0. \end{cases} \tag{5.2}$$

But $\tilde{f}(0) = 0$ and hence by the uniqueness result by Picard, $u_\lambda \equiv 0$, which is a contradiction. Hence $u_\lambda > 0$ in $(0,1)$.

Next we prove that the solution u_λ has only one interior maximum.

Lemma 15

Assume $(\tilde{C}_1)$, *and* $(\tilde{C}_2)$ *hold and let* u_λ *be the solution of* (5.1) *for* $\lambda \gg 1$. *Then* u_λ *has only one interior maximum.*

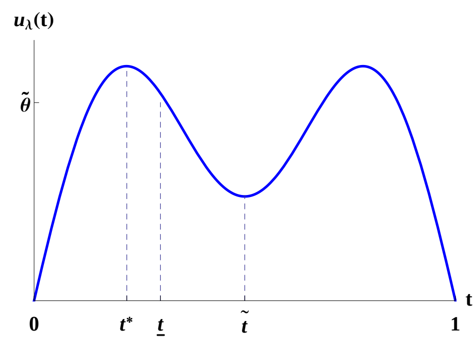

Figure 5.2

A solution with more than one maximum

Proof Let $\tilde{E}(t) := \lambda \tilde{F}(u_\lambda(t))h(t) + \frac{[u'_\lambda(t)]^2}{2}, t \in (0,1)$. Hence $\tilde{E}'(t) = \lambda \tilde{F}(u_\lambda(t))h'(t)$. Note that $h(s)$ decreases for $s > 0$. Let $\tilde{\theta}$ be such that $\tilde{\rho}_1 < \tilde{\theta} < \tilde{\rho}_2$ and $\tilde{F}(\tilde{\theta}) = 0$. Then $\tilde{E}(t)$ increases when $u_\lambda(t) < \tilde{\theta}$ and decreases when $u_\lambda(t) > \tilde{\theta}$. Let $t^* \in (0,1)$ be the first point at which u_λ has a local maximum, and assume $u_\lambda(t) \leq \tilde{\theta}, \forall t \leq t^*$. Integrating (1.11) from t to t^*, $t < t^*$, and using the integrability assumption on h,

$$u'_\lambda(t) \;\; = \;\; \lambda \int_t^{t^*} h(s)\tilde{f}(u_\lambda(s))ds \leq \lambda \frac{d\tilde{f}(\tilde{\theta})}{1-\alpha}(t^{*1-\alpha} - t^{1-\alpha}) \leq \lambda \frac{d\tilde{f}(\tilde{\theta})}{1-\alpha} \qquad (5.3)$$

where $d \geq c$ is such that $h(t) \leq \frac{d}{t^\alpha}$ for all $t \in (0,1)$ and $\alpha \in (0,1)$. Integrating (5.3) again from 0 to $t, t < t^*$, $u_\lambda(t) \leq \lambda M_0 t$ where $M_0 = \frac{d\tilde{f}(\tilde{\theta})}{1-\alpha}$. Since $\tilde{f}$ is continuous, there exists $k_0 > 0$ such that $|\tilde{F}(u_\lambda)| \leq k_0 u_\lambda$ for $u_\lambda \in [0, \tilde{\theta}]$. Hence

$$\lim_{t\to 0+} \lambda |\tilde{F}(u_\lambda(t))| h(t) \leq \lim_{t \to 0+} k_0 \lambda M_0 d t^{1-\alpha} = 0,$$

which implies $\lim_{t\to 0+} \tilde{E}(t) \geq 0$. Since $\tilde{E}(t)$ increases on $[0, t^*]$, $\tilde{E}(t^*) = \lambda \tilde{F}(u_\lambda(t^*))h(t^*)$ > 0, which is a contradiction if $u_\lambda(t^*) \leq \tilde{\theta}$. Hence $u_\lambda(t^*) > \tilde{\theta}$.

Now suppose u_λ has two interior maxima. Let $\tilde{t} \in (t^*, 1)$ be such that $u_\lambda'(\tilde{t}) = 0$ and $u_\lambda''(\tilde{t}) \geq 0$ (as in Figure 5.2). Since $u_\lambda''(\tilde{t}) = -\lambda h(\tilde{t})\tilde{f}(u_\lambda(\tilde{t})) \geq 0$, we see that $u_\lambda(\tilde{t}) \leq \tilde{\rho_1}$ and thus $\tilde{E}(\tilde{t}) < 0$. Let $\underline{t} \in (t^*, \tilde{t})$ be such that $u_\lambda(\underline{t}) = \tilde{\theta}$. Since $\tilde{E}(\underline{t}) \geq 0$ and $\tilde{E}$ increases in $(\underline{t}, \tilde{t})$, $\tilde{E}(\tilde{t}) > 0$ which is contradiction. Hence u_λ can have only one interior maximum and that maximum value is bigger than $\tilde{\theta}$.

Now we prove Theorem 8.

First modify f in $\mathbb{R} \setminus [0, \rho_2]$ as follows. Let $f(s) \leq 0$ for $s \in (\rho_2, \infty)$, $f(s) = 0$ for $s \in (-\infty, -1]$ and $\int_t^{\rho_2} f(s)\, ds > 0$ for $t \in [-1, 0)$ such that $f \in C^1$. By Lemma 14,

$$\begin{cases} -u''(s) = \mu h(s) f(u(s) - 1), \quad 0 < s < 1 \\ \\ \quad u(0) = u(1) = 0 \end{cases} \tag{5.4}$$

has a positive solution w for some μ large enough with $\max w \in (\rho_1 + 1, \rho_2 + 1]$. Define $v(t) = w(t) - 1$ for all $t \in (0,1)$.

By Lemma 15 v has only two zeros, say α_1, α_2 and $v > 0$ in (α_1, α_2). Extend v in $(1, \infty)$ such that $v(t) \leq -1$ and $v''(t) = 0$ for all $t \in (1, \infty)$. Also extend $h(t)$ as $h(t) = h(1)$ for all $t \in (1, \infty)$. Then $v(t)$ satisfies $-v''(s) = \mu h(s) f(v)$ in $(0, \infty)$ with $\max v \in (\rho_1, \rho_2]$. Now for a fixed $y_0 \in (0,1)$, define $\psi_{y_0}(\lambda, x) = v\Big((\frac{\lambda}{\mu})^{\frac{1}{2}} |x - y_0| + t^* \Big)$,

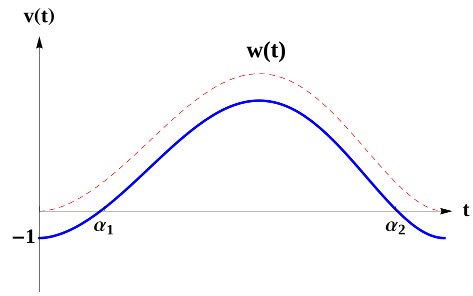

Figure 5.3

Graphs of v(t) and w(t)

where t^* is a point at which v has maximum. Let $\tilde{\Omega} = (0,1)$ and $d(y_0, \partial\tilde{\Omega})$ denote the distance from y_0 to the boundary of $\tilde{\Omega}$. If $\lambda > \mu(\alpha_2 - t^*)^2 d(y_0, \tilde{\Omega})^{-2} = \lambda^*$, $\psi_{y_0} < 0$ on $\partial\tilde{\Omega}$. Thus ψ_{y_0} is a subsolution of (1.11) for $\lambda > \lambda^*$. Clearly $Z = \rho_2$ is a supersolution of (1.11). Also the subsolution, $\psi_{y_0} \leq \rho_2$ for all λ. Thus (1.11) has a solution $u_{y_0} \in [\psi_{y_0}, \rho_2]$ if $\lambda > \lambda^*$.

Next we will show that $u_{y_0} > 0$, using the sweeping principle. For $y \in I_\lambda = \Big((\alpha_2 - t^*)(\frac{\mu}{\lambda})^{\frac{1}{2}}, 1 - (\alpha_2 - t^*)(\frac{\mu}{\lambda})^{\frac{1}{2}}\Big)$, define $\psi_y(\lambda, x) = v\Big((\frac{\lambda}{\mu})^{\frac{1}{2}}|x - y| + t^*\Big)$. Then $\{\psi_y, y \in I_\lambda\}$ is a family of subsolutions to the problem (1.11) with $\psi_y < 0$ on the $\partial\Omega$ and

- $y \to \psi_y(\lambda, x)$ is continuous with respect to $||.||_\infty$ and
- $y_0 \in I_\lambda$ and $u_{y_0} \geq \psi_{y_0}$ in $[0,1]$.

Thus by the sweeping principle, $u_{y_0}(x) > \psi_y(\lambda, x)$ for all $y \in I_\lambda$. For $x \in I_\lambda$, by choosing $y = x$, we see that $u_{y_0}(x) \geq v(t^*) > 0$. For $x \in (0,1) - I_\lambda$, we choose $y \in I_\lambda$ such that $d(x,y) < (\frac{\mu}{\lambda})^{\frac{1}{2}}(\alpha_2 - t^*)$. Since $t^* > \alpha_1$ and by the choice of y, $\alpha_1 < (\frac{\lambda}{\mu})^{\frac{1}{2}}|x - y| + t^* < \alpha_2$ which implies $\psi_y(\lambda, x) > 0$ for $x \in (0,1) - I_\lambda$. Hence $u_{y_0}(x) > 0$ for all $x \in (0,1)$.

5.2 Proof of Theorem 9

Let $F(s) = \int_0^s f(t)dt$. Note that there exist a positive real number θ such that $\rho_1 < \theta < \rho_2$ and $F(\theta) = 0$ (See Figure 5.4.)

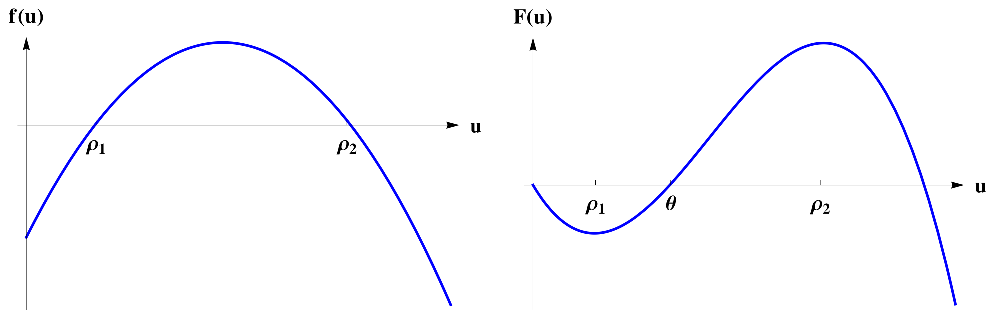

Figure 5.4

Graphs of $f(u)$ and $F(u)$

Let u denote a nonnegative solution of problem (1.11) for $\lambda \gg 1$ under the assumptions $(C_1) - (C_3)$. We first establish some properties of u, namely , Lemmas 16-19, which will help us to prove Theorem 9.

Lemma 16

u has only one interior maximum, say at t_0, and $u(t_0) > \theta$.

Proof. Follows by similar arguments as in the proof of Lemma 15.

Lemma 17

Let t_2 and $\hat{t_2} \in (0,1)$ be such that $t_2 < \hat{t_2}$ and $u(t_2) = u(\hat{t_2}) = \frac{\rho_1+\theta}{2}$, then $t_2, 1 - \hat{t_2} \leq O(\lambda^{-\frac{1}{2}})$.

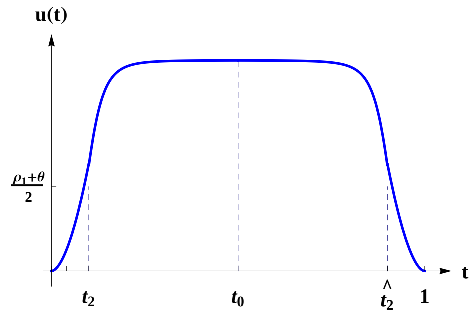

Figure 5.5

Graph of a solution

Proof. Let t_1 be the first point in $(0,1)$ such that $u(t_1) = \frac{\rho_1}{2}$. Integrating (1.11) from 0 to $t, t < t_1$,

$$u'(t) \;=\; u'(0) - \lambda \int_0^t h(s) f(u(s)) ds \geq \lambda \hat{h} t(-f(\frac{\rho_1}{2})). \tag{5.5}$$

Integrating (5.5) from 0 to t_1 yields $t_1 \leq \tilde{c}\lambda^{-\frac{1}{2}}$, where $\tilde{c} = \left(\frac{\rho_1}{\hat{h}(-f(\frac{\rho_1}{2}))}\right)^{\frac{1}{2}}$. Now let $E(t) := \lambda F(u)h(t) + \frac{[u'(t)]^2}{2}, t \in (0,1)$. As in the discussion in Lemma 15, $\lim_{t\to 0+} E(t) \geq 0$ and E is increasing if $u(t) < \theta$. Hence we have $E(t) \geq 0$ for all t $\in (0,1)$. This implies that

$$\frac{(u'(t))^2}{2} \geq \lambda(-F(u))h(t) \text{ for all t} \in (0,1).$$

For $t \in (t_1, t_2)$, $u'(t) \geq \lambda^{\frac{1}{2}} k_1$, where $k_1 = \min_{t \in (t_1,t_2)} \sqrt{-2F(u)h(t)} > 0$. Integrating this from t_1 to t_2, we obtain $(t_2 - t_1) \leq O(\lambda^{-\frac{1}{2}})$. Since $t_1 \leq \tilde{c}\lambda^{-\frac{1}{2}}$, this implies $t_2 \leq O(\lambda^{-\frac{1}{2}})$. Similarly $1 - \hat{t_2} \leq O(\lambda^{-\frac{1}{2}})$.

Lemma 18

$||u||_\infty \to \rho_2$ *as* $\lambda \to \infty$.

Proof. Suppose there exists $\epsilon > 0$ such that $||u||_\infty < \rho_2 - \epsilon$, for all $\lambda > 0$. Let $G(t,s)$ denote the Green's function of the operator $-u''$ with boundary condition $u(0) = 0 = u(1)$. Then for $t \in (0,1)$ we have

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)h(s)f(u(s))ds \\ &\geq \lambda \Big[\int_0^{t_2} G(t,s)h(s)f(u(s))ds + \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)h(s)f(u(s))ds \\ &\quad + \int_{\hat{t_2}}^1 G(t,s)h(s)f(u(s))ds \Big]. \end{aligned}$$

Since h is integrable, by Lemma 17 we have

$$u(t) \geq \frac{1}{2}\lambda \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)h(s)f(u(s))ds \text{ for } \lambda \gg 1.$$

By our assumption ($||u||_\infty < \rho_2 - \epsilon$), there exists $k_2 > 0$ such that $f(u(s)) > k_2$ in $[\frac{1}{4}, \frac{3}{4}]$. Then for all $t \in [\frac{1}{4}, \frac{3}{4}]$,

$$u(t) \geq \frac{1}{2}\lambda \hat{h} k_2 \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_{\frac{1}{4}}^{\frac{3}{4}} G(t,s)ds,$$

which is a contradiction, since all positive solutions of (1.11) are bounded above by ρ_2 . Hence $||u||_\infty \to \rho_2$ as $\lambda \to \infty$.

Lemma 19

Let $\tilde{\rho} \in (\rho_2 - \tau, \rho_2]$ and t_λ, $\hat{t_\lambda}$ be such that $u(t_\lambda) = u(\hat{t_\lambda}) = \tilde{\rho}$ with $t_\lambda < \hat{t_\lambda}$. Then $t_\lambda, (1 - \hat{t_\lambda}) \to 0$ as $\lambda \to \infty$.

Proof. By (C_3), $f'(s) < 0$ for $s \in [\tilde{\rho}, \rho_2]$ and by Lemma 18 there exists t_λ and $\hat{t_\lambda}$ such that $u(t_\lambda) = u(\hat{t_\lambda}) = \tilde{\rho}$ when λ is large. We first prove $t_\lambda \to 0$ as $\lambda \to \infty$. Suppose there exists $\gamma_1 > 0$ such that $t_\lambda > \gamma_1 > 0$ for all $\lambda > 0$. Then

$$\begin{aligned} u(t) &= \lambda \int_0^1 G(t,s)h(s)f(u(s))ds \\ &\geq \lambda \Big[\int_0^{t_2} G(t,s)h(s)f(u(s))ds + \int_{\gamma_0}^{\gamma_1} G(t,s)h(s)f(u(s))ds \\ &\quad + \int_{\hat{t_2}}^1 G(t,s)h(s)f(u(s))ds \Big], \end{aligned}$$

where γ_0 is such that $0 < t_2 < \gamma_0 < \gamma_1$. Now using Lemma 17, for t in say, $[\frac{1}{4}, \frac{3}{4}]$,

$$u(t) \geq \frac{1}{2} \hat{h} \lambda k_3 \inf_{t \in [\frac{1}{4}, \frac{3}{4}]} \int_{\gamma_0}^{\gamma_1} G(t,s)ds,$$

where $k_3 > 0$ is such that $f(u(s)) > k_3$ in $[\gamma_0, \gamma_1]$. This again contradicts the fact that solutions of (1.11) are bounded. Hence $t_\lambda \to 0$ as $\lambda \to \infty$. Similarly $(1 - \hat{t_\lambda}) \to 0$ as $\lambda \to \infty$.

Now we prove Theorem 9.

By Theorem 8, (1.11) has a positive solution for $\lambda \gg 1$. Note that (1.11) has a maximal solution, $\bar{u}$, since all positive solutions of (1.11) are bounded above by ρ_2, which is also a supersolution. To prove the uniqueness of the positive solution, u for $\lambda \gg 1$, we will show that $u \equiv \bar{u}$. Since $\bar{u}$ and u satisfy (1.11),

$$\begin{aligned} -(\bar{u} - u)''(t) &= \lambda h(t) \Big(f(\bar{u}(t)) - f(u(t)) \Big), \quad 0 < t < 1 \\ (\bar{u} - u)(0) &= (\bar{u} - u)(1) = 0. \end{aligned} \tag{5.6}$$

By the Mean Value Theorem there exists ξ such that $u \leq \xi \leq \bar{u}$ in $[0, 1]$ and

$$\begin{aligned} -(\bar{u} - u)''(t) &= \lambda h(t) f'(\xi)(\bar{u}(t) - u(t)), \quad 0 < t < 1 \\ (\bar{u} - u)(0) &= (\bar{u} - u)(1) = 0. \end{aligned} \tag{5.7}$$

Multiplying (1.11) by $(\bar{u} - u)$, (5.7) by u and integrating,

$$\lambda \int_0^1 \Big(f(u) - f'(u)u \Big) h(s)(\bar{u} - u)ds \leq 0. \tag{5.8}$$

Here we also used the concavity of f. Let $\tilde{\Omega}_+ = (t_\lambda, \hat{t_\lambda})$ and $\tilde{\Omega}_- = (0,1) - \tilde{\Omega}_+$, where t_λ is as in Lemma 19. Since $f'(s) \leq 0$ for $s > \tilde{\rho}$, there exists a constant $a > 0$ such that $f(z) - f'(z)z > a$ in $\tilde{\Omega}_+$. Also since f is concave, $f(z) - f'(z)z \geq f(0)$ for all $z > 0$. Thus

$$\int_{\tilde{\Omega}_+} ah(s)(\bar{u} - u)ds + \int_{\tilde{\Omega}_-} f(0)h(s)(\bar{u} - u)ds \leq 0. \tag{5.9}$$

By Lemma 19, $|\tilde{\Omega}_-| \to 0$ as $\lambda \to \infty$. Also using the facts that $(\bar{u} - u)$ is bounded and $h(s)$ is positive and integrable, we see that (5.9) is true only if $(\bar{u} - u) \equiv 0$.

CHAPTER 6

PROOFS OF THEOREMS 10-13

6.1 Proof of Theorem 10

We first construct a supersolution for (1.12). Let $Z = M_\lambda e_p$ where $M_\lambda \gg 1$ and e_p is the unique positive solution of

$$\begin{cases} -\Delta_p e_p = 1 & \text{in } \Omega \\ e_p = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Let $\tilde{f}(s) = \max_{t\in[0,s]} f(t)$. Then $f(s) \leq \tilde{f}(s)$, $\tilde{f}(s)$ is increasing and $\lim\limits_{u\to\infty} \frac{\tilde{f}(u)}{u^{p-1}} = \sigma$. Hence, we can choose $M_\lambda \gg 1$ such that

$$2\sigma \geq \frac{\tilde{f}(M_\lambda ||e_p||_\infty)}{(M_\lambda ||e_p||_\infty)^{p-1}}.$$

Now let $\hat{\lambda} = \frac{1}{2\sigma ||e_p||_\infty^{p-1}}$. For $\lambda \leq \hat{\lambda}$,

$$-\Delta_p Z = M_\lambda^{p-1} \geq \frac{\tilde{f}(M_\lambda ||e_p||_\infty)}{2\sigma ||e_p||_\infty^{p-1}} \geq \lambda \tilde{f}(M_\lambda e_p) \geq \lambda f(M_\lambda e_p) \geq \lambda f(Z) - \frac{1}{Z^\alpha}.$$

Hence Z is a supersolution of (1.12) if $\lambda \leq \hat{\lambda}$. Next we construct a subsolution. Consider the boundary value problem

$$\begin{cases} -\Delta_p z - \mu |z|^{p-2} z = -1 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega. \end{cases} \tag{6.2}$$

By the anti-maximum principle established in [37], there exists a constant $\xi = \xi(\Omega) > 0$ such that if $\mu \in (\mu_1, \mu_1 + \xi)$, where μ_1 is the principal eigenvalue of $-\Delta_p$ with Dirichlet boundary conditions, then the solution z of (6.2) is positive in Ω and $\frac{\partial z}{\partial \nu} < 0$ on $\partial\Omega$ where ν is the outer unit normal vector. Now fix $\mu \in (\mu_1, \mu_1 + \xi)$ and let z_μ denote the solution of (6.2). Since $z_\mu > 0$ in Ω and $\frac{\partial z_\mu}{\partial \nu} < 0$ on $\partial\Omega$, there exist $m > 0, A > 0, \delta > 0$ such that $|\nabla z_\mu| \geq m$ in Ω_δ and $z_\mu \geq A$ in $\Omega - \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega : d(x, \partial\Omega) < \delta\}$. Define $\psi = k_0 z_\mu^{\frac{p}{p-1+\alpha}}$ where $k_0 > 0$ is such that

$$\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{k k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2\sigma||e_p||_\infty^{p-1}}\Big) \leq \min\Big\{\frac{p^{p-1}(1-\alpha)(p-1)m^p}{(p-1+\alpha)^p}, \Big(\frac{p}{p-1+\alpha}\Big)^{p-1} A\Big\}. \quad (6.3)$$

Then

$$\begin{aligned}
\nabla\psi &= k_0\Big(\frac{p}{p-1+\alpha}\Big) z_\mu^{\frac{1-\alpha}{p-1+\alpha}} \nabla z_\mu,\\
-\Delta_p \psi &= -div(|\nabla\psi|^{p-2}\nabla\psi)\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} div\Big(z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} |\nabla z_\mu|^{p-2}\nabla z_\mu\Big)\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\Big\{(\nabla z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}).|\nabla z_\mu|^{p-2}\nabla z_\mu + z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \Delta_p z_\mu\Big\}\\
&= -k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\Big\{\frac{(1-\alpha)(p-1)}{p-1+\alpha} z_\mu^{\frac{-\alpha p}{p-1+\alpha}} |\nabla z_\mu|^p\\
&\qquad + z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} (1 - \mu z_\mu^{p-1})\Big\}\\
&= k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} \mu z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}}\\
&\qquad - \frac{k_0^{p-1}p^{p-1}(1-\alpha)(p-1)|\nabla z_\mu|^p}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}(p-1+\alpha)^p}. \qquad (6.4)
\end{aligned}$$

Now we let $s_0^*(\sigma, \Omega) = k_0||z_\mu^{\frac{p}{p-1+\alpha}}||_\infty$. If we can prove

$$-\Delta_p\psi \leq \lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}} - \lambda k - \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}, \qquad (6.5)$$

then (D_1) implies $-\Delta_p\psi \leq \lambda f(\psi) - \frac{1}{\psi^\alpha}$ and ψ will be a subsolution of (1.12). We will now prove (6.5) by comparing terms in (6.4) and (6.5). Let $\underline{\lambda} = \frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{\sigma_1}$. For $\lambda \geq \underline{\lambda}$,

$$k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1}\mu z_\mu^{\frac{p(p-1)}{p-1+\alpha}} \leq \lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{p(p-1)}{p-1+\alpha}}. \tag{6.6}$$

Also since $\lambda \leq \hat{\lambda} = \frac{1}{2\sigma||e_p||_\infty^{p-1}}$,

$$\begin{aligned}\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k &\leq \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \frac{k}{2\sigma||e_p||_\infty^{p-1}} \\ &= \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{k k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2\sigma||e_p||_\infty^{p-1}}\Big)\Big]. \end{aligned} \tag{6.7}$$

Now in Ω_δ we have $|\nabla z_\mu| \geq m$ and by (6.3),

$$\frac{1}{k_0^{p-1+\alpha}}(1 + \frac{k k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}}{2\sigma||e_p||_\infty^{p-1}})] \leq \frac{p^{p-1}(1-\alpha)(p-1)m^p}{(p-1+\alpha)^p}.$$

Hence

$$\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k \leq \frac{k_0^{p-1}p^{p-1}(1-\alpha)(p-1)|\nabla z_\mu|^p}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}(p-1+\alpha)^p} \text{ in } \Omega_\delta. \tag{6.8}$$

From (6.6) and (6.8) it can be seen that (6.5) holds in Ω_δ. We will now prove (6.5) holds also in $\Omega - \Omega_\delta$. Since $z_\mu \geq A$ in $\Omega - \Omega_\delta$ and by (6.3) and (6.7), we get

$$\begin{aligned}\frac{1}{k_0^\alpha z_\mu^{\frac{\alpha p}{p-1+\alpha}}} + \lambda k &\leq \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha p}{p-1+\alpha}}}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu \\ &= k_0^{p-1}\Big(\frac{p}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha)(p-1)}{p-1+\alpha}} \text{ in } \Omega - \Omega_\delta. \end{aligned} \tag{6.9}$$

From (6.6) and (6.9), (6.5) holds also in $\Omega - \Omega_\delta$. Thus ψ is a positive subsolution of (1.12) if $\lambda \in [\underline{\lambda}, \hat{\lambda}]$. We can now choose $M_\lambda \gg 1$ such that $\psi \leq Z$. Let $J(\Omega) = 2||e_p||_\infty^{p-1}\mu(\frac{p}{p-1+\alpha})^{p-1}$. If $\frac{\sigma_1}{\sigma} \geq J$ it is easy to see that $\underline{\lambda} \leq \hat{\lambda}$ and for $\lambda \in [\underline{\lambda}, \hat{\lambda}]$ we have a positive solution. This completes the proof of Theorem 10.

Remark. Note that in the proof the choice of k_0 can be adjusted easily to obtain a subsolution for all $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{\sigma})$ where $\underline{\lambda} = \frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{\sigma_1}$. Further, for the case when $p = 2$ using the asymptotically linear condition at ∞, a large enough supersolution can be created for all $\lambda \leq \frac{\lambda_1}{\sigma}$ (see [24] for details). Hence in the case $p = 2$, a positive solution exists for all $\lambda \in [\underline{\lambda}, \frac{\lambda_1}{\sigma})$.

6.2 Proof of Theorem 11

We first construct a supersolution for the system (1.13) when

$$\lambda \leq \frac{1}{(2\sigma)^{\frac{p-1}{p-1+\tau}}||e_p||_\infty^{p-1}} = \lambda_{**}.$$

Let $(Z_1, Z_2) = (M_\lambda e_p, [\lambda f_2(M_\lambda ||e_p||_\infty)]^{\frac{1}{p-1}} e_p)$, where e_p is as before and M_λ is a large positive constant. Since $\lim_{s\to\infty} \frac{f_1(f_2(s)^{p-1})}{s^{p-1}} = \sigma$, we can choose $M_\lambda \gg 1$ such that

$$2\sigma \geq \frac{f_1([f_2(M_\lambda ||e_p||_\infty)]^{\frac{1}{p-1}})}{(M_\lambda ||e_p||_\infty)^{p-1}}.$$

Then

$$-\Delta_p Z_1 = M_\lambda^{p-1} \geq \frac{f_1([f_2(M_\lambda ||e_p||_\infty)]^{\frac{1}{p-1}})}{||e_p||_\infty^{p-1} 2\sigma}.$$

Now since $\lambda \leq \lambda_{**}$ we have

$$\begin{aligned} -\Delta_p Z_1 &\geq \frac{\lambda^{\frac{p-1+\tau}{p-1}} ||e_p||_\infty^{p-1+\tau} f_1([f_2(M_\lambda ||e_p||_\infty)]^{\frac{1}{p-1}})}{||e_p||_\infty^{p-1}} \\ &= \lambda \lambda^{\frac{\tau}{p-1}} ||e_p||_\infty^\tau f_1([f_2(M_\lambda ||e_p||_\infty)]^{\frac{1}{p-1}}). \end{aligned}$$

Note that (D_4) implies $f_2(s) \to \infty$ as $s \to \infty$. Hence from (D_5) for $M_\lambda \gg 1$ we get

$$\begin{aligned} -\Delta_p Z_1 &\geq \lambda f_1(\lambda^{\frac{1}{p-1}} ||e_p||_\infty [f_2(M_\lambda ||e_p||_\infty)]^{\frac{1}{p-1}}) \\ &\geq \lambda f_1([\lambda f_2(M_\lambda ||e_p||_\infty)]^{\frac{1}{p-1}} ||e_p||_\infty) \geq \lambda f_1(Z_2) - \frac{1}{Z_1^\alpha}. \end{aligned} \tag{6.10}$$

Also,

$$-\Delta_p Z_2 = \lambda f_2(M_\lambda ||e_p||_\infty) \geq \lambda f_2(M_\lambda e_p) \geq \lambda f_2(Z_1) - \frac{1}{Z_2^\alpha}. \tag{6.11}$$

Hence, from (6.10) and (6.11) we see that (Z_1, Z_2) is a supersolution of (1.13) when $\lambda \leq \frac{1}{(2\sigma)^{\frac{p-1}{p-1+\tau}} ||e_p||_\infty^{p-1}}$. Next we let $\psi_1 = \psi_2 = k_0 z_\mu^{\frac{p}{p-1+\alpha}}$ where k_0 is as in (6.3) with $k = \max\{k_1, k_2\}$. Setting $s^* = k_0 ||z_\mu^{\frac{p}{p-1+\alpha}}||_\infty$ and following the steps in the proof of Theorem (10) it is now easy to see that (ψ_1, ψ_2) is a subsolution of (1.13) when $\lambda \in [\lambda_*, \lambda_{**}]$, where λ_{**} is as defined above and $\lambda_* = \frac{\mu(\frac{p}{p-1+\alpha})^{p-1}}{\min(\sigma_1, \sigma_2)}$. We now choose $M_\lambda \gg 1$ such that $\psi_1 \leq Z_1$ and $\psi_2 \leq Z_2$. Let $J^*(\Omega) = 2^{\frac{p-1}{p-1+\tau}} \mu(\frac{p}{p-1+\alpha})^{p-1} ||e_p||_\infty^{p-1}$. If $\frac{\min(\sigma_1,\sigma_2)}{\sigma^{\frac{p-1}{p-1+\tau}}} \geq J^*$, then the interval of λ for which we have positive solution is nonempty. Thus we have proven Theorem 11.

6.3 Proof of Theorem 12

We begin the proof by constructing a supersolution. Let $Z = M_\lambda e_p$ where $M_\lambda \gg 1$ and e_p is the unique positive solution of

$$\begin{cases} -(|e_p'|^{p-2} e_p')' = h(t) \quad \text{in } (0,1), \\ e_p(0) = 0 = e_p(1). \end{cases} \tag{6.12}$$

As in the proof of Theorem (10) it can be seen that Z is a supersolution of (1.14) when $\lambda \leq \hat{\lambda} = \frac{1}{2\sigma ||e_p||_\infty^{p-1}}$. Now consider the boundary value problem

$$\begin{cases} -(|z'|^{p-2} z')' - \mu |z|^{p-2} z = -1 \quad \text{in } (0,1), \\ z(0) = 0 = z(1). \end{cases} \tag{6.13}$$

By the anti-maximum principle established in [37], there exists a $\xi > 0$ such that if $\mu \in (\mu_1, \mu_1 + \xi),$ where μ_1 is the principal eigenvalue of

$$\begin{cases} -(|z'|^{p-2}z')' = \mu|z|^{p-2}z \quad \text{ in } (0,1), \\ z(0) = 0 = z(1), \end{cases} \tag{6.14}$$

then the solution z of (6.13) is positive in $(0,1)$ and $|z'| > 0$ at $s = 0,1$. Now fix a $\mu \in (\mu_1, \mu_1 + \xi)$ and let z_μ denote the solution of (6.13). Since $z_\mu > 0$ in $(0,1)$ and $|z'_\mu| > 0$ at $s = 0,1$, there exist $m > 0, A > 0, \epsilon > 0$ such that $|z'_\mu| \geq m$ in $(0,\epsilon] \cup [1-\epsilon, 1)$ and $z_\mu \geq A$ in $(\epsilon, 1-\epsilon)$ where $\epsilon < \epsilon_1$. Also note that there exists a $c > 0$ such that $0 < z_\mu(s) \leq cs(1-s)$ for all $s \in (0,1)$. Define $\psi = k_0 z_\mu^{\frac{p-\beta}{p-1+\alpha}}$, where $k_0 > 0$ is such that

$$\begin{aligned} &\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{kk_0^\alpha z_{\lambda^*}^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma||e_p||_\infty^{p-1}}\Big) \\ &\leq \min\Big\{\frac{(p-\beta)^{p-1}(1-\alpha-\beta)(p-1)m^p}{(p-1+\alpha)^p dc^\beta}, \Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1}\frac{A^{1-\beta}}{\bar{c}}\Big\}, \end{aligned} \tag{6.15}$$

where $\bar{c}$ is such that $h(s) \leq \bar{c}$ for all $s \in (\epsilon, 1-\epsilon)$. Then

$$\begin{aligned} &-(|\psi'|^{p-2}\psi')' = k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1}\mu z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}} \\ &- k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha-\beta)(p-1)}{p-1+\alpha}} - \frac{k_0^{p-1}(p-\beta)^{p-1}(1-\alpha-\beta)(p-1)|z'_\mu|^p}{z_\mu^{\frac{\alpha p+\beta p-\beta}{p-1+\alpha}}(p-1+\alpha)^p}. \end{aligned} \tag{6.16}$$

Let $s^*(\sigma,\Omega) = k_0||z_\mu^{\frac{p-\beta}{p-1+\alpha}}||_\infty$. If we can prove

$$-(|\psi'|^{p-2}\psi')' \leq h(s)\Big[\lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}} - \lambda k - \frac{1}{k_0^\alpha z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big], \tag{6.17}$$

then by (D_1), ψ will be a subsolution of (1.14). Now we compare the terms in (6.16) and (6.17) to see that (6.17) holds in $(0,1)$. Let $\underline{\tilde{\lambda}} = \frac{\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}}{\sigma_1 \hat{h}}$ where $\hat{h} = \inf_{s\in(0,1)} h(s) > 0$. For $\lambda \geq \underline{\tilde{\lambda}}$,

$$k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1}\mu z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}} \leq h(s)\lambda\sigma_1 k_0^{p-1} z_\mu^{\frac{(p-\beta)(p-1)}{p-1+\alpha}}. \tag{6.18}$$

Also, since $\lambda \leq \hat{\lambda}$,

$$h(s)\Big[\frac{1}{k_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] \leq h(s)\Big[\frac{1}{k_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \frac{k}{2\sigma||e_p||_\infty^{p-1}}\Big]$$

$$= \frac{h(s)k_0^{p-1}}{z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{kk_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma||e_p||_\infty^{p-1}}\Big)\Big]. \tag{6.19}$$

Now in $(0, \epsilon]$ we have $h(s) \leq \frac{d}{t^\beta}$ and $z_\mu \leq ct$. Hence

$$h(s)\Big[\frac{1}{k_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] \leq \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{kk_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma||e_p||_\infty^{p-1}}\Big)\Big]\frac{d}{t^\beta}$$

$$\leq \frac{k_0^{p-1}}{z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}\Big[\frac{1}{k_0^{p-1+\alpha}}\Big(1 + \frac{kk_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}}{2\sigma||e_p||_\infty^{p-1}}\Big)\Big]\frac{dc^\beta}{z_\mu^\beta}.$$

Also in $(0, \epsilon], |{z_\mu}'| \geq m$ and thus by (6.15) we have

$$h(s)\Big[\frac{1}{k_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] \leq \frac{k_0^{p-1}(p-\beta)^{p-1}(1-\alpha-\beta)(p-1)|z_\mu'|^p}{z_\mu^{\frac{\alpha p+\beta p-\beta}{p-1+\alpha}}(p-1+\alpha)^p}. \tag{6.20}$$

From (6.18) and (6.20) we see that (6.17) holds in $(0, \epsilon]$. Proving that (6.17) holds in $[1-\epsilon, 1)$ is easier since h is not singular at $s = 1$. Next we prove (6.17) holds also in $(\epsilon, 1-\epsilon)$. Since $z_\mu \geq A, h(s) \leq \bar{c}$ for all $s \in (\epsilon, 1-\epsilon)$ and by (6.15) and (6.19) we get

$$h(s)\Big[\frac{1}{k_0^{\alpha} z_\mu^{\frac{\alpha(p-\beta)}{p-1+\alpha}}} + \lambda k\Big] \leq k_0^{p-1}\Big(\frac{p-\beta}{p-1+\alpha}\Big)^{p-1} z_\mu^{\frac{(1-\alpha-\beta)(p-1)}{p-1+\alpha}}. \tag{6.21}$$

Thus (6.17) holds also in $(\epsilon, 1-\epsilon)$ and ψ is a subsolution of (1.14). Now we can choose $M_\lambda \gg 1$ such that $\psi \leq Z$. Hence (1.14) has a positive solution when $\lambda \in [\tilde{\underline{\lambda}}, \hat{\lambda}]$. Let $\bar{J}(\Omega) = \frac{2\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}||e_p||_\infty^{p-1}}{\hat{h}}$. It is clear that if $\frac{\sigma_1}{\sigma} \geq \bar{J}$ we have a nonempty interval of λ where (1.14) has a positive solution.

6.4 Proof of Theorem 13

The proof of Theorem 13 follows using similar arguments as in the proof of Theorem 11 with the necessary adjustments to overcome the singularity from $h(s)$ (as done in the proof of Theorem 12). Here, $s^* = k_0||z_\mu^{\frac{p-\beta}{p-1+\alpha}}||_\infty,\ \bar{J}^*(\Omega) = \frac{2^{\frac{p-1}{p-1+\tau}}\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}||e_p||_\infty^{p-1}}{\hat{h}},\ \tilde{\lambda}_* = \frac{\mu(\frac{p-\beta}{p-1+\alpha})^{p-1}}{\min(\sigma_1,\sigma_2)\hat{h}}$ and $\lambda_{**} = \frac{1}{\sigma^{\frac{p-1}{p-1+\tau}}||e_p||_\infty^{p-1}}$.

CHAPTER 7

PROOFS OF THEOREMS 14-16

7.1 Proof of Theorem 14

We first construct a subsolution. Consider the eigenvalue problem $-\Delta_p \phi = \lambda |\phi|^{p-2}\phi$ in Ω, $\phi = 0$ on $\partial\Omega$. Let ϕ_1 be an eigenfunction corresponding to the first eigenvalue λ_1 such that $\phi_1 > 0$ and $||\phi_1||_\infty = 1$. Also let $\delta, m, \mu > 0$ be such that $|\nabla \phi_1| \geq m$ in Ω_δ and $\phi_1 \geq \mu$ in $\Omega - \Omega_\delta$, where $\Omega_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. Let $\beta \in (1, \frac{p}{p-1+\alpha})$ be fixed. Here note that since $\alpha \in (0,1)$, $\frac{p}{p-1+\alpha} > 1$. Choose a $k > 0$ such that $2bk^{\gamma-p} + \beta^{p-1}\lambda_1 k^\alpha \leq a$. Define $c_1 = \min\left\{k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p, \frac{1}{2}k^{p-1}\mu^{\beta(p-1)}\left(a - \beta^{p-1}\lambda_1 k^\alpha\right)\right\}$. Note that $c_1 > 0$ by the choice of k and β. Let $\psi = k\phi_1^\beta$. Then

$$-\Delta_p \psi \;=\; k^{p-1}\beta^{p-1}\lambda_1 \phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla \phi_1|^p}{\phi_1^{p-\beta(p-1)}}.$$

To prove that ψ is a subsolution we need to establish:

$$\begin{aligned} &k^{p-1}\beta^{p-1}\lambda_1 \phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla \phi_1|^p}{\phi_1^{p-\beta(p-1)}} \\ &\leq ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)} - \frac{c}{k^\alpha \phi_1^{\alpha\beta}} \end{aligned} \tag{7.1}$$

in Ω if $c < c_1$. To achieve this, we split the term $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)}$ into three, namely, $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} = ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - \frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\left(a - k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\right)$

$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big)$. Now to prove (7.1) holds in Ω, it is enough to show the following three inequalities.

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big)\leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}, \text{ in } \Omega, \tag{7.2}$$

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big)\leq -\frac{c}{k^\alpha\phi_1^{\alpha\beta}}, \text{ in } \Omega-\Omega_\delta, \tag{7.3}$$

$$-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}}\leq -\frac{c}{k^\alpha\phi_1^{\alpha\beta}}, \text{ in } \Omega_\delta. \tag{7.4}$$

From the choice of k,

$-(a-\beta^{p-1}\lambda_1k^\alpha)\leq -2bk^{\gamma-p}$, hence

$$\begin{aligned}-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big) &\leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\\ &\leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}.\end{aligned} \tag{7.5}$$

Using $\phi_1\geq\mu$ in $\Omega-\Omega_\delta$ and $c<\frac{1}{2}k^{p-1}\mu^{\beta(p-1)}(a-\beta^{p-1}\lambda_1k^\alpha)$,

$$\begin{aligned}-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big) &\leq \frac{-k^{p-1}\phi_1^{\beta(p-1)}\big(a-k^\alpha\lambda_1\beta^{p-1}\big)}{2k^\alpha\phi_1^{\alpha\beta}}\\ &\leq \frac{-c}{k^\alpha\phi_1^{\alpha\beta}}.\end{aligned} \tag{7.6}$$

Finally, since $|\nabla\phi_1|\geq m$, in Ω_δ and $c<k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p$,

$$\begin{aligned}-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}} &\leq \frac{-k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{k^\alpha\phi_1^{\alpha\beta}\phi_1^{p-\beta(p-1)-\alpha\beta}}\\ &\leq \frac{-c}{k^\alpha\phi_1^{\alpha\beta}\phi_1^{p-\beta(p-1+\alpha)}}.\end{aligned}$$

Since $p-\beta(p-1+\alpha)>0$,

$$-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}}\leq\frac{-c}{k^\alpha\phi_1^{\alpha\beta}}. \tag{7.7}$$

From (7.5), (7.6) and (7.7) we see that equation (7.1) holds in Ω, if $c < c_1$. Next we construct a supersolution. Let e be the solution of $-\Delta_p e = 1$ in $\Omega, e = 0$ on $\partial\Omega$. Choose $\bar{M} > 0$ such that $\frac{au^{p-1} - bu^{\gamma-1} - c}{u^\alpha} \leq \bar{M}^{p-1}\ \forall u > 0$ and $\bar{M}e \geq \psi$. Define $Z = \bar{M}e$. Then Z is a supersolution of (1.2). Thus Theorem (14) is proven.

7.2 Proof of Theorem 15

We begin the proof by constructing a subsolution. Consider

$$\begin{aligned} -(|\phi'|^{p-2}\phi')' &= \lambda|\phi|^{p-2}\phi,\ t \in (0,1), \\ \phi(0) &= \phi(1) = 0. \end{aligned} \tag{7.8}$$

Let ϕ_1 be an eigenfunction corresponding to the first eigenvalue of (7.8) such that $\phi_1 > 0$ and $||\phi_1||_\infty = 1$. Then there exist $d_1 > 0$ such that $0 < \phi_1(t) \leq d_1 t(1-t)$ for $t \in (0,1)$. Also let $\epsilon < \epsilon_1$ and $m, \mu > 0$ be such that $|\phi_1'| \geq m$ in $(0,\epsilon] \cup [1-\epsilon, 1)$ and $\phi_1 \geq \mu$ in $(\epsilon, 1-\epsilon)$. Let $\beta \in (1, \frac{p-\rho}{p-1+\alpha})$ be fixed and choose $k > 0$ such that $2bk^{\gamma-p} + \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}} \leq a$. Define $c_2 = \min\left\{\frac{k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{d_1^\rho}, \frac{1}{2}k^{p-1}\mu^{\beta(p-1)}(a - \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}})\right\}$. Then, $c_2 > 0$ by the choice of k and β. Let $\psi = k\phi_1^\beta$. This implies that:

$$-(|\psi'|^{p-2}\psi')' = k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}}.$$

To prove that ψ is a subsolution, we need to establish:

$$\begin{aligned} &k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{\phi_1'^{\,p}}{\phi_1^{p-\beta(p-1)}} \\ &\leq h(t)(ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)} - \frac{c}{k^\alpha\phi_1^{\alpha\beta}}). \end{aligned} \tag{7.9}$$

Here, we note that $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} = \frac{\hat{h}k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)}}{\hat{h}} \leq h(t)\big(ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}$ $-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\big) - \frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a - \frac{k^\alpha\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\big)\big)$ where

$\hat{h} = \inf_{s\in(0,1)} h(s)$. Now to prove (7.9) holds in $(0,1)$, it is enough to show the following three inequalities.

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a - \frac{k^\alpha \phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\big) \le -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}, \text{in } (0,1), \tag{7.10}$$

$$-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a - \frac{k^\alpha \phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\big) \le -\frac{c}{k^\alpha\phi_1^{\alpha\beta}}, \text{in } (\epsilon, 1-\epsilon), \tag{7.11}$$

$$-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}} \le -\frac{ch(t)}{k^\alpha\phi_1^{\alpha\beta}}, \text{in } (0,\epsilon]\cup[1-\epsilon,1). \tag{7.12}$$

From the choice of k,

$-(a - \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}}) \le -2bk^{\gamma-p}$, hence,

$$\begin{aligned}-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a - \frac{k^\alpha \phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\big) &\le -bk^{\gamma-1-\alpha}\phi_1^{\beta(p-1-\alpha)} \\ &\le -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}.\end{aligned} \tag{7.13}$$

Using $\phi_1 \ge \mu$ in $(\epsilon, 1-\epsilon)$ and $c < \frac{1}{2}k^{p-1}\mu^{\beta(p-1)}\big(a - \frac{\beta^{p-1}\lambda_1 k^\alpha}{\hat{h}}\big)$,

$$\begin{aligned}-\frac{1}{2}k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a - \frac{k^\alpha \phi_1^{\alpha\beta}\beta^{p-1}\lambda_1}{\hat{h}}\big) &\le \frac{-k^{p-1}\phi_1^{\beta(p-1)}\big(a - \frac{k^\alpha\lambda_1\beta^{p-1}}{\hat{h}}\big)}{2k^\alpha\phi_1^{\alpha\beta}} \\ &\le \frac{-c}{k^\alpha\phi_1^{\alpha\beta}}.\end{aligned} \tag{7.14}$$

Next we prove (7.12) holds in $(0,\epsilon]$. Since $|\phi_1'| \ge m$ and $p - \beta(p-1) > \alpha\beta + \rho$,

$$\begin{aligned}-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}} &\le \frac{-k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{k^\alpha\phi_1^{\alpha\beta}\phi_1^\rho} \\ &\le \frac{-k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{k^\alpha\phi_1^{\alpha\beta}d_1^\rho t^\rho}.\end{aligned}$$

Since $h(t) \le \frac{1}{t^\rho}$ in $(0,\epsilon]$, and $c < \frac{k^{p-1+\alpha}\beta^{p-1}(\beta-1)(p-1)m^p}{d_1^\rho}$,

$$-k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\phi_1'|^p}{\phi_1^{p-\beta(p-1)}} \le \frac{-ch(t)}{k^\alpha\phi_1^{\alpha\beta}}. \tag{7.15}$$

Proving (7.12) holds in $[1-\epsilon, 1)$ is straight forward since h is not singular at $t=1$. Thus from equations (7.13), (7.14) and (7.15), we see that (7.9) holds in $(0,1)$. Hence ψ is a subsolution. Let $Z=\bar{M}e$ where e satisfies $-(|e'|^{p-2}e')'=h(t)$ in $(0,1), e(0)=e(1)=0$ and $\bar{M}$ is such that $\frac{au^{p-1}-bu^{\gamma-1}-c}{u^{\alpha}} \leq \bar{M}^{p-1}\ \forall u>0$ and $\bar{M}e \geq \psi$. Then Z is a supersolution of (1.17) and there exists a solution u of (1.17) such that $u \in [\psi, Z]$. Thus Theorem (15) is proven.

7.3 Proof of Theorem 16

We first prove (1.18) has a positive solution for every $a>0$. We begin by constructing a subsolution. Let ϕ_1 be as in the proof of Theorem 14. Let $\beta \in (1, \frac{p}{p-1})$, and choose a $k>0$ such that $bk^{\gamma-p}+\beta^{p-1}\lambda_1 k^{\alpha} \leq a$. Let $\psi=k\phi_1^{\beta}$. Then,

$$-\Delta_p\psi \;=\; k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} - k^{p-1}\beta^{p-1}(\beta-1)(p-1)\frac{|\nabla\phi_1|^p}{\phi_1^{p-\beta(p-1)}}.$$

To prove that ψ is a subsolution, we will establish:

$$k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} \leq ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)} \tag{7.16}$$

in Ω. To achieve this, we rewrite the term $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)}$ as $k^{p-1}\beta^{p-1}\lambda_1\phi_1^{\beta(p-1)} = ak^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)} - k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^{\alpha}\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big)$. Now to prove (7.16) holds in Ω, it is enough to show $-k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^{\alpha}\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big) \leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}$. From the choice of k, $-(a-\beta^{p-1}\lambda_1k^{\alpha}) \leq -bk^{\gamma-p}$, hence

$$\begin{aligned} -k^{p-1-\alpha}\phi_1^{\beta(p-1-\alpha)}\big(a-k^{\alpha}\phi_1^{\alpha\beta}\beta^{p-1}\lambda_1\big) \;&\leq\; -bk^{\gamma-1-\alpha}\phi_1^{\beta(p-1-\alpha)} \\ &\leq -bk^{\gamma-1-\alpha}\phi_1^{\beta(\gamma-1-\alpha)}. \end{aligned}$$

Thus ψ is a subsolution. It is easy to see that $Z = (\frac{a}{b})^{\frac{1}{\gamma - p}}$ is a supersolution of (1.18). Since k can be chosen small enough, $\psi \leq Z$. Thus (1.18) has a positive solution for every $a > 0$. Also all positive solutions are bounded above by Z. Hence when a is close to 0, every positive solution of (1.18) approaches 0. Also $u \equiv 0$ is a solution for every a. This implies that we have a branch of positive solutions bifurcating from the trivial branch of solutions $(a, 0)$ at $(0, 0)$.

CHAPTER 8

COMPUTATIONAL RESULTS

8.1 Computational results for (1.12) in the one dimensional case

Here we consider the boundary value problem

$$\begin{cases} -u''(x) = \lambda f(u) - \frac{1}{u^{\alpha}}, \quad x \in (0,1), \\ \\ u(0) = 0 = u(1), \end{cases} \tag{8.1}$$

where $f(s) = s^{p-1} + m_0 s^{\frac{1}{2}} - 2; m_0 > 0$ and $\alpha \in (0,1)$. Using the quadrature method (see [27]), it follows that the bifurcation diagram of positive solutions of (8.1) is given by

$$G(\rho, \lambda) = \int_0^{\rho} \frac{ds}{\sqrt{[2\lambda(F(\rho) - F(s)) - (\frac{\rho^{1-\alpha} - s^{1-\alpha}}{1-\alpha})]}} = \frac{1}{2}, \tag{8.2}$$

where $F(s) := \int_0^s f(t)dt$ and $\rho = u(\frac{1}{2}) = ||u||_{\infty}$. Now we use Mathematica to plot (8.2) and provide the exact bifurcation diagrams when $m_0 = 10$ and $m_0 = 5000$ (See Figure 8.1).

8.2 Computational results for (1.16) and (1.18) in the one dimensional case

Consider the boundary value problem

$$\begin{cases} -u''(x) = \frac{au - bu^2 - c}{u^{\alpha}}, \quad x \in (0,1), \\ \\ u(0) = 0 = u(1), \end{cases} \tag{8.3}$$

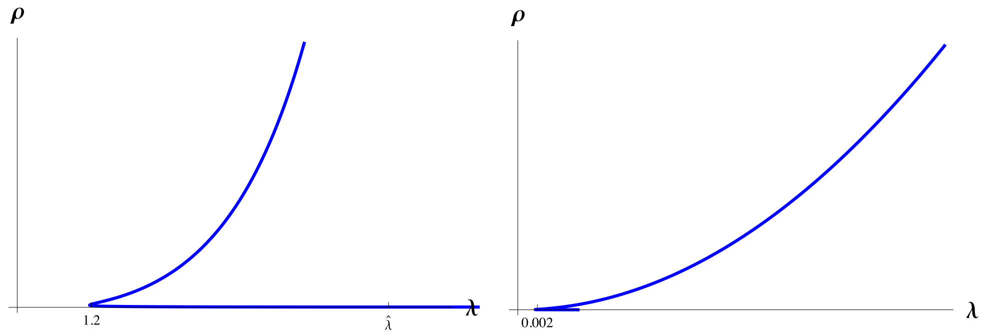

Figure 8.1

Bifurcation diagrams with $m_0 = 10$, $m_0 = 5000$ respectively

where $a, b > 0, c \geq 0$ and $\alpha \in (0, 1)$. Using the quadrature method (See [27]) the bifurcation diagram of positive solutions of (8.3) is given by

$$G(\rho, c) = \int_0^\rho \frac{ds}{\sqrt{[2(F(\rho) - F(s))]}} = \frac{1}{2}, \tag{8.4}$$

where $F(s) := \int_0^s f(t)dt$ where $f(t) = \frac{at - bt^2 - c}{t^\alpha}$ and $\rho = u(\frac{1}{2}) = ||u||_\infty$. We plot the exact bifurcation diagram of positive solutions of (8.3) using Mathematica. Figure 8.2 shows bifurcation diagrams of positive solutions of (8.3) when $a = 8$ $(< \lambda_1)$ and $b = 1$ for different values of α.

Bifurcation diagrams of positive solutions of (8.3) when $a = 15$ $(> \lambda_1)$ and $b = 1$ for different values of α is shown in Figure 8.3.

Finally, we provide the exact bifurcation diagram for the case when $p = 2, \Omega = (0, 1)$ and $c = 0$. Consider,

$$\begin{cases} -u''(x) = \frac{au - bu^2}{u^\alpha}, \quad x \in (0, 1), \\ \\ u(0) = 0 = u(1), \end{cases} \tag{8.5}$$

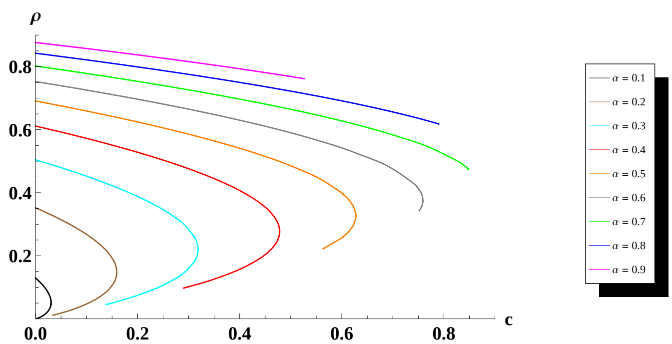

Figure 8.2

Bifurcation diagrams, c vs ρ for (8.3) with $a = 8$, $b = 1$

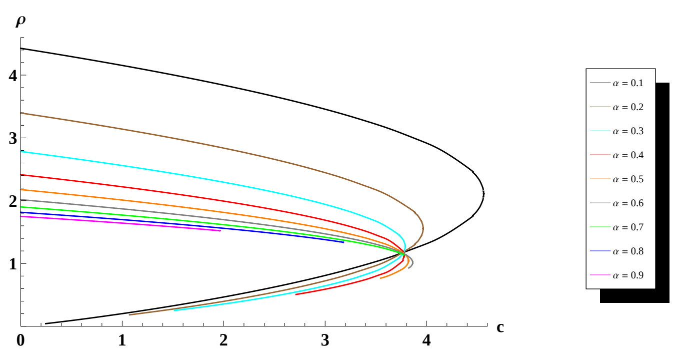

Figure 8.3

Bifurcation diagrams, c vs ρ for (8.3) with $a = 15$, $b = 1$

where $a, b > 0$ and $\alpha \in (0, 1)$. The bifurcation diagram of positive solutions of (8.5) is given by

$$\tilde{G}(\rho, a) = \int_0^{\rho} \frac{ds}{\sqrt{[2(\tilde{F}(\rho) - \tilde{F}(s))]}} = \frac{1}{2}, \tag{8.6}$$

where $\tilde{F}(s) := \int_0^s \tilde{f}(t)dt$ with $\tilde{f}(t) = \frac{at - bt^2}{t^{\alpha}}$ and $\rho = u(\frac{1}{2}) = ||u||_{\infty}$. The bifurcation diagram of positive solutions of (8.5) as well as the trivial solution branch are shown in Figure 8.4 when $\alpha = 0.5$ and $b = 1$.

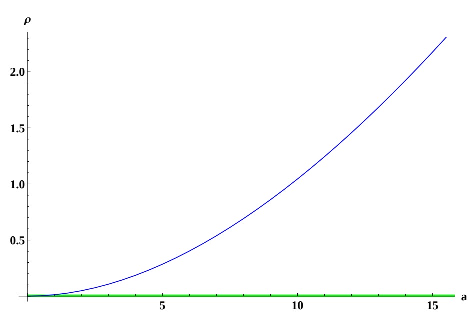

Figure 8.4

Bifurcation diagram, a vs ρ for (8.5) with $\alpha = 0.5$, $b = 1$

This bifurcation diagram (Figure 8.4) indicates that $(0, 0)$ is a bifurcation point of (8.5) as in Theorem 16.

CHAPTER 9

CONCLUSIONS AND FUTURE DIRECTIONS

9.1 Conclusions

In this thesis, we have extended the theory of semipositone problems to exterior domains, including problems involving the $p-$ Laplacian operator as well as systems, and to the case of infinite semipositone problems. We have also established new results in the bounded domain.

9.2 Future directions

We plan to continue and expand the theory of infinite semipositone problems. In the near future, we will study the following open problems.

- Consider
$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ u \quad = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.1}$$
where λ is a positive parameter, $\Delta u = \mathrm{div}\big(\nabla u\big)$ is the Laplacian of u, Ω is a smooth bounded domain in $\mathbb{R}^n, n \geq 1$, and $g : (0,\infty) \to \mathbb{R}$ is a C^1 function such that $\lim_{u\to 0^+} g(u) = -\infty$, and satisfies a sublinear growth condition ($\lim_{s\to\infty} \frac{g(s)}{s} = 0$). We will aim to prove uniqueness results for large values of parameter λ.

- Consider
$$\begin{cases} -u''(t) = \lambda h(t) g(u), & 0 < t < 1 \\ \quad u(0) = u(1) = 0, \end{cases} \tag{9.2}$$
where λ is a positive parameter, $g : (0,\infty) \to \mathbb{R}$ is a C^1 function such that $\lim_{u\to 0^+} g(u) = -\infty$, and satisfies $\lim_{s\to\infty} \frac{g(s)}{s} = 0$, and $h \in C((0,1],(0,\infty))$ is singular at $t = 0$. We will aim to prove uniqueness results for large values of parameter λ.

- Consider
$$\begin{cases} -\Delta u = \lambda g(u) & \text{in } \Omega \\ \ u \quad = 0 & \text{on } \partial\Omega, \end{cases} \tag{9.3}$$
where λ is a positive parameter, $\Delta u = \text{div}\big(\nabla u\big)$ is the Laplacian of u, Ω is a smooth bounded domain in $\mathbb{R}^n, n \geq 1$, and $g : (0, \infty) \to \mathbb{R}$ is a C^1 function such that $\lim_{u \to 0^+} g(u) = -\infty$, and $\lim_{s \to \infty} \frac{g(s)}{s} = \infty$ (superlinear grwoth condition). We will aim to prove existence and uniqueness of positive solutions when $\lambda \approx 0$.

- Consider
$$\begin{cases} -u''(t) = \lambda h(t) g(u), \quad 0 < t < 1 \\ \quad u(0) = u(1) = 0, \end{cases} \tag{9.4}$$
where λ is a positive parameter, $g : (0, \infty) \to \mathbb{R}$ is a C^1 function such that $\lim_{u \to 0^+} g(u) = -\infty$, and $\lim_{s \to \infty} \frac{g(s)}{s} = \infty$ and $h \in C((0, 1], (0, \infty))$ is singular at $t = 0$. We will aim to prove existence and uniqueness results for $\lambda \approx 0$.

- We will also aim to extend the analysis of the above open problems to the case of systems, and to problems involving the $p-$Laplacian operator.

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