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Infinite semipositone systems

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INFINITE SEMIPOSITONE SYSTEMS

By

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We study positive solutions to classes of nonlinear elliptic singular problems of the form:

$$\begin{aligned} -\Delta_p u &= \lambda \frac{g(u)}{u^\alpha} \quad && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$, λ is a positive parameter, $\alpha \in (0,1)$, $\Delta_p u := div(|\nabla u|^{p-2} \nabla u); p > 1$ is the $p-$Laplacian operator, and g is a smooth function. Such elliptic problems naturally arise in the study of steady state reaction diffusion processes.

In particular, we will be interested in the challenging new class of problems when $g(0) < 0$ (hence $lim_{s \to 0^+} \frac{g(s)}{s^\alpha} = -\infty$) which we refer to as infinite semipositone problems. Our focus is on existence results. We obtain results for the single equation case as well as to the case of systems. We use the method of sub-super solutions to prove our results.

The results in this dissertation provide a solid foundation for the analysis of such infinite semipositone problems.

Key words: Infinite semipositone systems, sub-super solutions, positive solutions

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LIST OF SYMBOLS, ABBREVIATIONS AND NOMENCLATURE

Ω bounded domain of $\mathbb{R}^N$.

$\partial\Omega$ boundary of Ω.

$\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) < \delta\}$.

$C((0,\infty))$ is the space of continuous real valued functions on $(0,\infty)$.

$C(\Omega)$ is the space of continuous real valued functions on Ω.

$C^m(\Omega)$ is the space of continuously $m-$times differentiable functions on Ω.

$C^\infty(\Omega) = \bigcap_{k=0}^\infty C^k(\Omega)$.

$C_0^\infty(\Omega)$ is the space of functions in $C^\infty(\Omega)$ with compact support in Ω.

$W^{m,p}(\Omega)$ is the Sobolev space of order m for $1 \leq p \leq \infty$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Δu Laplacian of u, i.e., $\Delta u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_Nx_N}$.

$\Delta_p u$ $p-$Laplacian of u, i.e., $\Delta_p u = div(|\nabla u|^{p-2} \nabla u)$.

CHAPTER 1

INTRODUCTION

Nonlinear eigenvalue problems of the form:

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where λ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N, N \geq 1$ with smooth boundary $\partial\Omega$ (C^∞ boundary), Δ is a Laplacian operator, and $f : [0, \infty) \to \mathbb{R}$ is a continuous function, arise in the study of steady state reaction diffusion processes, in particular, nonlinear heat generation, combustion theory, chemical reactor theory and population dynamics (see Parks [Pk], Sattinger [Sa], Parter [Pt], Tam [Ta], Aris [Ar] and Roberts [RS]). In the case when $f(0) > 0$ (positone problems) there is a very rich history (spanning over 50 years) on the study of positive solutions (see Amann [Am], Brown [BIS], Cohen [CK], Crandall [CR], de Figueiredo [DLN], Gidas [GNN], Joseph [JL], Kazdan [KW], Laetsch [La], and Rabinowitz [Ra]).

The case when $f(0) < 0$ (semipositone problems) is mathematically more challenging as pointed out by P. L. Lions [Li]. See also Castro [CMS]. However, in the past 20 years, there has been considerable progress on the study of semipositone problems (see [AAB], [ACS], [AHS], [AZ], [BCS], [BS], [CG], [CGS1-3], [CGSh], [CHS], [CS1-3], [Te]). One of the main tools used in these studies is the method of sub-super solutions. By a sub-solution of (1.1) we mean a function $\psi \in C^2(\Omega)\bigcap C(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta\psi &\leq \lambda f(\psi) \qquad \text{in } \Omega \\ \psi &\leq 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}$$

and by a super-solution of (1.1) we mean a function $Z \in C^2(\Omega)\bigcap C(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta Z &\geq \lambda f(Z) \qquad \text{in } \Omega \\ Z &\geq 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

Then it is well known (see [Am], [DS]) that if there exists a sub-solution ψ and a super-solution Z such that $\psi \leq Z$ in Ω then (1.1) has a solution u such that $\psi \leq u \leq Z$. In applying this method to obtain positive solutions, it is essential that we are able to construct non-negative strict sub-solutions. By a strict sub-solution we mean a sub-solution that is not a solution.

We will first briefly review some important results obtained by this method of sub-super solutions in the cases: $f(0) > 0$ (finite), $f(0) = 0$ and $f(0) < 0$ (finite).

In the case when $f(0) > 0$ (positone problems), it is trivial to see that $\psi = 0$ is a strict sub-solution for every $\lambda > 0$. We consider the problem (1.1) under the following assumptions:

$(H1)$ $f(0) > 0$.

$(H2)$ $lim_{s\to\infty}\frac{f(s)}{s} = 0$.

Then the following result holds:

Lemma 1

Assume $(H1)$ and $(H2)$. Then (1.1) has a positive solution for all $\lambda > 0$.

Proof: It is clear that $\psi = 0$ is a strict sub-solution since $f(0) > 0$. Let $\widetilde{f}(s) := \max_{t\in[0,s]} f(t)$. Then $f(s) \leq \widetilde{f}(s)$, $\widetilde{f}$ is nondecreasing and $lim_{s\to\infty}\frac{\widetilde{f}(s)}{s} = 0$. Hence we can choose $m(\lambda) \gg 1$ such that

$$\frac{1}{\|e\|_\infty \lambda} \geq \frac{\widetilde{f}(m(\lambda)\|e\|_\infty)}{m(\lambda)\|e\|_\infty}$$

where e is the solution of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$. Let $Z := m(\lambda)e$. Then

$$-\Delta Z = m(\lambda) \geq \lambda \widetilde{f}(m(\lambda)\|e\|_\infty) \geq \lambda \widetilde{f}(m(\lambda)e) \geq \lambda f(m(\lambda)e).$$

Thus Z is a super-solution. Hence (1.1) has a positive solution. □

The case when $f(0) = 0$ also causes considerable problems in the construction of positive sub-solutions, specially in the case when we have no other information at the origin. Here we consider the problem (1.1) under the following assumption:

$(H3)$ $f(0) = 0$ and $f'(0) > 0$.

Lemma 2

Assume $(H2)$ and $(H3)$. Then (1.1) has a positive solution for $\lambda > \frac{\lambda_1}{f'(0)}$.

Proof: Since $f'(0) > \frac{\lambda_1}{\lambda}$ we know that there exist $\widetilde{m} = \widetilde{m}(\lambda) > 0$ such that

$$f(s) > \frac{\lambda_1}{\lambda} s \qquad \text{for all } s \in (0, \widetilde{m}). \tag{1.2}$$

Let $\psi := \widetilde{m}\phi$ where $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary condition and ϕ is the corresponding eigenfunction satisfying $\phi > 0$ in Ω, $\frac{\partial \phi}{\partial \nu} <$

0 on $\partial\Omega$, where ν is the outward normal vector on $\partial\Omega$ and $\|\phi\|_\infty = 1$. Note that λ_1 and ϕ satisfy:

$$\begin{cases} -\Delta\phi = \lambda_1\phi & \text{in } \Omega \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then

$$-\Delta\psi = \lambda_1 \widetilde{m}\phi \leq \lambda f(\widetilde{m}\phi) = \lambda f(\psi).$$

Thus ψ is a positive sub-solution of (1.1). Next let $Z = m(\lambda)e$ where e is as defined in the proof of Lemma 1. Then as in the proof of Lemma 1 we can choose $m(\lambda) \gg 1$ so that Z is a super-solution. Further, since $e > 0$; Ω, and $\frac{\partial e}{\partial \nu} < 0$; $\partial\Omega$, where ν is the outward normal vector on $\partial\Omega$, we can choose $m(\lambda) \gg 1$ so that $Z \geq \psi$ in Ω. Thus (1.1) has a positive solution $u \in [\psi, Z]$ for $\lambda > \frac{\lambda_1}{f'(0)}$. □

For the case $f(0) = 0$, we can also study (1.1) when f does not satisfy $(H2)$ but satisfies:

$(H4)$ There exists $r_0 > 0$ such that $f(s) > 0$ for $s \in (0, r_0)$ and $f(r_0) = 0$.

Lemma 3

Assume $(H4)$. Then (1.1) has a positive solution for $\lambda \gg 1$.

Proof: Fix $\sigma \in (0, r_0)$ and let $\psi := \frac{\sigma}{2}\phi^2$ where ϕ is as defined in the proof of Lemma 2. Then

$$-\Delta\psi = -\sigma\{|\nabla\phi|^2 - \lambda_1\phi^2\}.$$

Let $\delta > 0, \mu > 0, m > 0$ be such that $\mid \nabla\phi \mid^2 - \lambda_1\phi^2 \geq m$ in $\overline{\Omega_\delta}$ and $\mu \leq \phi \leq 1$ in $\Omega \setminus \overline{\Omega_\delta}$ where $\overline{\Omega_\delta} := \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $\mid \nabla\phi \mid \neq 0$ on $\partial\Omega$. We can choose $\lambda \gg 1$ such that

$$\sigma\lambda_1 < \lambda \min_{s \in [\frac{\sigma}{2}\mu^2, \sigma]} f(s).$$

Thus in $\Omega \setminus \overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$-\Delta\psi \leq \sigma\lambda_1 < \lambda \min_{s \in [\frac{\sigma}{2}\mu^2, \sigma]} f(s) \leq \lambda f(\psi). \tag{1.3}$$

On the other hand, in $\overline{\Omega}_\delta$,

$$-\Delta\psi < -\sigma m \leq \lambda f(\psi), \tag{1.4}$$

since $\lambda f(\psi) \geq 0$. Combining (1.3) and (1.4), if $\lambda \gg 1$ we see that ψ is a positive sub-solution of (1.1). Next, it is easy to check that the constant function $Z := r_0$ is a super-solution of (1.1) with $Z \geq \psi$. Hence for $\lambda \gg 1$, (1.1) has a positive solution and Lemma 3 is proven. □

The case when $f(0) < 0$ (semipositone problems) is non-trivial and quite challenging. Here our test functions for positive sub-solutions must come from positive functions ψ such that $-\Delta\psi < 0$ near $\partial\Omega$ while $-\Delta\psi > 0$ in a large part of the interior of Ω.

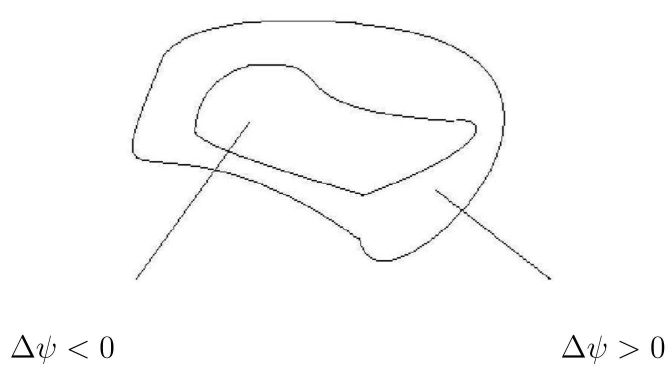

Figure 1.1

Here we discuss two results for the problem (1.1). First, we assume:

$(H5)$ There exists $K_0 > 0$ such that $f(s) \geq -K_0$ for all $s > 0$.

$(H6)$ $\lim_{s\to\infty} f(s) = \infty$.

Note that $(H5)$ includes the case $f(0) < 0$.

Lemma 4

Assume $(H2)$, $(H5)$ and $(H6)$. Then (1.1) has a positive solution for $\lambda \gg 1$.

Proof: Let λ_1, ϕ be as defined in the proof of Lemma 2 and let $\delta > 0, \mu > 0, m > 0, \Omega_\delta$ be as before (see the proof of Lemma 3). Let $\psi := \frac{K_0\lambda}{2m}\phi^2$ for $K_0 > 0$. Then

$$\nabla\psi = \frac{K_0\lambda}{m}\phi\nabla\phi$$

and

$$\begin{aligned}-\Delta\psi = -div(\nabla\psi) &= -\frac{K_0\lambda}{m}\{\phi\Delta\phi + |\nabla\phi|^2\}\\ &= -\frac{K_0\lambda}{m}\{|\nabla\phi|^2 - \lambda_1\phi^2\}.\end{aligned}$$

Then in $\overline{\Omega_\delta}$,

$$-\Delta\psi \leq -K_0\lambda \leq \lambda f(\psi). \tag{1.5}$$

Next in $\Omega \setminus \overline{\Omega_\delta}$, $\phi^2 \geq \mu^2$ and hence by $(H6)$ for $\lambda \gg 1$ we have

$$-\Delta\psi \leq \frac{K_0\lambda\lambda_1}{m} \leq \lambda f(\frac{K_0\lambda}{2m}\phi^2) = \lambda f(\psi). \tag{1.6}$$

Combining (1.5) and (1.6), if $\lambda \gg 1$ we see that

$$-\Delta\psi \leq \lambda f(\psi) \qquad \text{in } \Omega.$$

Thus ψ is a positive sub-solution of (1.1). By the same argument as in the proof of Lemma 2, we can find a super-solution Z of (1.1) with $Z \geq \psi$. Thus (1.1) has a positive solution $u \in [\psi, Z]$ for $\lambda \gg 1$. □

We also recall another semipositone problem where $f(u) < 0$ for $u \gg 1$. Namely,

$$\begin{cases} -\Delta u = au - bu^2 - c \quad \text{in } \Omega \\ \\ \qquad u = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1.7}$$

Note that $(H6)$ is not satisfied here. This equation arises in the study of population dynamics of one species with u representing the concentration of the species and c representing the rate of harvesting. To get a positive sub-solution, one can use here the anti-maximum principle by Clement and Peletier ([CP]), and establish the following result:

Lemma 5

Suppose that $a > \lambda_1$ and $b > 0$. Then there exists $c_1 = c_1(a, b)$ such that for $0 < c < c_1$, (1.7) has a positive solution. (see [OSS])

Proof: Consider the boundary value problem

$$\begin{cases} -\Delta z - \lambda z = -1 \qquad \text{in } \Omega, \\ \\ \qquad z = 0 \qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{1.8}$$

By the anti-maximum principle, there exist a $\delta_1 = \delta_1(\Omega) > 0$ such that if $\lambda \in (\lambda_1, \lambda_1 + \delta_1)$ then (1.8) has a solution $z = z_\lambda$ which is positive in Ω and $\frac{\partial z_\lambda}{\partial \nu} < 0$ on $\partial\Omega$. Fix $\lambda^* \in (\lambda_1, \min\{a, \lambda_1 + \delta_1\})$. Let z_{λ^*} be the solution of (1.8) when $\lambda = \lambda^*$ and $\alpha := \|z_{\lambda^*}\|_\infty$.

Define $\psi = Kcz_{\lambda^*}$, where $K \geq 1$ is to be determined later. We will choose $K \geq 1$ and $c > 0$ appropriately so that ψ is a sub-solution. We know

$$-\Delta\psi = Kc(-\Delta z_{\lambda^*}) = Kc(\lambda^* z_{\lambda^*}) - Kc.$$

Thus if we prove

$$(a - \lambda^*)Kz_{\lambda^*} - bc(Kz_{\lambda^*})^2 + K - 1 \geq 0, \tag{1.9}$$

then

$$\begin{aligned} -\Delta\psi &= Kc(\lambda^* z_{\lambda^*}) - Kc \\ &\leq a(Kcz_{\lambda^*}) - b(Kcz_{\lambda^*})^2 - c, \end{aligned}$$

and $\psi = Kcz_{\lambda^*}$ will be a sub-solution of (1.7). To show (1.9), define $H(y) := (a - \lambda^*)y - bcy^2 + (K-1)$. If $H(y) \geq 0$ for all $y \in [0, K\alpha]$, then (1.9) is true. Since $a > \lambda^*$, if $K \geq 1$, then it suffice to show that $H(K\alpha) = (a - \lambda^*)K\alpha - bc(K\alpha)^2 + (K-1) \geq 0$, which is equivalent to

$$c \leq \frac{(a - \lambda^*)K\alpha + (K-1)}{b(K\alpha)^2}.$$

Thus if we define

$$c_1 = c_1(a, b) := \sup_{K \geq 1} \frac{(a - \lambda^*)K\alpha + (K-1)}{b(K\alpha)^2},$$

then we know that when $0 < c < c_1$, there exist $\tilde{K} \geq 1$ such that $\psi = \tilde{K}cz_{\lambda^*}$ is a sub-solution. It is obvious that $Z = M$ where **M** is a sufficiently large constant is a super-solution of (1.7) with $Z \geq \psi$. Thus Lemma 5 is proven. $\square$

Now we describe our main contributions to this thesis. We will establish new results for the challenging case when f is a class of continuous functions on $(0, \infty)$ such that

$$\boxed{lim_{s\to 0^+} f(s) = -\infty \text{ (infinite semipositone problems).}}$$

This singular problem causes considerable difficulty in producing positive solutions. However, if we can produce a sub-solution ψ such that $\psi > 0;\ \Omega$, then the method of sub-super solutions can be extended to such singular problems. We will discuss this extension in Section 2.7. In the case of infinite semipositone problems, we not only need to produce sub-solutions such that $\psi > 0;\ \Omega, \psi = 0;\ \partial\Omega$ but also they must satisfy $lim_{x\to\partial\Omega}(-\Delta\psi) = -\infty$. This is required since $lim_{s\to 0} f(s) = -\infty$ and a positive sub-solution ψ has to satisfy $-\Delta\psi \leq \lambda f(\psi);\ \Omega, \psi = 0;\ \partial\Omega$.

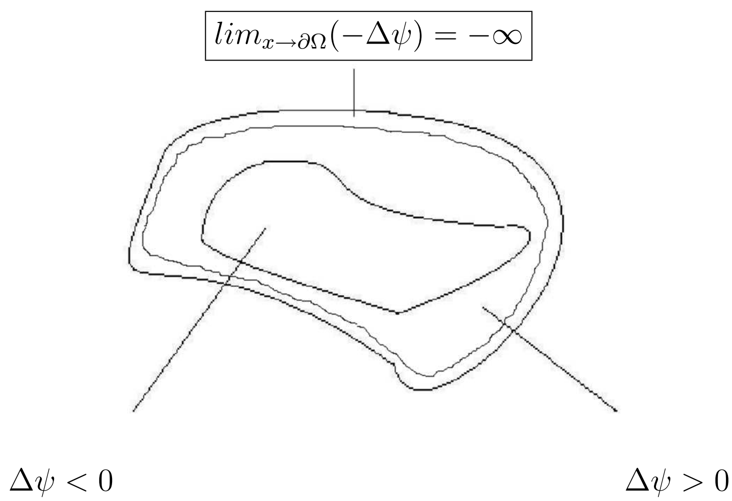

Figure 1.2

In the rest of this chapter (Sections 1.1-1.5) we will discuss all our main results and in Section 1.6 we will discuss some examples that satisfy our theorems. In Chapter 2, some preliminary results will be discussed. In Chapter 3, proofs of our existence results (Theorems 1-3 in Section 1.1) for infinite semipositone Laplacian equations will be discussed. In Chapter 4, proofs of our existence results (Theorems 4-8 in Section 1.2) for infinite semipositone p-Laplacian systems will be discussed. In Chapter 5, proofs of our existence results (Theorems 9-11 in Section 1.3) for infinite semipositone pq-Laplacian systems will be discussed. In Chapter 6, proofs of our existence results (Theorems 12-15 in Section 1.4) for Laplacian and p-Laplacian $n \times n$ sysyems will be discussed. In Chapter 7, proofs of our existence results (Theorems 16-17 in Section 1.5) for infinite semipositone problems with falling zeros will be discussed. Conclusions and future directions will be discussed in Chapter 8.

1.1 Infinite semipositone Laplacian equations ($lim_{s\to 0^+} f(s) = -\infty$)

Our interest here is to discuss the existence of positive solutions to infinite semipositone problem in the case of a single equation. Namely

$$\begin{cases} -\Delta u = \lambda \frac{g(u)}{u^\alpha} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where $\alpha \in (0,1)$, $g : [0,\infty) \to \mathbb{R}$ is continuous and $g(0) < 0$. We introduce the following hypotheses:

$(H7)$ There exists $\gamma > 0$ and $B > 0$ such that $\alpha \le \gamma < \alpha + 1$ and $g(s) \le Bs^\gamma$ for $s \ge 0$.

$(H8)$ There exists $\beta > 0$ and $A > 0$ such that $g(s) \ge As^\beta$ for $s \gg 1$.

Then we establish:

Theorem 1

Assume $(H7)$ and $(H8)$ and g is nondecreasing. Then (1.10) has a positive solution for $\lambda \gg 1$.

We also study the following infinite semipositone problem:

$$\begin{cases} -\Delta u = \lambda[g(u) - 1/u^{\alpha}]; x \in \Omega \\ \quad u = 0;\ x \in \partial\Omega, \end{cases} \tag{1.11}$$

where $\alpha \in (0, 1)$. We assume:

$(H9)$ $g \in C^1(0, \infty), g^{'}(s) > 0; s > 0$ and $\lim_{s\to\infty} \frac{g(s)}{s} = 0$,

and establish the following theorems:

Theorem 2

Assume $(H9)$ and $g(0) > 0$. Then there exists positive constants μ_1, μ_2 such that $\mu_1 < \mu_2$ and (1.11) has no positive solution for $\lambda < \mu_1$ and has at least one positive solution for $\lambda > \mu_2$.

Theorem 3

Assume $(H9)$, $g(0) = 0$ and $\lim_{s\to 0} \frac{g(s)}{s^{\beta}} = k > 0\,(0 < \beta \leq 1)$. Then there exists positive constants μ_1, μ_2 such that $\mu_1 < \mu_2$ and (1.11) has no positive solution for $\lambda < \mu_1$ and has at least one positive solution for $\lambda > \mu_2$.

1.2 Infinite semipositone p-Laplacian systems

In this section we study infinite semipositone p-Laplacian systems:

$$\begin{cases} -\Delta_p u = \lambda[g_1(v) - \frac{1}{u^{\alpha_1}}] & \text{in } \Omega \\ -\Delta_p v = \lambda[g_2(u) - \frac{1}{v^{\alpha_2}}] & \text{in } \Omega \\ \quad u = 0 \ = \ v & \text{on } \partial\Omega \end{cases} \tag{1.12}$$

and

$$\begin{cases} -\Delta_p u = \lambda[g_1(v) - \frac{1}{v^\alpha}] & \text{in } \Omega \\ -\Delta_p v = \lambda[g_2(u) - \frac{1}{u^\alpha}] & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{1.13}$$

where $\alpha,\ \alpha_i \in (0,1)$, $g_i \in C([0,\infty))$, $g_i(0) > 0$, g_i is nondecreasing for $i = 1,2$, and $\Delta_p u := div(|\nabla u|^{p-2}\nabla u), p > 1$. We assume:

$(H10)$ $\lim_{s\to\infty} g_2(s) = \infty$ $\big($ or $\lim_{s\to\infty} g_1(s) = \infty$ $\big)$ and

$$\lim_{s\to\infty} \frac{g_1(M(g_2(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0,\ \ \forall M > 0.$$

We establish the following theorems:

Theorem 4

Assume $(H10)$. Then (1.12) has a positive solution for $\lambda \gg 1$.

Theorem 5

Assume $(H10)$. Then (1.13) has a positive solution for $\lambda \gg 1$.

We also study infinite semipositone p-Laplacian systems of the form:

$$\begin{cases} -\Delta_p u = \lambda \frac{g_1(v)}{u^{\alpha_1}} & \text{in } \Omega \\ -\Delta_p v = \lambda \frac{g_2(u)}{v^{\alpha_2}} & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{1.14}$$

and

$$\begin{cases} -\Delta_p u = \lambda \frac{g_1(v)}{v^{\alpha}} & \text{in } \Omega \\ -\Delta_p v = \lambda \frac{g_2(u)}{u^{\alpha}} & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{1.15}$$

where $\alpha,\ \alpha_i \in (0,1),\ g_i \in C([0,\infty))$ and $g_i(0) < 0$ for $i = 1,2$. We introduce the following hypotheses:

$(H11)$ There exist $\sigma_i > 0$ and $A_i > 0, i = 1,2$ such that $0 < \alpha_1 - \sigma_1 < \alpha_2, 0 < \alpha_2 - \sigma_2 < \alpha_1$ and $g_i(s) > A_i s^{\sigma_i}$, for $s \gg 1$.

$(H12)$ $\lim_{s\to\infty} \frac{g_1(M(g_2(s))^{\frac{1}{p-1}})}{s^{p-1+\alpha_1}} = 0$ $\big($ or $\lim_{s\to\infty} \frac{g_2(M(g_1(s))^{\frac{1}{p-1}})}{s^{p-1+\alpha_2}} = 0, \forall M > 0.\big)$

$(H13)$ There exists $r_i > 0$ and $B_i > 0$, $i = 1.2$ such that $\alpha_i < r_i < \alpha_i + (p-1)$, and $g_i(s) \leq B_i s^{r_i}$, for all $s \geq 0$.

$(H14)$ $\lim_{s\to\infty} \tilde{g}_2(s) = \infty$ $\big($ or $\lim_{s\to\infty} \tilde{g}_1(s) = \infty$ $\big)$ and

$$\lim_{s\to\infty} \frac{\tilde{g}_1(M(\tilde{g}_2(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0,\ \ \forall\, M > 0,$$

where $\tilde{g}_i(s) = \frac{g_i(s)}{s^{\alpha}}$, for $i = 1,2$.

We establish the following theorems:

Theorem 6

Assume $(H11)$, $(H12)$ and g_i be nondecreasing for $i = 1,2$. Then (1.14) has a positive solution for $\lambda \gg 1$.

Theorem 7

Assume $(H11)$, $(H13)$ and g_i be nondecreasing for $i = 1,2$. Then (1.14) has a positive solution for $\lambda \gg 1$.

Theorem 8

Assume $(H11)$ with $\alpha = \alpha_1 = \alpha_2$, $(H14)$ and $\frac{g_i(s)}{s^{\alpha}}$ be nondecreasing for $i = 1,2$. Then (1.15) has a positive solution for $\lambda \gg 1$.

1.3 Infinite semipositone pq-Laplacian systems

In this section we study infinite semipositone pq-Laplacian systems:

$$\begin{cases} -\Delta_p u = \lambda[g_1(u,v) - \frac{1}{u^{\alpha_1}}] & \text{in } \Omega \\ -\Delta_q v = \lambda[g_2(u,v) - \frac{1}{v^{\alpha_2}}] & \text{in } \Omega \\ \quad u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{1.16}$$

where $\alpha_i \in (0,1)$, $g_i \in C([0,\infty) \times [0,\infty))$, $g_i(0,0) > 0$, g_i is nondecreasing for both u and v, $i = 1, 2$, and $\Delta_r u := div(\mid \nabla u \mid^{r-2} \nabla u)$, $r = p, q > 1$. We assume:

$(H15)$ $\lim_{s\to\infty} g_2(s,s) = \infty, \lim_{s\to\infty} \frac{g_2(s,s)}{s^{q-1}} = 0$ and

$$\lim_{s\to\infty} \frac{g_1(s, (Mg_2(s,s))^{\frac{1}{q-1}})}{s^{p-1}} = 0, \quad \forall M > 0.$$

$\big($ or $\lim_{s\to\infty} g_1(s,s) = \infty, \lim_{s\to\infty} \frac{g_1(s,s)}{s^{p-1}} = 0$ and

$$\lim_{s\to\infty} \frac{g_2((Mg_1(s,s))^{\frac{1}{p-1}}, s)}{s^{q-1}} = 0, \quad \forall M > 0.\big)$$

We establish the following theorem:

Theorem 9

Assume $(H15)$. Then (1.16) has a positive solution for $\lambda \gg 1$.

We also obtain results to the following infinite semipositone pq-Laplacian system:

$$\begin{cases} -\Delta_p u = \lambda \frac{g_1(u,v)}{u^{\alpha_1}} & \text{in } \Omega \\ -\Delta_q v = \lambda \frac{g_2(u,v)}{v^{\alpha_2}} & \text{in } \Omega \\ \quad u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{1.17}$$

where $\alpha_i \in (0,1)$, $g_i \in C([0,\infty) \times [0,\infty))$, $g_i(0,0) < 0$ and g_i is nondecreasing for both u and v, $i = 1, 2$. We assume the following hypotheses:

$(H16)$ There exist $A_1 > 0$ and $A_2 > 0$ such that $g_1(s,t) > A_1 s^{\alpha_1}$ (or $g_1(s,t) > A_1 t^{\alpha_1}$) and $g_2(s,t) > A_2 s^{\alpha_2}$ (or $g_2(s,t) > A_2 t^{\alpha_2}$) for $s \gg 1$ and $t \gg 1$.

$(H17)$ There exists $r_i > 0, \theta_i > 0, C_i > 0$ and $B_i > 0; i = 1,2$ such that

$$\alpha_1 < r_i < \alpha_1 + (p-1), \;\; \alpha_2 < \theta_i < \alpha_2 + (q-1)$$

and $g_1(s,t) \le B_1 s^{r_1} + B_2 t^{r_2}$, $g_2(s,t) \le C_1 s^{\theta_1} + C_2 t^{\theta_2}$, for all $(s,t) \ge (0,0)$.

We establish the following result:

Theorem 10

Assume $(H16)$ and $(H17)$. Then (1.17) has a positive solution for $\lambda \gg 1$.

We can also prove such an existence result under the following hypothesis:

$(H18)$ $\lim_{s\to\infty} \frac{g_2(s,s)}{s^{q-1}} = 0$ and $\lim_{s\to\infty} \frac{g_1(s,M(g_2(s,s))^{\frac{1}{q-1}})}{s^{\alpha_1+p-1}} = 0$ for all $M > 0$.
$\big($ or $\lim_{s\to\infty} \frac{g_1(s,s)}{s^{p-1}} = 0$ and $\lim_{s\to\infty} \frac{g_2(M(g_1(s,s))^{\frac{1}{p-1}},s)}{s^{\alpha_2+q-1}} = 0$ for all $M > 0$. $\big)$

Theorem 11

Assume $(H16)$ and $(H18)$. Then (1.17) has a positive solution for $\lambda \gg 1$.

1.4 Infinite semipositone Laplacian and p-Laplacian $n \times n$ systems

We first study Laplacian $n \times n$ systems of the form:

$$\begin{cases} -\Delta u_1 = \lambda \frac{g_1(u_2)}{u_1^{\alpha_1}} & \text{in } \Omega \\ -\Delta u_2 = \lambda \frac{g_2(u_3)}{u_2^{\alpha_2}} & \text{in } \Omega \\ \vdots & \\ -\Delta u_{n-1} = \lambda \frac{g_{n-1}(u_n)}{u_{n-1}^{\alpha_{n-1}}} & \text{in } \Omega \\ -\Delta u_n = \lambda \frac{g_n(u_1)}{u_n^{\alpha_n}} & \text{in } \Omega \\ u_1 = u_2 = \cdots = u_n = 0 & \text{on } \partial\Omega \end{cases} \tag{1.18}$$

and

$$\begin{cases} -\Delta u_1 = \lambda \frac{g_1(u_2)}{u_2^{\alpha}} & \text{in } \Omega \\ -\Delta u_2 = \lambda \frac{g_2(u_3)}{u_3^{\alpha}} & \text{in } \Omega \\ \quad \vdots \\ -\Delta u_{n-1} = \lambda \frac{g_{n-1}(u_n)}{u_n^{\alpha}} & \text{in } \Omega \\ -\Delta u_n = \lambda \frac{g_n(u_1)}{u_1^{\alpha}} & \text{in } \Omega \\ u_1 = u_2 = \cdots = u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.19}$$

where $g_i \in C([0,\infty))$, $g_i(0) < 0$ and α, $\alpha_i \in (0,1)$, for $i = 1, \cdots, n$.

To state our results precisely we introduce the following hypotheses:

$(H19)$ There exist $\sigma > 0$ and $A > 0$ such that $\overline{\alpha} - \underline{\alpha} < \sigma < \overline{\alpha}$ and $g_i(s) > As^{\sigma}$ for $s \gg 1$, $\forall\ i = 1, \cdots, n$ where $\overline{\alpha} = \max_{i=1,\cdots,n}\{\alpha_i\}$ and $\underline{\alpha} = \min_{i=1,\cdots,n}\{\alpha_i\}$.

$(H20)$ There exist $\sigma > 0$ and $A > 0$ such that $0 < \sigma < \alpha$ and $g_i(s) > As^{\sigma}$ for $s \gg 1$, $\forall$ $i = 1, \cdots, n$.

$(H21)$

$$\lim_{s\to\infty} \frac{g_1^{[M]} \circ g_2^{[M]} \circ \cdots \circ g_{n-1}^{[M]} \circ g_n(s)}{s^{1+\alpha_1}} = 0, \qquad \forall M > 0,$$

where $g_i^{[M]}(s) = g_i(Ms)$, $\forall\ i = 1, \cdots, n$.

$(H22)$ $\lim_{s\to\infty} \tilde{g}_j(s) = \infty$ for $j = 2, 3, \cdots, n$ and

$$\lim_{s\to\infty} \frac{\tilde{g}_1^{[M]} \circ \tilde{g}_2^{[M]} \circ \cdots \circ \tilde{g}_{n-1}^{[M]} \circ \tilde{g}_n(s)}{s} = 0, \qquad \forall M > 0,$$

where $\tilde{g}_i(s) = \frac{g_i(s)}{s^{\alpha}}$ and $\tilde{g}_i^{[M]}(s) = \tilde{g}_i(Ms)$, $\forall\ i = 1, \cdots, n$.

We establish the following results:

Theorem 12

Assume $(H19)$, $(H21)$ hold and g_i is nondecreasing for $i = 1, \cdots, n$. Then (1.18) has a positive solution for $\lambda \gg 1$.

Theorem 13

Assume $(H20)$, $(H22)$ hold and $\frac{g_i(s)}{s^\alpha}$ is nondecreasing for $i = 1, \cdots, n$. Then (1.19) has a positive solution for $\lambda \gg 1$.

1.4.1 p-Laplacian systems

In this section we state the extensions of Theorems 12-13 to the following p-Laplacian $n \times n$ systems:

$$\begin{cases} -\Delta_p u_1 = \lambda \frac{g_1(u_2)}{u_1^{\alpha_1}} & \text{in } \Omega \\ -\Delta_p u_2 = \lambda \frac{g_2(u_3)}{u_2^{\alpha_2}} & \text{in } \Omega \\ \quad\vdots & \\ -\Delta_p u_{n-1} = \lambda \frac{g_{n-1}(u_n)}{u_{n-1}^{\alpha_{n-1}}} & \text{in } \Omega \\ -\Delta_p u_n = \lambda \frac{g_n(u_1)}{u_n^{\alpha_n}} & \text{in } \Omega \\ u_1 = u_2 = \cdots = u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.20}$$

and

$$\begin{cases} -\Delta_p u_1 = \lambda \frac{g_1(u_2)}{u_2^{\alpha}} & \text{in } \Omega \\ -\Delta_p u_2 = \lambda \frac{g_2(u_3)}{u_3^{\alpha}} & \text{in } \Omega \\ \quad\vdots & \\ -\Delta_p u_{n-1} = \lambda \frac{g_{n-1}(u_n)}{u_n^{\alpha}} & \text{in } \Omega \\ -\Delta_p u_n = \lambda \frac{g_n(u_1)}{u_1^{\alpha}} & \text{in } \Omega \\ u_1 = u_2 = \cdots = u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Here $g_i \in C([0,\infty))$, $g_i(0) < 0$ and α, $\alpha_i \in (0,1)$ for $i = 1, \cdots, n$.

To state our results for these p-Laplacian systems, we introduce the following hypotheses:

$(H23)$
$$\lim_{s\to\infty} \frac{\left[g_1^{[M]} \circ g_2^{[M]} \circ \cdots \circ g_{n-1}^{[M]} \circ (g_n(s))^{\frac{1}{p-1}}\right]^{p-1}}{s^{p-1+\alpha_1}} = 0, \qquad \forall M > 0,$$
where $g_i^{[M]}(s) = (g_i(Ms))^{\frac{1}{p-1}}$.

$(H24)$] $\lim_{s\to\infty} \tilde{g}_j(s) = \infty$ for $j = 2, \cdots, n$ and
$$\lim_{s\to\infty} \frac{\left[\tilde{g}_1^{[M]} \circ \tilde{g}_2^{[M]} \circ \cdots \circ \tilde{g}_{n-1}^{[M]} \circ (\tilde{g}_n(s))^{\frac{1}{p-1}}\right]^{p-1}}{s^{p-1}} = 0, \qquad \forall M > 0,$$
where $\tilde{g}_i(s) = \frac{g_i(s)}{s^\alpha}$ and $\tilde{g}_i^{[M]}(s) = (\tilde{g}_i(Ms))^{\frac{1}{p-1}}$.

We establish the following results:

Theorem 14

Assume $(H19)$, $(H23)$ and g_i is nondecreasing for $i = 1, \cdots, n$. Then (1.20) has a positive solution for $\lambda \gg 1$.

Theorem 15

Assume $(H20)$, $(H24)$ and $\frac{g_i(s)}{s^\alpha}$ is nondecreasing for $i = 1, \cdots, n$. Then (1.21) has a positive solution for $\lambda \gg 1$.

1.5 Infinite semipositone problems with falling zeros

In this section we study positive solutions to the boundary value problem:

$$\begin{cases} -\Delta u = au - g(u) - \frac{c}{u^\alpha} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.22}$$

where $0 < \alpha < 1$, $a > 0$ and $c > 0$ are constants, and $g : [0,\infty) \to \mathbb{R}$ is a continuous function. We consider problem (1.22) under the following assumptions:

$(H25)$ There exist $A > 0$ and $p > 1$ such that $g(u) \leq Au^p$, for all $u \in [0, \infty)$.

$(H26)$ There exists a constant $M > 0$ such that $g(u) \geq au - M$, for all $u \in [0, \infty)$.

We establish:

Theorem 16

Assume $(H25)$ and $(H26)$. If $a > \frac{2\lambda_1}{1+\alpha}$, then there exists $c^* = c^*(a, \Omega, A, p, \alpha)$ such that for $c < c^*$, (1.22) has a positive solution. Here λ_1 is the first eigenvalue of operator $-\Delta$ with Dirichlet boundary conditions.

We also study the following system:

$$\begin{cases} -\Delta u = a_1 u - g_1(u) - \frac{c_1}{v^\alpha} & \text{in } \Omega \\ -\Delta v = a_2 v - g_2(v) - \frac{c_2}{u^\alpha} & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{1.23}$$

where $0 < \alpha < 1$, $a_1 > 0$, $a_2 > 0$, $c_1 > 0$ and $c_2 > 0$ are constants, and $g_i : [0, \infty) \to \mathbb{R}$ are continuous functions for $i = 1, 2$. We consider the following assumptions:

$(H27)$ There exist $A > 0$ and $p > 1$ such that $g_i(u) \leq Au^p$; $i = 1, 2$ for all $u \in [0, \infty)$.

$(H28)$ There exists a constant $M > 0$ such that $g_i(u) \geq a_i u - M$; $i = 1, 2$ for all $u \in [0, \infty)$.

We establish:

Theorem 17

Assume $(H27)$ and $(H28)$. If $\min\{a_1, a_2\} > \frac{2\lambda_1}{1+\alpha}$, then there exists $c^* = c^*(a_1, a_2, \Omega, A, p, \alpha)$ such that for $\max\{c_1, c_2\} < c^*$, (1.23) has a positive solution.

1.6 Some examples

Example 1

$g_1(s) = e^{\frac{s}{s+1}}$, $g_2(s) = e^s$. Here $g_i \in C([0,\infty))$, $g_i(0) > 0$, g_i is nondecreasing for $i = 1, 2$, $\lim_{s\to\infty} g_2(s) = \infty$ and

$$\lim_{s\to\infty} \frac{g_1(M(g_2(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0, \ \ \forall M > 0.$$

Therefore this example satisfies all the hypotheses in Theorems 4 and 5.

Example 2

$g_1(s) = s^4 - 1$, $g_2(s) = s - 1$, $p = 3$, $\alpha_i = \frac{1}{2}$, $\ i = 1, 2$. Here $g_i \in C([0,\infty))$, $g_i(0) < 0$, g_i is nondecreasing for $i = 1, 2$, satisfies $(H11)$ and

$$\lim_{s\to\infty} \frac{g_1(M(g_2(s))^{\frac{1}{2}})}{s^{\frac{5}{2}}} = 0, \ \ \forall M > 0.$$

Therefore this example satisfies all the hypotheses in Theorem 6.

Example 3

$g_1(s) = s^2 - 1$, $g_2(s) = s - 1$, $p = 3$, $\alpha_i = \frac{1}{2}$, $\ i = 1, 2$. Here $g_i \in C([0,\infty))$, $g_i(0) < 0$, g_i is nondecreasing for $i = 1, 2$, satisfies $(H11)$ and

$$g_1(s) \le s^{\frac{9}{4}} \ \text{ and } \ g_2(s) \le s^2.$$

Therefore this example satisfies all the hypotheses in Theorem 7.

Example 4

$g_1(s) = s^4 - 1$, $g_2(s) = s - 1$, $p = 3$, $\alpha = \frac{1}{2}$. Here $g_i \in C([0,\infty))$, $g_i(0) < 0$ for $i = 1, 2$ and $(H11)$ is satisfied. Also $\tilde{g}_1(s) = s^{\frac{7}{2}} - s^{-\frac{1}{2}}$, $\tilde{g}_2(s) = s^{\frac{1}{2}} - s^{-\frac{1}{2}}$ and

$$\lim_{s\to\infty} \frac{\tilde{g}_1(M(\tilde{g}_2(s))^{\frac{1}{2}})}{s^2} = 0, \ \ \forall M > 0.$$

Therefore this example satisfies all the hypotheses in Theorem 8.

Example 5

$g_1(u,v) = e^{\frac{u}{u+1}+\frac{v}{v+1}}, g_2(u,v) = u+v+1, p = 2, q = 3$. Here $g_i \in C([0,\infty)\times[0,\infty))$, $g_i(0,0) > 0$, g_i is nondecreasing for both u and v $i = 1,2$, $\lim_{s\to\infty} g_2(s,s) = \infty, \lim_{s\to\infty} \frac{g_2(s,s)}{s^2} = 0$ and

$$\lim_{s\to\infty} \frac{g_1(s,(Mg_2(s,s))^{\frac{1}{2}})}{s} = 0, \ \ \forall M > 0.$$

Therefore this example satisfies all the hypotheses in Theorem 9.

Example 6

$g_1(u,v) = u^2 + v^{\frac{49}{20}} - 1$, $g_2(u,v) = u^{\frac{1}{2}} + v^{\frac{25}{8}} - 1$, $p = 3$, $q = 4$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4}$. Here $g_i \in C([0,\infty)\times[0,\infty))$, $g_i(0,0) < 0$, $g_{iu} > 0$, $g_{iv} > 0$ and g_i satisfies $(H16)$ and $(H17)$ for $i = 1,2$. Therefore this example satisfies all the hypotheses in Theorem 10 but we can see that this example does not satisfy $(H18)$.

Example 7

$g_1(u,v) = u^2 + (uv)^{\frac{3}{2}} - 1$, $g_2(u,v) = u^{\frac{1}{2}} + (uv)^{\frac{1}{4}} - 1$, $p = 3$, $q = 4$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4}$. Here $g_i \in C([0,\infty)\times[0,\infty))$, $g_i(0,0) < 0$, $g_{iu} > 0$, $g_{iv} > 0$ and g_i satisfies $(H16)$ for $i = 1,2$,

$$\lim_{s\to\infty} \frac{g_1(s, M(g_2(s,s))^{\frac{1}{3}})}{s^{\frac{5}{2}}} = 0, \ \ \forall M > 0$$

and $\lim_{s\to\infty} \frac{g_2(s,s)}{s^3} = 0$. Therefore this example satisfies all the hypotheses in Theorem 11 but we can see that this example does not satisfy $(H17)$.

CHAPTER 2

PRELIMINARIES

In this chapter we will discuss some preliminary results that will be used to establish our results. In particular, we will discuss maximum principles, anti-maximum principles, comparison principles and the method of sub-super solutions.

2.1 Linear elliptic boundary value problems

Consider the following Laplacian equation:

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$ with smooth boundary $\partial\Omega$. Let $C^{m+r}(\Omega)$, $0 < r < 1$ be the space of m-times continuously differentiable functions whose m^{th} derivative are Hölder continuous on Ω with Hölder exponent r. We shall consider classical solutions of (2.1), that is $C^2(\Omega) \bigcap C^1(\overline{\Omega})$ functions satisfying (2.1) pointwise. Let $f \in C^\alpha(\overline{\Omega})$ with $\alpha = 0$ if $N = 1$ and $0 < \alpha < 1$ if $N \geq 2$. Then it is well known that (2.1) has a solution $u = Kf$ where $K : C^\alpha(\overline{\Omega}) \to C^{2+\alpha}(\overline{\Omega})$ is a solution operator whose kernel is the Green's function $G(x, y)$ for (2.1), that is, $Kf(x) := \int_\Omega G(x, y)f(y)dy$.

2.2 Maximum, anti-maximum and comparison principles

Here we recall the classical maximum principle, the Hopf maximum principle, the anti-maximum principle and two comparison principles.

Lemma 6 (Maximum principle, see [PW] and [GT])

Let $\Delta u \geq 0$ in Ω. If u attains its maximum at any interior point in Ω, then $u = M$ in Ω.

Lemma 7 (Hopf Maximum Principle, see [PW] and [GT])

Let $\Delta u \geq 0$ in Ω. Suppose that $u \leq M$ in Ω and that $u = M$ at some $p \in \partial\Omega$. Then $\frac{\partial u}{\partial \nu} > 0$ at $p \in \partial\Omega$ unless $u = M$. Here $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative.

Lemma 8 (Anti-maximum principle, See [CP])

There exists $\delta = \delta(\Omega) > 0$ such that for $\alpha \in (\lambda_1, \lambda_1 + \delta)$ the problem

$$-\Delta z_\alpha - \alpha z_\alpha = -1 \quad \text{in } \Omega; \quad z_\alpha = 0 \quad \text{on } \partial\Omega$$

has a $C^1(\overline{\Omega})$ solution z_α such that $z_\alpha > 0$ in Ω and $\frac{\partial z_\alpha}{\partial \nu} < 0$ on $\partial\Omega$, where λ_1 is the first eigenvalue of the operator $-\Delta$ with Dirichlet boundary conditions.

Lemma 9 (Weak comparison principle)

Assume that $\Delta u \geq \Delta v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\overline{\Omega}$.

Lemma 10 (Strong comparison principle)

Assume that $\Delta u > \Delta v$ in Ω and $u = v$ on $\partial\Omega$. Then $u < v$ in Ω and $\frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}$ on $\partial\Omega$.

2.3 Sub-super solutions

In this section we discuss the classical method of sub-super solutions. Consider

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where f is a continuous function.

By a sub-solution of (2.2) we mean a function $\psi \in C^2(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta\psi &\leq \lambda f(\psi) \qquad \text{in } \Omega \\ \psi &\leq 0 \qquad\qquad \text{on } \partial\Omega, \end{aligned}$$

and by a super-solution of (2.2) we mean a function $Z \in C^2(\overline{\Omega})$ that satisfies:

$$\begin{aligned} -\Delta Z &\geq \lambda f(Z) \qquad \text{in } \Omega \\ Z &\geq 0 \qquad\qquad \text{on } \partial\Omega. \end{aligned}$$

Then we have the following result:

Lemma 11 (See [Am], [DS])

Suppose there exists a sub-solution ψ and a super-solution Z for the problem (2.2) satisfying $\psi \leq Z$ in Ω, then there exists a solution u such that $\psi \leq u \leq Z$.

2.4 Laplacian systems

In this section, we discuss a similar result for Laplacian systems of the form:

$$\begin{cases} -\Delta\underline{u} = \underline{F}(\underline{u}) & \text{in } \Omega \\ \underline{u} = \underline{0} & \text{on } \partial\Omega, \end{cases} \tag{2.3}$$

where $\underline{u} = (u_1, u_2, ..., u_n)$ and $\underline{F}(\underline{u}) = (F_1, F_2, ..., F_n)$: $[C([0,\infty)^n)]^n \to \mathbb{R}^n$. Such a system as (2.3) is called quasi-monotone if for fixed u_i, $F_i(u_1, u_2, ..., u_n)$ is nondecreasing in $u_j, \forall j \neq i$, $i = 1, 2, ..., n$. We say $\underline{u} \leq \underline{v}$ if $u_i \leq v_i, i = 1, 2, ..., n$.

For simplicity we present the result for the 2×2 systems case. Consider the following quasi-monotone system:

$$\begin{cases} -\Delta u_1 = F_1(u_1, u_2) & \text{in } \Omega \\ -\Delta u_2 = F_2(u_1, u_2) & \text{in } \Omega \\ \quad u_1 = 0 = u_2 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

By a sub-solution of (2.4) we mean a pair of functions $(\psi_1, \psi_2) \in [C^2(\overline{\Omega})]^2$ that satisfy:

$$\begin{cases} -\Delta \psi_1 \leq F_1(\psi_1, \psi_2) & \text{in } \Omega \\ -\Delta \psi_2 \leq F_2(\psi_1, \psi_2) & \text{in } \Omega \\ (\psi_1, \psi_2) \leq (0, 0) & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

and by a super-solution of (2.4) we mean a pair of functions $(Z_1, Z_2) \in [C^2(\overline{\Omega})]^2$ that satisfy:

$$\begin{cases} -\Delta Z_1 \geq F_1(Z_1, Z_2) & \text{in } \Omega \\ -\Delta Z_2 \geq F_2(Z_1, Z_2) & \text{in } \Omega \\ (Z_1, Z_2) \geq (0, 0) & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

Then we have the following result:

Lemma 12 (See [Am], [Ma], [MS])

Suppose there exists a sub-solution (ψ_1, ψ_2) and a super-solution (Z_1, Z_2) for the problem (2.4) satisfying $(\psi_1, \psi_2) \leq (Z_1, Z_2)$ in Ω, then there exists a solution (u_1, u_2) such that $(\psi_1, \psi_2) \leq (u_1, u_2) \leq (Z_1, Z_2)$.

2.5 p-Laplacian equations

Consider the following p-Laplacian equation of the form:

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.7}$$

By a weak solution of (2.7), we mean a function $u \in W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega \mid \nabla u \mid^{p-2} \nabla u \cdot \nabla w = \int_\Omega f(x) w, \forall w \in C_0^\infty(\Omega).$$

2.5.1 Regularity result

We here state the following regularity result:

Lemma 13 (See [DKT])

Let $u \in W_0^{1,p}(\Omega)$ be any weak solution of the Dirichlet problem (2.7). If $f(x) \in L^\infty(\Omega)$, then $u \in C^1(\overline{\Omega})$.

2.5.2 Comparison principles

Consider the following equations:

$$-\Delta_p u = f(x) \text{ in } \Omega; \;\; u = 0 \text{ on } \partial\Omega,$$

$$-\Delta_p v = g(x) \text{ in } \Omega; \;\; v = 0 \text{ on } \partial\Omega.$$

Now we state the weak and strong comparison principles for the above problem (see [DKT], [FT], [CT] and [PS]).

Lemma 14

Let $f, g \in L^\infty(\Omega)$ satisfy $f \leq g$ in Ω. Then $u \leq v$ in Ω.

Lemma 15

Let $f, g \in L^\infty(\Omega)$ satisfy $0 \leq f \leq g$ and $f \neq g$ in Ω. Then $u < v$ in Ω and $0 \geq \frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}$ on $\partial\Omega$.

2.5.3 Sub-super solutions

Consider the following p-Laplacian equation of the form:

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

where $p > 1$ and f is a continuous function.

We say a function $\psi \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$ is a sub-solution and $Z \in W^{1,p}(\Omega) \bigcap C(\overline{\Omega})$ is a super-solution of (2.8) if ψ and Z satisfy $\psi \leq 0$ on $\partial\Omega$, $Z \geq 0$ on $\partial\Omega$,

$$\int_\Omega \mid \nabla\psi \mid^{p-2} \nabla\psi \cdot \nabla w \leq \int_\Omega f(\psi) w, \forall w \in W$$

and

$$\int_\Omega \mid \nabla Z \mid^{p-2} \nabla Z \cdot \nabla w \geq \int_\Omega f(Z) w, \forall w \in W$$

where $W := \{\xi \in C_0^\infty(\Omega) \mid \xi \geq 0 \text{ in } \Omega\}$.

Lemma 16 (See [Ma], [MS])

Suppose there exists a sub-solution ψ and a super-solution Z for the problem (2.8) satisfying $\psi \leq Z$ in Ω, then there exists a solution u of (2.8) such that $\psi \leq u \leq Z$.

2.6 p-Laplacian systems

In this section, we consider p-Laplacian $n \times n$ systems of the form:

$$\begin{cases} -\Delta_{\underline{p}} \underline{u} = \underline{F}(\underline{u}) & \text{in } \Omega \\ \\ \underline{u} = \underline{0} & \text{on } \partial\Omega, \end{cases} \tag{2.9}$$

where $-\Delta_{\underline{p}} \underline{u} = (-\Delta_{p_1} u_1, -\Delta_{p_2} u_2, ..., -\Delta_{p_n} u_n); p_i > 1, i = 1, 2, ..., n$ and $\underline{F}(\underline{u}) = (F_1(u_1, u_2, ..., u_n), F_2(u_1, u_2, ..., u_n), ..., F_n(u_1, u_2, ..., u_n))$. We further assume that $\underline{F}$ is quasi-monotone.

For simplicity we will consider the pq-Laplacian 2×2 system of the form:

$$\begin{cases} -\Delta_p u_1 = F_1(u_1, u_2) & \text{in } \Omega \\ \\ -\Delta_q u_2 = F_2(u_1, u_2) & \text{in } \Omega \\ \\ u_1 = 0 = u_2 & \text{on } \partial\Omega. \end{cases} \tag{2.10}$$

We say a function $(\psi_1, \psi_2) \in (W^{1,p}(\Omega) \bigcap C(\overline{\Omega})) \times (W^{1,q}(\Omega) \bigcap C(\overline{\Omega}))$ is a sub-solution and $(Z_1, Z_2) \in (W^{1,p}(\Omega) \bigcap C(\overline{\Omega})) \times (W^{1,q}(\Omega) \bigcap C(\overline{\Omega}))$ is a super-solution of (2.10) if (ψ_1, ψ_2) and (Z_1, Z_2) satisfy $(\psi_1, \psi_2) \leq (0, 0)$ on $\partial\Omega$, $(Z_1, Z_2) \geq (0, 0)$ on $\partial\Omega$,

$$\begin{cases} \int_\Omega \mid \nabla\psi_1 \mid^{p-2} \nabla\psi_1 \cdot \nabla w \leq \int_\Omega F_1(\psi_1, \psi_2) w \\ \\ \int_\Omega \mid \nabla\psi_2 \mid^{q-2} \nabla\psi_2 \cdot \nabla w \leq \int_\Omega F_2(\psi_1, \psi_2) w \end{cases} \tag{2.11}$$

and

$$\begin{cases} \int_\Omega \mid \nabla Z_1 \mid^{p-2} \nabla Z_1 \cdot \nabla w \geq \int_\Omega F_1(Z_1, Z_2) w \\ \\ \int_\Omega \mid \nabla Z_2 \mid^{q-2} \nabla Z_2 \cdot \nabla w \geq \int_\Omega F_2(Z_1, Z_2) w \end{cases} \tag{2.12}$$

for all $w \in W$.

Lemma 17 (See [Ma], [MS])

Suppose there exists a sub-solution (ψ_1, ψ_2) and a super-solution (Z_1, Z_2) for the problem (2.10) satisfying $(\psi_1, \psi_2) \le (Z_1, Z_2)$ in Ω, then there exists a solution (u_1, u_2) of (2.10) such that $(\psi_1, \psi_2) \le (u_1, u_2) \le (Z_1, Z_2)$.

2.7 Singular systems

In this section we first recall a sub-super solution lemma for singular semilinear elliptic boundary problems from [Cu]. Consider:

$$\begin{cases} Lu + f(x, u, Du) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.13}$$

where Ω is a bounded domain in $\mathbb{R}^N$, L is a second-order elliptic partial differential operator on Ω, ϕ is a nonnegative function defined on $\partial\Omega$, and $f(x, u, \eta)$ is a continuous function defined on $\Omega \times (0, \infty) \times \mathbb{R}^N$, and satisfies the following conditions:

$(D1)$ $f(x, u, \eta)$ is locally Hölder continuous in $\Omega \times (0, \infty) \times \mathbb{R}^N$ and continuously differentiable with respect to the variable u and η.

$(D2)$ For any $\Omega' \subset\subset \Omega$ and any $a, b \in (0, \infty)$, $(a < b)$, there exists a corresponding constant $c = c(\overline{\Omega'}, a, b) > 0$ such that

$$|f(x, u, \eta)| \le c(1+ \mid \eta \mid^2), \qquad \forall (x, u) \in \overline{\Omega'} \times [a, b], \forall \eta \in \mathbb{R}^N.$$

By a sub-solution of (2.13) we mean a function $\psi \in C^2(\Omega) \bigcap C(\overline{\Omega})$ that satisfies:

$$\begin{cases} L\psi + f(x, \psi, D\psi) \leq 0 & \text{in } \Omega \\ \quad \psi > 0 & \text{in } \Omega \\ \quad \psi = \phi & \text{on } \partial\Omega, \end{cases} \tag{2.14}$$

and by a super-solution of (2.13) we mean a function $Z \in C^2(\Omega) \bigcap C(\overline{\Omega})$ that satisfies:

$$\begin{cases} LZ + f(x, Z, DZ) \geq 0 & \text{in } \Omega \\ \quad Z > 0 & \text{in } \Omega \\ \quad Z = \phi & \text{on } \partial\Omega. \end{cases} \tag{2.15}$$

Then we have the following result:

Lemma 18

Suppose that the function f satisfies conditions $(D1)$ and $(D2)$. Suppose further that problem (2.13) has a pair of sub-super solutions ψ and Z such that $\psi \leq Z$ on $\overline{\Omega}$, then (2.13) has at least one solution $u \in C^2(\Omega) \bigcap C(\overline{\Omega})$ satisfying $\psi \leq u \leq Z$ on $\overline{\Omega}$.

We extend this lemma to the p-Laplacian equation of the form:

$$\begin{cases} -\Delta_p u = h(x, u) & \text{in } \Omega \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

where $h : \Omega \times (0, \infty) \to \mathbb{R}$ is continuous.

By a sub-solution of (2.16) we mean a function $\psi : \overline{\Omega} \to \mathbb{R}$ such that $\psi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and satisfies

$$\begin{cases} -\Delta_p \psi \leq h(x,\psi) & \text{in } \Omega \\ \psi > 0 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega, \end{cases}$$

and by a super-solution of (2.16) we mean a function $Z : \overline{\Omega} \to \mathbb{R}$ such that $Z \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ and satisfies

$$\begin{cases} -\Delta_p Z \geq h(x,Z) & \text{in } \Omega \\ Z > 0 & \text{in } \Omega \\ Z = 0 & \text{on } \partial\Omega. \end{cases}$$

Lemma 19

Suppose there exists a sub-solution ψ and a super-solution Z of (2.16) such that $\psi \leq Z$ on $\overline{\Omega}$. Then (2.16) has at least one solution $u \in [\psi, Z]$.

Proof of Lemma

First note that for any $\Omega' \subset\subset \Omega$ and any $a, b \in (0,\infty)$ $(a < b)$, there exists $c = c(\overline{\Omega'}, a, b) > 0$ such that

$$|h(x,u)| \leq c, \qquad \text{for all } (x,u) \in \overline{\Omega'} \times [a,b].$$

Take a sequence of sub-domains of Ω with C^∞-boundaries, say $\{\Omega_j\}_{j=1}^{\infty}$, such that

$$\Omega_1 \subset\subset \Omega_2 \subset\subset \cdots \subset\subset \Omega_j \subset\subset \Omega_{j+1} \subset\subset \cdots$$

and $\cup_{j=1}^{\infty}\Omega_j = \Omega$. For each $j = 1, 2, \cdots$, consider the following problem

$$\begin{cases} -\Delta_p u = h(x, u(x)), & x \in \Omega_j, \\ u(x) = \psi(x), & x \in \partial\Omega_j. \end{cases} \tag{2.17}$$

Let $a_j = \min_{\overline{\Omega}_j} \psi$ and $b_j = \max_{\overline{\Omega}_j} Z$ and define $\tilde{h} : \Omega_j \times (0, \infty) \to \mathbb{R}$ by

$$\tilde{h}(x, u) = \begin{cases} h(x, a_j), & u < a_j, \\ h(x, u), & a_j \le u \le b_j, \\ h(x, b_j), & b_j < u. \end{cases}$$

Then $\tilde{h}$ is bounded in $\Omega_j \times (0, \infty)$ and obviously, the restrictions of the functions ψ and Z on Ω_j are the sub- and super-solutions of the following problem respectively.

$$\begin{cases} -\Delta_p u = \tilde{h}(x, u(x)), & x \in \Omega_j, \\ u(x) = \psi(x), & x \in \partial\Omega_j. \end{cases} \tag{2.18}$$

Then (2.18) has a solution $u_j \in W_0^{1,p}(\Omega_j) \times C(\overline{\Omega}_j)$ such that

$$\psi(x) \le u_j(x) \le Z(x), \qquad \text{for all } x \in \Omega_j$$

and this is also a solution of (2.17). First, we claim that for fixed k, there exists $d_k > 0$ such that $\|u_j\|_{C^{1,\beta}(\overline{\Omega}_k)} \le d_k$, for all $j \ge k+1$. In fact, take Q_k such that $\Omega_k \subset\subset Q_k \subset\subset \Omega_{k+1}$. Denote $h_j(x) = h(x, u_j(x))$. Then

$$-\Delta_p u = h_j, \qquad \text{on } Q_k.$$

Since $\{u_j\}_{j \ge k+1}$ are uniformly bounded on $\overline{\Omega}_{k+1}$, we know that there is $c_k > 0$ such that

$$\|h_j\|_{C(\overline{Q}_k)} < c_k, \qquad \text{for all } j \ge k+1.$$

Using (Prop. 3.7, pp. 806 in [To]), we see that there exists $d_k > 0$ such that

$$\|u\|_{C^{1,\beta}(\overline{\Omega}_k)} < d_k, \qquad \text{for all } j \geq k+1,$$

for some $\beta \in (0,1)$. So our claim is proven. Next, since the embedding $C^{1,\beta}(\overline{\Omega}_k) \hookrightarrow C^1(\overline{\Omega}_k)$ is compact, for each k, the sequence $\{u_j\}_{j=1}^{\infty}$ has a subsequence, renamed $\{u_j\}_{j=1}^{\infty}$, which converges to u in $C^1(\overline{\Omega}_k)$. This implies that $u \in C^1(\overline{\Omega}_k)$, for every k. Consequently, $u \in C^1(\Omega)$. Moreover, for each k, we have

$$-\int_{\Omega_k} |\nabla u_j|^{p-2} \nabla u_j \nabla \varphi = \int_{\Omega_k} h(x, u_j)\varphi,$$

for all $\varphi \in C_0^{\infty}(\Omega_k)$ and $j \geq k+1$. By taking the limit of the sequence converging in $C^1(\overline{\Omega}_k)$,

$$-\int_{\Omega_k} |\nabla u|^{p-2} \nabla u \nabla \varphi = \int_{\Omega_k} h(x, u)\varphi,$$

for all $\varphi \in C_0^{\infty}(\Omega_k)$ and $j \geq k+1$. Thus,

$$-\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \varphi = \int_{\Omega} h(x, u)\varphi,$$

for all $\varphi \in C_0^{\infty}(\Omega)$. Also, since $\psi(x) \leq u_j(x) \leq Z(x)$ for all j, we have $\psi(x) \leq u(x) \leq Z(x)$ for all $x \in \Omega$. Thus from $\psi(x) = Z(x) = 0$ on $x \in \partial\Omega$, we know that $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. This completes the proof of Lemma 19. □

Our Lemma 19 can be extended to the following p-Laplacian quasi-monotone system of the form:

$$\begin{cases} -\Delta_p u = h_1(x, u, v) & \text{in } \Omega \\ -\Delta_p v = h_2(x, u, v) & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega, \end{cases} \tag{2.19}$$

where $h_i : \Omega \times (0, \infty) \times (0, \infty) \to \mathbb{R}$ is continuous for $i = 1, 2$.

By a sub-solution we mean a pair of functions $(\psi_1, \psi_2) : \overline{\Omega} \to \mathbb{R}^2$ such that $(\psi_1, \psi_2) \in (W^{1,p}(\Omega) \cap C(\overline{\Omega})) \times (W^{1,p}(\Omega) \cap C(\overline{\Omega}))$ and satisfy

$$\begin{cases} -\Delta_p \psi_1 \leq h_1(x, \psi_1, \psi_2) & \text{in } \Omega \\ -\Delta_p \psi_2 \leq h_2(x, \psi_1, \psi_2) & \text{in } \Omega \\ \quad \psi_1 > 0, \psi_2 > 0 & \text{in } \Omega \\ \quad \psi_1 = 0 = \psi_2 & \text{on } \partial\Omega. \end{cases}$$

By a super-solution we mean a pair of functions $(Z_1, Z_2) : \overline{\Omega} \to \mathbb{R}^2$ such that $(Z_1, Z_2) \in (W^{1,p}(\Omega) \cap C(\overline{\Omega})) \times (W^{1,p}(\Omega) \cap C(\overline{\Omega}))$ and satisfy

$$\begin{cases} -\Delta_p Z_1 \geq h_1(x, Z_1, Z_2) & \text{in } \Omega \\ -\Delta_p Z_2 \geq h_2(x, Z_1, Z_2) & \text{in } \Omega \\ \quad Z_1 > 0, Z_2 > 0 & \text{in } \Omega \\ \quad Z_1 = 0 = Z_2 & \text{on } \partial\Omega. \end{cases}$$

Then if there exists a sub-solution (ψ_1, ψ_2) and a super-solution (Z_1, Z_2) for (2.19) such that $(\psi_1, \psi_2) \leq (Z_1, Z_2)$ on $\overline{\Omega}$, then there exists at least one solution (u, v) satisfying $(\psi_1, \psi_2) \leq (u, v) \leq (Z_1, Z_2)$ on $\overline{\Omega}$. This follows from the natural extension of Lemma 19.

Finally we note that this result extends also for pq-Laplacian quasi-monotone systems as well as Laplacian and p-Laplacian $n \times n$ quasi-monotone systems.

CHAPTER 3

INFINITE SEMIPOSITONE LAPLACIAN EQUATIONS

In this chapter proofs of Theorems 1-3 will be presented. In Section 3.1, Theorem 1 will be proved. In Section 3.2, proofs of Theorem 2 and 3 will be presented.

3.1 Proof of Theorem 1

Let λ_1, ϕ be as defined in the proof of Lemma 2. Let $\psi := \lambda^r \phi^{\frac{2}{1+\alpha}}, r \in (\frac{1}{1+\alpha}, \frac{1}{1+\alpha-\beta})$. Then

$$\nabla\psi = \lambda^r (\frac{2}{1+\alpha}) \phi^{\frac{1-\alpha}{1+\alpha}} \nabla\phi$$

and

$$\begin{aligned} -\Delta\psi &= -\lambda^r (\frac{2}{1+\alpha}) \{\phi^{\frac{1-\alpha}{1+\alpha}} \Delta\phi + (\frac{1-\alpha}{1+\alpha}) \phi^{-\frac{2\alpha}{1+\alpha}} \mid \nabla\phi \mid^2\} \\ &= \lambda^r (\frac{2}{1+\alpha}) \frac{1}{(\phi^{\frac{2}{1+\alpha}})^\alpha} \{\lambda_1 \phi^2 - (\frac{1-\alpha}{1+\alpha}) \mid \nabla\phi \mid^2\}. \end{aligned}$$

Let $\delta > 0, \mu > 0, m > 0$ be such that

$$(\frac{2}{1+\alpha}) \{(\frac{1-\alpha}{1+\alpha}) \mid \nabla\phi \mid^2 - \lambda_1 \phi^2\} \geq m \quad \text{in } \overline{\Omega_\delta},$$

and $\phi \in [\mu, 1]$ in $\Omega \setminus \overline{\Omega_\delta}$, where $\overline{\Omega_\delta} := \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$. This is possible since $\mid \nabla\phi \mid \neq 0; \partial\Omega$. Then in $\overline{\Omega_\delta}$, if $\lambda \gg 1$ then

$$(\frac{2}{1+\alpha}) \{\lambda_1 \phi^2 - (\frac{1-\alpha}{1+\alpha}) \mid \nabla\phi \mid^2\} \leq -m \leq \frac{\lambda g(0)}{\lambda^r \lambda^{r\alpha}} = \lambda^{[1-r-r\alpha]} g(0)$$

since $g(0) < 0$ and $1 - r - r\alpha < 0$. Hence in $\overline{\Omega_\delta}$, if $\lambda \gg 1$ then

$$-\Delta\psi = \lambda^r(\frac{2}{1+\alpha})\frac{1}{(\phi^{\frac{2}{1+\alpha}})^\alpha}\{\lambda_1\phi^2 - (\frac{1-\alpha}{1+\alpha}) \mid \nabla\phi \mid^2\} \leq \lambda\frac{g(\lambda^r\phi^{\frac{2}{1+\alpha}})}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha}. \qquad (3.1)$$

Next, in $\Omega \setminus \overline{\Omega_\delta}$, since $\phi \geq \mu$, from $(H8)$,

$$g(\lambda^r\phi^{\frac{2}{1+\alpha}}) \geq A(\lambda^r\phi^{\frac{2}{1+\alpha}})^\beta,$$

for $\lambda \gg 1$. Also since $0 < \mu \leq \phi < 1$ and $1 + r(\beta - \alpha) - r > 0$,

$$(\frac{2\lambda_1}{1+\alpha})[\lambda^r\phi^{\frac{2}{1+\alpha}}] \leq \lambda A(\lambda^r\phi^{\frac{2}{1+\alpha}})^{\beta-\alpha} = \lambda\frac{A(\lambda^r\phi^{\frac{2}{1+\alpha}})^\beta}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha},$$

for $\lambda \gg 1$. Hence in $\Omega \setminus \overline{\Omega_\delta}$, for $\lambda \gg 1$,

$$-\Delta\psi \leq \lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \leq \lambda\frac{A(\lambda^r\phi^{\frac{2}{1+\alpha}})^\beta}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha} \leq \lambda\frac{g(\lambda^r\phi^{\frac{2}{1+\alpha}})}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha}. \qquad (3.2)$$

Combining (3.1) and (3.2), we see that

$$-\Delta\psi \leq \lambda\frac{g(\psi)}{\psi^\alpha} \qquad \text{in } \Omega$$

for $\lambda \gg 1$. Thus ψ is a positive sub-solution.

Now we construct a super-solution $Z \geq \psi$. Since $1 + \alpha - \gamma > 0$ and $\gamma - \alpha > 0$, we can choose $m(\lambda) \gg 1$ such that

$$m(\lambda)^{1+\alpha-\gamma} \geq \lambda B e^{\gamma-\alpha}$$

where e is as defined in the proof of Lemma 1. Hence for $m(\lambda) \gg 1$

$$m(\lambda) \geq \frac{\lambda B(m(\lambda)e)^\gamma}{(m(\lambda)e)^\alpha}.$$

Let $Z := m(\lambda)e$. Then by $(H7)$ we have

$$-\Delta Z = m(\lambda) \geq \frac{\lambda g(m(\lambda)e)}{(m(\lambda)e)^\alpha}.$$

Thus Z is a super-solution. Further $m(\lambda)$ can be chosen large enough so that $Z \geq \psi$ in $\overline{\Omega}$. This is possible since $e > 0$ in Ω and $\frac{\partial e}{\partial \nu} < 0$ on $\partial\Omega$, where ν is outward normal vector on $\partial\Omega$. Hence for $\lambda \gg 1$, (1.10) has a positive solution and Theorem 1 is proven. $\square$

3.2 Proofs of Theorems 2-3

3.2.1 Proof of Theorem 2

Since $\lim_{s\to\infty} \frac{g(s)}{s} = 0$, there exists constants $a > 0,\ b > 0$ such that $g(s) - \frac{1}{s^\alpha} < as - b$. Let λ_1, ϕ be as defined in the proof of Lemma 2. Suppose $u > 0; \Omega$ is a positive solution of (1.11). Then

$$\int_\Omega (-\Delta u)\phi dx \leq \lambda \int_\Omega (au - b)\phi dx.$$

But

$$\int_\Omega (-\Delta u)\phi dx = \int_\Omega u(-\Delta\phi)dx = \int_\Omega \lambda_1 u\phi dx.$$

Thus

$$\int_\Omega (\lambda_1 - \lambda a)u\phi dx \leq \int_\Omega (-\lambda b)\phi dx.$$

This is impossible if $\lambda < \lambda_1/a$ and the first part of Theorem 2 is proven.

Next, let $\psi := \lambda^r \phi^{\frac{2}{1+\alpha}}$, where the parameter $r \in (\frac{1}{1+\alpha}, 1)$. Then

$$\nabla\psi = \lambda^r (\frac{2}{1+\alpha})\phi^{\frac{1-\alpha}{1+\alpha}}\nabla\phi$$

and

$$\begin{aligned}\Delta\psi &= \lambda^r(\frac{2}{1+\alpha})\{\phi^{\frac{1-\alpha}{1+\alpha}}\Delta\phi + \frac{1-\alpha}{1+\alpha}\phi^{-\frac{2\alpha}{1+\alpha}} \mid \nabla\phi \mid^2\} \\ &= \lambda^r(\frac{2}{1+\alpha})\{-\lambda_1\phi^{\frac{2}{1+\alpha}} + \frac{1-\alpha}{1+\alpha}\frac{\mid \nabla\phi \mid^2}{\phi^{\frac{2\alpha}{1+\alpha}}}\}.\end{aligned}$$

Thus

$$-\Delta\psi = \lambda^r(\frac{2}{1+\alpha})\{\lambda_1\phi^{\frac{2}{1+\alpha}} - \frac{1-\alpha}{1+\alpha}\frac{|\nabla\phi|^2}{\phi^{\frac{2\alpha}{1+\alpha}}}\}.$$

Let $\delta > 0, \mu > 0, m > 0$ be such that $|\nabla\phi|^2 \geq m$, in $\overline{\Omega_\delta}$, and $\phi^{\frac{2}{1+\alpha}} \in [\mu, 1]$ in $\Omega \setminus \overline{\Omega_\delta}$, where $\overline{\Omega_\delta} := \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla\phi| \neq 0; \partial\Omega$. Then in $\overline{\Omega_\delta}$, if $\lambda \gg 1$

$$-\lambda^r(\frac{2}{1+\alpha})\frac{1-\alpha}{1+\alpha}\frac{|\nabla\phi|^2}{\phi^{\frac{2\alpha}{1+\alpha}}} \leq \lambda[-\frac{1}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha}]$$

since $1 - r - r\alpha < 0$. Also in $\overline{\Omega_\delta}$ (in fact in Ω),

$$\lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \leq \lambda g(0) \leq \lambda g(\lambda^r\phi^{\frac{2}{1+\alpha}}) \text{ if } \lambda \gg 1.$$

Hence in $\overline{\Omega_\delta}$,

$$\begin{aligned} -\Delta\psi &\leq \lambda[g(\lambda^r\phi^{\frac{2}{1+\alpha}}) - \frac{1}{(\lambda^r\phi^{\frac{2}{1+\alpha}})^\alpha}] \\ &= \lambda[g(\psi) - \frac{1}{\psi^\alpha}]. \end{aligned} \tag{3.3}$$

Next, in $\Omega\setminus\overline{\Omega_\delta}$, since $\phi^{\frac{2}{1+\alpha}} \geq \mu, \lambda[g(\psi) - \frac{1}{\psi^\alpha}] \geq \lambda[g(\lambda^r\mu) - \frac{1}{(\lambda^r\mu)^\alpha}]$. But if $\lambda \gg 1, -\Delta\psi \leq \lambda^r\frac{2}{(1+\alpha)}\lambda_1 \leq \lambda[g(\lambda^r\mu) - \frac{1}{(\lambda^r\mu)^\alpha}]$, since $r < 1$. Hence if $\lambda \gg 1$, in $\Omega \setminus \overline{\Omega_\delta}$, we have

$$-\Delta\psi \leq \lambda[g(\psi) - \frac{1}{\psi^\alpha}]. \tag{3.4}$$

Combining (3.3) and (3.4) we see that $\psi = \lambda^r\phi^{\frac{2}{1+\alpha}}$ is a positive sub-solution of (1.11).

Now we construct a super-solution $Z \geq \psi$. Since $\lim_{s\to\infty}\frac{g(s)}{s} = 0, \forall\lambda > 0, \exists m(\lambda) > 0$

such that $m(\lambda) \geq \lambda g(m(\lambda) \parallel e \parallel_\infty)$, where e is as defined in the proof of Lemma 1. Let $Z := m(\lambda)e$. Then

$$\begin{aligned} -\Delta Z &= m(\lambda) \\ &\geq \lambda g(m(\lambda) \parallel e \parallel_\infty) \\ &\geq \lambda g(m(\lambda)e) \\ &= \lambda g(Z) \end{aligned}$$

Thus Z is a super-solution. Further, $m(\lambda)$ can be chosen large enough so that $Z = m(\lambda)e \geq \psi$ in $\overline{\Omega}$. Hence for $\lambda \gg 1$, (1.11) has a positive solution $u \in [\psi, Z]$ and the second part of Theorem 2 is proven. □

3.2.2 Proof of Theorem 3

Let $\psi = \lambda^r \phi^{\frac{2}{1+\alpha}}$ as earlier. In the proof of Theorem 2, when proving ψ is a sub-solution of (1.11) for $\lambda \gg 1$, we used the fact that $g(0) > 0$ to show $\lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \leq \lambda g(\lambda^r\phi^{\frac{2}{1+\alpha}})$ in $\overline{\Omega_\delta}$. Here with $g(0) = 0$ we establish the above inequality by using $\lim_{s\to 0}\frac{g(s)}{s^\beta} = k > 0$. (The rest of the proof of Theorem 3 is exactly as the proof of Theorem 2.) Let $A > 0$ be such that $g(x) \geq \frac{k}{2}x^\beta$ for $x \in [0, A]$. Choose $\lambda^* \gg 1$ such that

$$\lambda^r(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}} \leq \lambda\{\frac{k}{2}\lambda^{r\beta}\phi^{\frac{2\beta}{1+\alpha}}\} \text{ holds } \forall\lambda \geq \lambda^*.$$

Hence, if $(\lambda^*)^r\phi^{\frac{2}{1+\alpha}} \leq A$, then

$$g((\lambda^*)^r\phi^{\frac{2}{1+\alpha}}) \geq (\lambda^*)^{r-1}(\frac{2}{1+\alpha})\lambda_1\phi^{\frac{2}{1+\alpha}}.$$

Further, for $\lambda \geq \lambda^*$ we have

$$
\begin{aligned}
g(\lambda^r \phi^{\frac{2}{1+\alpha}}) &\geq g((\lambda^*)^r \phi^{\frac{2}{1+\alpha}}) \\
&\geq (\lambda^*)^{r-1} (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}} \\
&\geq \lambda^{r-1} (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}.
\end{aligned}
$$

Multiplying by λ we get

$$
\lambda g(\psi) \geq \lambda^r (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}.
$$

Next, if $(\lambda^*)^r \phi^{\frac{2}{1+\alpha}} > A$, then $\forall \lambda (\geq \lambda^*)$ and sufficiently large we have

$$
\lambda g(\psi) \geq \lambda g(A) \geq \lambda^r (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}.
$$

Hence here for $\lambda \gg 1$

$$
\lambda g(\psi) \geq \lambda^r (\frac{2}{1+\alpha}) \lambda_1 \phi^{\frac{2}{1+\alpha}}
$$

must hold in $\overline{\Omega_\delta}$ (in fact throughout Ω). Thus Theorem 3 is proven. $\square$

CHAPTER 4

INFINITE SEMIPOSITONE p-LAPLACIAN SYSTEMS

In this chapter we will prove Theorems 4-8. We prove Theorem 4 in Section 4.1, Theorem 5 in Section 4.2, Theorems 6-7 in Section 4.3 and Theorem 8 in Section 4.4.

4.1 Proof of Theorem 4

Let $\phi_1 > 0$ be the eigenfunction corresponding to the first eigenvalue λ_1 of the operator $-\Delta_p$ with Dirichlet boundary condition i.e. ϕ_1 satisfies:

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1 \phi_1^{p-1} & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

We also choose ϕ_1 such that $\| \phi_1 \|_\infty = 1$. Let $(\psi_1, \psi_2) := (\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_2}})$, where $\gamma \in (\frac{1}{p-1+\alpha_1}, \frac{1}{p-1}) \cap (\frac{1}{p-1+\alpha_2}, \frac{1}{p-1})$. Then

$$\nabla \psi_1 = \lambda^\gamma (\frac{p}{p-1+\alpha_1}) \phi_1^{\frac{1-\alpha_1}{p-1+\alpha_1}} \nabla \phi_1$$

and

$$\begin{aligned}
\Delta_p\psi_1 &= div(\mid \nabla\psi_1 \mid^{p-2} \nabla\psi_1) \\
&= \lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1} div(\phi_1^{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}} \mid \nabla\phi_1 \mid^{p-2} \nabla\phi_1) \\
&= \lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\{\nabla(\phi_1^{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}}) \cdot \mid \nabla\phi_1 \mid^{p-2} \nabla\phi_1 + \phi_1^{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}} \Delta_p\phi_1\} \\
&= \lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}\phi_1^{-\frac{\alpha_1 p}{p-1+\alpha_1}} \mid \nabla\phi_1 \mid^{p} - \lambda_1\phi_1^{\frac{p(p-1)}{p-1+\alpha_1}}\}.
\end{aligned}$$

Thus

$$-\Delta_p\psi_1 = \lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\{\lambda_1\phi_1^{\frac{p(p-1)}{p-1+\alpha_1}} - \frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}\frac{\mid \nabla\phi_1 \mid^p}{\phi_1^{\frac{\alpha_1 p}{p-1+\alpha_1}}}\}.$$

Similarly

$$-\Delta_p\psi_2 = \lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_2})^{p-1}\{\lambda_1\phi_1^{\frac{p(p-1)}{p-1+\alpha_2}} - \frac{(1-\alpha_2)(p-1)}{p-1+\alpha_2}\frac{\mid \nabla\phi_1 \mid^p}{\phi_1^{\frac{\alpha_2 p}{p-1+\alpha_2}}}\}.$$

Let $\delta > 0, \mu > 0, m > 0$ be such that $\mid \nabla\phi_1 \mid^p \geq m$ in $\overline{\Omega_\delta}$, and $\phi_1^{\frac{p}{p-1+\alpha_1}}, \phi_1^{\frac{p}{p-1+\alpha_2}} \in [\mu, 1]$ in $\Omega \setminus \overline{\Omega_\delta}$, where $\overline{\Omega_\delta} := \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $\mid \nabla\phi_1 \mid \neq 0$ on $\partial\Omega$. Then in $\overline{\Omega_\delta}$, if $\lambda \gg 1$,

$$\big[-\lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}\frac{\mid \nabla\phi_1 \mid^p}{\phi_1^{\frac{\alpha_1 p}{p-1+\alpha_1}}}\big] \leq \lambda[-\frac{1}{(\lambda^\gamma\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}}]$$

since $\gamma(p-1) > 1-\alpha_1\gamma$. Also in $\overline{\Omega_\delta}$ (in fact in Ω), if $\lambda \gg 1$,

$$\lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1\phi_1^{\frac{p(p-1)}{p-1+\alpha_1}} \leq \lambda g_1(0) \leq \lambda g_1(\lambda^\gamma\phi_1^{\frac{p}{p-1+\alpha_2}}),$$

since $\gamma(p-1) < 1$. Hence in $\overline{\Omega_\delta}$, if $\lambda \gg 1$,

$$\begin{aligned}
-\Delta_p\psi_1 &\leq \lambda[g_1(\lambda^\gamma\phi_1^{\frac{p}{p-1+\alpha_2}}) - \frac{1}{(\lambda^\gamma\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}}] \\
&= \lambda[g_1(\psi_2) - \frac{1}{\psi_1^{\alpha_1}}]. \qquad (4.1)
\end{aligned}$$

Next, in $\Omega \setminus \overline{\Omega_\delta}$, since $\phi_1^{\frac{p}{p-1+\alpha_1}} \geq \mu, \phi_1^{\frac{p}{p-1+\alpha_2}} \geq \mu$,

$$\lambda[g_1(\psi_2) - \frac{1}{\psi_1^{\alpha_1}}] \geq \lambda[g_1(\lambda^\gamma \mu) - \frac{1}{(\lambda^\gamma \mu)^{\alpha_1}}].$$

But if $\lambda \gg 1$,

$$-\Delta_p \psi_1 \leq \lambda^{\gamma(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1 \phi_1^{\frac{p(p-1)}{p-1+\alpha_1}} \leq \lambda[g_1(\lambda^\gamma \mu) - \frac{1}{(\lambda^\gamma \mu)^{\alpha_1}}],$$

since $\gamma(p-1) < 1$. Hence in $\Omega \setminus \overline{\Omega_\delta}$, if $\lambda \gg 1$, we have

$$-\Delta_p \psi_1 \leq \lambda[g_1(\psi_2) - \frac{1}{\psi_1^{\alpha_1}}]. \tag{4.2}$$

Combining (4.1) and (4.2) if $\lambda \gg 1$ we see that

$$-\Delta_p \psi_1 \leq \lambda[g_1(\psi_2) - \frac{1}{\psi_1^{\alpha_1}}] \qquad \text{in } \Omega.$$

Similarly for $\lambda \gg 1$ we get

$$-\Delta_p \psi_2 \leq \lambda[g_2(\psi_1) - \frac{1}{\psi_2^{\alpha_2}}] \qquad \text{in } \Omega.$$

Thus (ψ_1, ψ_2) is a positive sub-solution of (1.12).

Now we construct a super-solution $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. In the hypothesis $(H10)$, we will consider the case $\lim_{s\to\infty} g_2(s) = \infty$. (If $g_2(s) \nrightarrow \infty$ as $s \to \infty$, then a similar proof can be given by using $\lim_{s\to\infty} g_1(s) = \infty$ and the equivalent combined p-sublinear condition). Since $\lim_{s\to\infty} \frac{g_1(M(g_2(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0$ for all $M > 0$, there exists $m(\lambda) > 0$ such that

$$\frac{1}{\lambda} \geq \frac{g_1((\lambda g_2(m(\lambda) \parallel e_p \parallel_\infty))^{\frac{1}{p-1}} \parallel e_p \parallel_\infty)}{m(\lambda)^{p-1}},$$

where e_p is the unique positive solution of $-\Delta_p e_p = 1$ in Ω, $e_p = 0$ on $\partial\Omega$.

Let $(Z_1, Z_2) := (m(\lambda)e_p, (\lambda g_2(m(\lambda) \parallel e_p \parallel_\infty))^{\frac{1}{p-1}} e_p)$. Then

$$\begin{aligned}-\Delta_p Z_1 &= m(\lambda)^{p-1} \\ &\geq \lambda g_1((\lambda g_2(m(\lambda) \parallel e_p \parallel_\infty))^{\frac{1}{p-1}} e_p) \\ &\geq \lambda(g_1(Z_2) - \frac{1}{Z_1^{\alpha_1}}).\end{aligned}$$

Also

$$\begin{aligned}-\Delta_p Z_2 &= \lambda(g_2(m(\lambda) \parallel e_p \parallel_\infty)) \\ &\geq \lambda(g_2(m(\lambda)e_p)) \\ &\geq \lambda(g_2(Z_1) - \frac{1}{Z_2^{\alpha_2}}).\end{aligned}$$

Thus (Z_1, Z_2) is a super-solution. Further, $m(\lambda)$ can be chosen large enough so that $(Z_1, Z_2) \geq (\psi_1, \psi_2)$ in $\overline{\Omega}$. This is possible since $e_p > 0; \Omega, \frac{\partial e_p}{\partial \nu} < 0; \partial\Omega$, where ν is the outward normal vector at $\partial\Omega$ and $\lim_{s\to\infty} g_2(s) = \infty$. Hence for $\lambda \gg 1$, (1.12) has a positive solution (u, v) and Theorem 4 is proven. □

4.2 Proof of Theorem 5

Let $(\psi_1, \psi_2) := (\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha}}, \lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha}})$ for the same γ and ϕ_1 introduced in the proof of Theorem 4. Then since $\psi_1 \equiv \psi_2$, we can easily show that (ψ_1, ψ_2) is a sub-solution of (1.13) for $\lambda \gg 1$ by the same argument as in the proof of Theorem 4. And the construction of super-solution (Z_1, Z_2) with $(Z_1, Z_2) \geq (\psi_1, \psi_2)$ can be achieved by the same way as in the proof of Theorem 4. □

4.3 Proofs of Theorems 6-7

4.3.1 Proof of Theorem 6

Let $I := \left(\frac{1}{(p-1)+\alpha_1}, \frac{1}{(p-1)+(\alpha_1-\sigma_1)}\right) \cap \left(\frac{1}{(p-1)+\alpha_2}, \frac{1}{(p-1)+(\alpha_2-\sigma_2)}\right)$. Then from $(H11)$, we know $I \neq \varnothing$. For $\gamma \in I$, let $(\psi_1, \psi_2) = \left(\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_2}}\right)$, where ϕ_1 is as defined in the proof of Theorem 4. Then

$$-\Delta_p \psi_1 = \left(\lambda^\gamma \frac{p}{p-1+\alpha_1}\right)^{p-1} \phi_1^{\frac{-p\alpha_1}{p-1+\alpha_1}} \left[\lambda_1 \phi_1^p - \frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1} |\nabla \phi_1|^p\right].$$

Let $\delta > 0$, $m > 0$ and $\mu > 0$ be such that

$$\frac{(1-\alpha_i)(p-1)}{p-1+\alpha_i} |\nabla \phi_1|^{p-1} - \lambda_1 \phi_1^p \geq m, \quad i = 1, 2$$

in $\overline{\Omega}_\delta$, and $\phi_1 \geq \mu > 0$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla \phi_1| \neq 0\,;\partial\Omega$.

First, in $\overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$(\lambda^\gamma \frac{p}{p-1+\alpha_1})^{p-1} \left[\lambda_1 \phi_1^p - \frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1} |\nabla \phi_1|^p\right] \leq \lambda \frac{g_1(0)}{(\lambda^\gamma)^{\alpha_1}},$$

since $1 - (p-1)\gamma - \alpha_1 \gamma < 0$ and $g_1(0) < 0$.

Hence, in $\overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$-\Delta_p \psi_1 \leq \lambda \frac{g_1(0)}{(\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}} \leq \lambda \frac{g_1(\psi_2)}{\psi_1^{\alpha_1}}. \tag{4.3}$$

Next, in $\Omega \setminus \overline{\Omega}_\delta$, since $\phi_1 \geq \mu > 0$, from $(H11)$, we know that

$$g_1(\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_2}}) \geq A_1 (\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_2}})^{\sigma_1},$$

for $\lambda \gg 1$. Also, since $0 < \mu \leq \phi_1 < 1$ and $1 + (\sigma_1 - \alpha_1)\gamma - (p-1)\gamma > 0$,

$$(\lambda^\gamma \frac{p}{p-1+\alpha_1})^{p-1} \lambda_1 \phi_1^p \leq \lambda \frac{A_1 (\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_2}})^{\sigma_1}}{\lambda^{\gamma\alpha_1}},$$

for $\lambda \gg 1$. Then in $\Omega \setminus \overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$-\Delta_p \psi_1 \leq (\lambda^\gamma \frac{p}{p-1+\alpha_1})^{p-1} \lambda_1 \phi_1^{\frac{-p\alpha_1}{p-1+\alpha_1}+p} \leq \lambda \frac{g_1(\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_2}})}{(\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}} = \lambda \frac{g_1(\psi_2)}{\psi_1^{\alpha_1}}. \tag{4.4}$$

Combining (4.3) and (4.4), we see that for $\lambda \gg 1$,

$$-\Delta_p \psi_1 \leq \lambda \frac{g_1(\psi_2)}{\psi_1^{\alpha_1}} \quad \text{in } \Omega.$$

Similarly, we can show that for $\lambda \gg 1$,

$$-\Delta_p \psi_2 \leq \lambda \frac{g_2(\psi_1)}{\psi_2^{\alpha_2}} \quad \text{in } \Omega.$$

Thus $(\psi_1, \psi_2) = \left(\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_2}}\right)$ is a positive sub-solution of (1.14).

Now we construct a super-solution $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. From [GST], we know that there exist functions w_1 and w_2 in $W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta_p w_1 = \frac{1}{w_1^{\alpha_1}} & \text{in } \Omega \\ w_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p w_2 = \frac{1}{w_2^{\alpha_2}} & \text{in } \Omega \\ w_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

and satisfying $w_1 \geq \varepsilon\phi_1$ and $w_2 \geq \varepsilon\phi_1$ for some $\varepsilon > 0$, where ϕ_1 is as defined in the proof of Theorem 4.

In $(H12)$, we will first work with the assumption

$$\lim_{s\to\infty} \frac{g_1((Mg_2(s))^{\frac{1}{p-1}})}{s^{p-1+\alpha_1}} = 0, \ \ \forall \ M > 0.$$

Let $(Z_1, Z_2) = (m(\lambda)w_1, (g_2(m(\lambda)\|w_1\|_\infty))^{\frac{1}{p-1}} w_2)$. Then, we know that for $m(\lambda) \gg 1$,

$$\frac{1}{\lambda} \geq \frac{g_1(\|w_2\|_\infty (g_2(m(\lambda)\|w_1\|_\infty))^{\frac{1}{p-1}})}{m(\lambda)^{p-1+\alpha_1}}.$$

Then for such $m(\lambda)$,

$$-\Delta_p Z_1 = \frac{m(\lambda)^{p-1}}{w_1^{\alpha_1}} \geq \lambda \frac{g_1((g_2(m(\lambda)\|w_1\|_\infty))^{\frac{1}{p-1}} w_2)}{(m(\lambda)w_1)^{\alpha_1}} = \lambda \frac{g_1(Z_2)}{Z_1^{\alpha_1}}.$$

From $(H11)$, we know that $g_2(s) \to \infty$ as $s \to \infty$. Thus for $m(\lambda) \gg 1$,

$$\frac{\lambda}{g_2(m(\lambda)\|w_1\|_\infty)^{\frac{\alpha_2}{p-1}}} \leq 1.$$

Hence, for $m(\lambda) \gg 1$,

$$-\Delta_p Z_2 = \frac{g_2(m(\lambda)\|w_1\|_\infty)}{w_2^{\alpha_2}} \geq \lambda \frac{g_2(m(\lambda)w_1)}{(g_2(m(\lambda)\|w_1\|_\infty)^{\frac{1}{p-1}})^{\alpha_2} w_2^{\alpha_2}} = \lambda \frac{g_2(Z_1)}{Z_2^{\alpha_2}}.$$

In $(H12)$, if we consider the assumption $\lim_{s\to\infty} \frac{g_2((Mg_1(s))^{\frac{1}{p-1}})}{s^{p-1+\alpha_2}} = 0,\ \forall\ M > 0$, let $(Z_1, Z_2) = ((g_1(m(\lambda)\|w_2\|_\infty))^{\frac{1}{p-1}} w_1, m(\lambda)w_2)$. Then, by using the same argument as above, we can show that (Z_1, Z_2) is a super-solution of (1.14). Further, $m(\lambda)$ can be chosen large enough so that $(Z_1, Z_2) \geq (\psi_1, \psi_2)$ in $\overline{\Omega}$, since $g_1(s) \to \infty$ and $g_2(s) \to \infty$ as $s \to \infty$. Therefore, problem (1.14) has a positive solution $(u, v) \in [(\psi_1, \psi_2), (Z_1, Z_2)]$. $\square$

4.3.2 Proof of Theorem 7

Here, the construction of the sub-solution (ψ_1, ψ_2) is the same as in the proof of Theorem 6. For the construction of the super-solution, we let $(Z_1, Z_2) = (m(\lambda)e_p, m(\lambda)e_p)$, where e_p is as defined in the proof of Theorem 4. Then for r_1 and B_1 described in $(H13)$, for $m(\lambda) \gg 1$, we have

$$(m(\lambda))^{(p-1)+\alpha_1-r_1} \geq B_1 e_p^{r_1-\alpha_1} > 0.$$

Hence for $m(\lambda) \gg 1$,

$$-\Delta_p Z_1 = (m(\lambda))^{p-1} \geq \lambda \frac{B_1(m(\lambda)e_p)^{r_1}}{(m(\lambda)e_p)^{\alpha_1}} \geq \lambda \frac{g_1(m(\lambda)e_p)}{(m(\lambda)e_p)^{\alpha_1}} = \lambda \frac{g_1(Z_2)}{Z_1^{\alpha_1}}.$$

Similarly, for $m(\lambda) \gg 1$, we can show that $-\Delta_p Z_2 \geq \lambda \frac{g_2(Z_1)}{Z_2^{\alpha_2}}$. Then (Z_1, Z_2) is a positive super-solution of (1.14) and by choosing $m(\lambda)$ large enough, we will have $(Z_1, Z_2) \geq$

(ψ_1, ψ_2) in $\overline{\Omega}$. Therefore, problem (1.14) has a positive solution $(u, v) \in [(\psi_1, \psi_2), (Z_1, Z_2)]$.

$\square$

4.4 Proof of Theorem 8

In the hypothesis $(H14)$, we will consider the case $\lim_{s\to\infty} \tilde{g}_2(s) = \infty$(If $\tilde{g}_2(s) \nrightarrow \infty$ as $s \to \infty$, then a similar proof can be given by using $\lim_{s\to\infty} \tilde{g}_1(s) = \infty$ and the equivalent combined p-sublinear condition). Let $(\psi_1, \psi_2) = \left(\lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha}}, \lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha}}\right)$ for the same γ and ϕ_1 as in the proof of Theorem 4. Then since $\psi_1 \equiv \psi_2$, we can easily show that (ψ_1, ψ_2) is a sub-solution of (1.15). Now, we construct a super-solution $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. By $(H14)$, since

$$\lim_{s\to\infty} \frac{\tilde{g}_1(M(\tilde{g}_2(s))^{\frac{1}{p-1}})}{s^{p-1}} = 0, \ \ \forall\, M > 0,$$

there exists $m(\lambda) \gg 1$ such that

$$\frac{1}{\lambda} \geq \frac{\tilde{g}_1((\lambda \tilde{g}_2(m(\lambda)\|e_p\|_\infty))^{\frac{1}{p-1}}\|e_p\|_\infty)}{m(\lambda)^{p-1}},$$

where e_p is as defined in the proof of Theorem 4.

Let $(Z_1, Z_2) := (m(\lambda)e_p, (\lambda \tilde{g}_2(m(\lambda)\|e_p\|_\infty))^{\frac{1}{p-1}} e_p)$. Then

$$\begin{aligned}
-\Delta_p Z_1 &= m(\lambda)^{p-1} \\
&\geq \lambda \tilde{g}_1((\lambda \tilde{g}_2(m(\lambda)\|e_p\|_\infty))^{\frac{1}{p-1}}\|e_p\|_\infty) \\
&\geq \lambda \tilde{g}_1((\lambda \tilde{g}_2(m(\lambda)\|e_p\|_\infty))^{\frac{1}{p-1}} e_p) \\
&= \lambda \tilde{g}_1(Z_2)
\end{aligned}$$

and

$$-\Delta_p Z_2 = \lambda \tilde{g}_2(m(\lambda)\|e_p\|_\infty)$$

$$\geq \lambda \tilde{g}_2(Z_1).$$

Thus (Z_1, Z_2) is a super-solution of (1.15) and $m(\lambda)$ can be chosen large enough so that $(Z_1, Z_2) \geq (\psi_1, \psi_2)$ in $\overline{\Omega}$. Therefore, Theorem 8 is proven. $\square$

CHAPTER 5

INFINITE SEMIPOSITONE $pq-$LAPLACIAN SYSTEMS

In this chapter proofs of Theorems 9-11 will be presented. In Section 5.1 we will prove Theorem 9. Theorems 10-11 will be proved in Section 5.2.

5.1 Proof of Theorem 9

Let $\phi_1 > 0, \phi_2 > 0$ be the eigenfunctions corresponding to the first eigenvalues $\lambda_1(p)$, $\lambda_1(q)$ of the operators $-\Delta_p$ and $-\Delta_q$ with Dirichlet boundary conditions respectively, i.e. ϕ_1, ϕ_2 satisfy:

$$\begin{cases} -\Delta_p \phi_1 = \lambda_1(p)\phi_1^{p-1} & \text{in } \Omega \\ \phi_1 = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta_q \phi_2 = \lambda_1(q)\phi_2^{q-1} & \text{in } \Omega \\ \phi_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

We also choose ϕ_1, ϕ_2 such that $\|\phi_1\|_\infty = 1$, $\|\phi_2\|_\infty = 1$.

Let $(\psi_1, \psi_2) := (\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}})$, where $\gamma_1 \in (\frac{1}{p-1+\alpha_1}, \frac{1}{p-1})$, $\gamma_2 \in (\frac{1}{q-1+\alpha_2}, \frac{1}{q-1})$.

Then

$$\nabla\psi_1 = \lambda^{\gamma_1}(\frac{p}{p-1+\alpha_1})\phi_1^{\frac{1-\alpha_1}{p-1+\alpha_1}}\nabla\phi_1$$

and

$$\Delta_p\psi_1 = div(\mid \nabla\psi_1 \mid^{p-2} \nabla\psi_1)$$

$$= \lambda^{\gamma_1(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1} div(\phi_1^{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}} \mid \nabla\phi_1 \mid^{p-2} \nabla\phi_1)$$

$$= \lambda^{\gamma_1(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\{\nabla(\phi_1^{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}}) \cdot \mid \nabla\phi_1 \mid^{p-2} \nabla\phi_1 + \phi_1^{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}} \Delta_p\phi_1\}$$

$$= \lambda^{\gamma_1(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\{\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}\phi_1^{-\frac{\alpha_1 p}{p-1+\alpha_1}} \mid \nabla\phi_1 \mid^p - \lambda_1(p)\phi_1^{\frac{p(p-1)}{p-1+\alpha_1}}\}.$$

Thus

$$-\Delta_p\psi_1 = \lambda^{\gamma_1(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\{\lambda_1(p)\phi_1^{\frac{p(p-1)}{p-1+\alpha_1}} - \frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}\frac{\mid \nabla\phi_1 \mid^p}{\phi_1^{\frac{\alpha_1 p}{p-1+\alpha_1}}}\}.$$

Similarly

$$-\Delta_q\psi_2 = \lambda^{\gamma_2(q-1)}(\frac{q}{q-1+\alpha_2})^{q-1}\{\lambda_1(q)\phi_2^{\frac{q(q-1)}{q-1+\alpha_2}} - \frac{(1-\alpha_2)(q-1)}{q-1+\alpha_2}\frac{\mid \nabla\phi_2 \mid^q}{\phi_2^{\frac{\alpha_2 q}{q-1+\alpha_2}}}\}.$$

Let $\delta > 0, \mu > 0, m > 0$ be such that $\mid \nabla\phi_1 \mid^p \geq m, \mid \nabla\phi_2 \mid^q \geq m$ in $\overline{\Omega_\delta}$, and $\phi_1^{\frac{p}{p-1+\alpha_1}}, \phi_2^{\frac{q}{q-1+\alpha_2}} \in [\mu, 1]$ in $\Omega \setminus \overline{\Omega_\delta}$, where $\overline{\Omega_\delta} := \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $\mid \nabla\phi_1 \mid \neq 0, \mid \nabla\phi_2 \mid \neq 0$ on $\partial\Omega$. Then in $\overline{\Omega_\delta}$, since $1 - \alpha_1\gamma_1 < \gamma_1(p-1)$, if $\lambda \gg 1$,

$$-\lambda^{\gamma_1(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}\frac{\mid \nabla\phi_1 \mid^p}{\phi_1^{\frac{\alpha_1 p}{p-1+\alpha_1}}} \leq \lambda[-\frac{1}{(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}}].$$

Also in $\overline{\Omega_\delta}$ (in fact in Ω), since $\gamma_1(p-1) < 1$, if $\lambda \gg 1$,

$$\lambda^{\gamma_1(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1(p)\phi_1^{\frac{p(p-1)}{p-1+\alpha_1}} \leq \lambda g_1(0,0) \leq \lambda g_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}).$$

Hence in $\overline{\Omega_\delta}$, if $\lambda \gg 1$,

$$-\Delta_p\psi_1 \leq \lambda[g_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}) - \frac{1}{(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}}]$$

$$= \lambda[g_1(\psi_1, \psi_2) - \frac{1}{\psi_1^{\alpha_1}}]. \tag{5.1}$$

Next, in $\Omega \setminus \overline{\Omega_\delta}$, since $\phi_1^{\frac{p}{p-1+\alpha_1}} \geq \mu, \phi_2^{\frac{q}{q-1+\alpha_2}} \geq \mu$,

$$\lambda[g_1(\psi_1, \psi_2) - \frac{1}{\psi_1^{\alpha_1}}] \geq \lambda[g_1(\lambda^{\gamma_1}\mu, \lambda^{\gamma_2}\mu) - \frac{1}{(\lambda^{\gamma_1}\mu)^{\alpha_1}}].$$

Also since $\gamma_1(p-1) < 1$, if $\lambda \gg 1$,

$$-\Delta_p \psi_1 \leq \lambda^{\gamma_1(p-1)}(\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1(p)\phi_1^{\frac{p(p-1)}{p-1+\alpha_1}} \leq \lambda[g_1(\lambda^{\gamma_1}\mu, \lambda^{\gamma_2}\mu) - \frac{1}{(\lambda^{\gamma_1}\mu)^{\alpha_1}}].$$

Hence in $\Omega \setminus \overline{\Omega_\delta}$, if $\lambda \gg 1$, we have

$$-\Delta_p \psi_1 \leq \lambda[g_1(\psi_1, \psi_2) - \frac{1}{\psi_1^{\alpha_1}}]. \tag{5.2}$$

Combining (5.1) and (5.2), if $\lambda \gg 1$, we see that

$$-\Delta_p \psi_1 \leq \lambda[g_1(\psi_1, \psi_2) - \frac{1}{\psi_1^{\alpha_1}}] \qquad \text{in } \Omega.$$

Similarly, for $\lambda \gg 1$ we obtain

$$-\Delta_q \psi_2 \leq \lambda[g_2(\psi_1, \psi_2) - \frac{1}{\psi_2^{\alpha_2}}] \qquad \text{in } \Omega.$$

Thus (ψ_1, ψ_2) is a positive sub-solution of (1.16).

Now we construct a super-solution $(Z_1, Z_2) \geq (\psi_1, \psi_2)$. From the hypothesis $(H15)$, we have $\lim_{s\to\infty} \frac{g_1(s,(Mg_2(s,s))^{\frac{1}{q-1}})}{s^{p-1}} = 0$ for all $M > 0$. Then there exists $m(\lambda) > 0$ such that

$$\frac{1}{\lambda} \geq \frac{g_1(m(\lambda) \parallel e_p \parallel_\infty, (\lambda g_2(m(\lambda) \parallel e_p \parallel_\infty, m(\lambda) \parallel e_p \parallel_\infty))^{\frac{1}{q-1}} \parallel e_q \parallel_\infty)}{m(\lambda)^{p-1}},$$

where e_r is the unique positive solution of $-\Delta_r e = 1$ in Ω, $e = 0$ on $\partial\Omega$ for $r = p, q$. Let $(Z_1, Z_2) := (m(\lambda)e_p, (\lambda g_2(m(\lambda) \parallel e_p \parallel_\infty, m(\lambda) \parallel e_p \parallel_\infty))^{\frac{1}{q-1}} e_q)$. Then

$$\begin{aligned} -\Delta_p Z_1 &= m(\lambda)^{p-1} \\ &\geq \lambda g_1(m(\lambda)e_p, (\lambda g_2(m(\lambda) \parallel e_p \parallel_\infty, m(\lambda) \parallel e_p \parallel_\infty))^{\frac{1}{q-1}} e_q) \\ &\geq \lambda(g_1(Z_1, Z_2) - \frac{1}{Z_1^{\alpha_1}}). \end{aligned}$$

Also from the hypothesis $(H15)$, we have $\lim_{s\to\infty}\frac{g_2(s,s)}{s^{q-1}}=0$. Hence we can choose $m(\lambda)$ large enough such that

$$\frac{1}{\lambda\parallel e_q\parallel_\infty^{q-1}}\geq\frac{g_2(m(\lambda)\parallel e_p\parallel_\infty,m(\lambda)\parallel e_p\parallel_\infty)}{(m(\lambda)\parallel e_p\parallel_\infty)^{q-1}}.$$

Then

$$m(\lambda)\parallel e_p\parallel_\infty\geq(\lambda g_2(m(\lambda)\parallel e_p\parallel_\infty,m(\lambda)\parallel e_p\parallel_\infty))^{\frac{1}{q-1}}\parallel e_q\parallel_\infty$$

and

$$\begin{aligned}-\Delta_q Z_2&=\lambda g_2(m(\lambda)\parallel e_p\parallel_\infty,m(\lambda)\parallel e_p\parallel_\infty)\\&\geq\lambda(g_2(m(\lambda)\parallel e_p\parallel,m(\lambda)\parallel e_p\parallel)-\frac{1}{Z_2^{\alpha_2}})\\&\geq\lambda(g_2(Z_1,Z_2)-\frac{1}{Z_2^{\alpha_2}}).\end{aligned}$$

Thus (Z_1,Z_2) is a super-solution of (1.16). Further, $m(\lambda)$ can be chosen large enough so that $(Z_1,Z_2)\geq(\psi_1,\psi_2)$ in $\overline{\Omega}$. This is possible since $e_r>0$ in Ω and $\frac{\partial e_r}{\partial\nu}<0$ on $\partial\Omega$ for $r=p,q$, where ν is the outward normal vector on $\partial\Omega$ and $\lim_{s\to\infty}g_2(s,s)=\infty$. Hence (1.16) has a positive solution (u,v) for $\lambda\gg1$ and Theorem 9 is proven. $\square$

5.2 Proofs of Theorems 10-11

5.2.1 Proof of Theorem 10

To construct our positive sub-solution, let $\delta>0$, $m>0$ and $\mu>0$ be such that

$$\frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}|\nabla\phi_1|^{p-1}-\lambda_1(p)\phi_1^p\geq m$$

and

$$\frac{(1-\alpha_2)(q-1)}{q-1+\alpha_2}|\nabla\phi_2|^{q-1}-\lambda_1(q)\phi_2^q\geq m$$

in $\overline{\Omega_\delta}$, and $\phi_1 \geq \mu > 0$, $\phi_2 \geq \mu > 0$ in $\Omega \setminus \overline{\Omega_\delta}$. Here ϕ_1, ϕ_2 are as defined in the proof of Theorem 9 and $\overline{\Omega_\delta} = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $\mid \nabla\phi_1 \mid \neq 0, \mid \nabla\phi_2 \mid \neq 0$ on $\partial\Omega$.

In the assumption $(H16)$, we first consider the case when

$$g_1(s,t) \geq A_1 s^{\alpha_1} \text{ and } g_2(s,t) \geq A_2 t^{\alpha_2} \text{ for } s,t \gg 1. \tag{5.3}$$

Choose γ_1, γ_2 such that

$$\frac{1}{(p-1)+\alpha_1} < \gamma_1 < \frac{1}{p-1}, \quad \frac{1}{(q-1)+\alpha_2} < \gamma_2 < \frac{1}{q-1}.$$

Let $(\psi_1, \psi_2) = \big(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}\big)$. Then

$$-\Delta_p\psi_1 = \big(\lambda^{\gamma_1}\frac{p}{p-1+\alpha_1}\big)^{p-1}\phi_1^{\frac{-p\alpha_1}{p-1+\alpha_1}}\big[\lambda_1(p)\phi_1^p - \frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}|\nabla\phi_1|^p\big].$$

First, in $\overline{\Omega_\delta}$, since $1-\alpha_1\gamma_1 < (p-1)\gamma_1$, if $\lambda \gg 1$,

$$(\lambda^{\gamma_1}\frac{p}{p-1+\alpha_1})^{p-1}\big[\lambda_1(p)\phi_1^p - \frac{(1-\alpha_1)(p-1)}{p-1+\alpha_1}|\nabla\phi_1|^p\big] \leq \lambda\frac{g_1(0,0)}{(\lambda^{\gamma_1})^{\alpha_1}}.$$

Hence, in $\overline{\Omega_\delta}$, if $\lambda \gg 1$,

$$-\Delta_p\psi_1 \leq \lambda\frac{g_1(0,0)}{(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}} \leq \lambda\frac{g_1(\psi_1,\psi_2)}{\psi_1^{\alpha_1}}. \tag{5.4}$$

Similarly, we can show that in $\overline{\Omega_\delta}$, since $1-\alpha_2\gamma_2 < (q-1)\gamma_2$, for $\lambda \gg 1$,

$$-\Delta_q\psi_2 \leq \lambda\frac{g_2(\psi_1,\psi_2)}{\psi_2^{\alpha_2}}. \tag{5.5}$$

Next, in $\Omega \setminus \overline{\Omega_\delta}$, we know $\phi_i \geq \mu > 0$ for $i = 1, 2$. Then from (5.3), we know that in $\Omega \setminus \overline{\Omega_\delta}$, for $\lambda \gg 1$,

$$g_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}) \geq A_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}$$

and

$$g_2(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}) \geq A_2(\lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}})^{\alpha_2}.$$

Also, since $\gamma_1(p-1) < 1$ and $\gamma_2(q-1) < 1$, if $\lambda \gg 1$,

$$(\lambda^{\gamma_1}\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1(p)\phi_1^p \leq \lambda\frac{A_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}}{\lambda^{\gamma_1\alpha_1}}$$

and

$$(\lambda^{\gamma_2}\frac{q}{q-1+\alpha_2})^{q-1}\lambda_1(q)\phi_2^q \leq \lambda\frac{A_2(\lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}})^{\alpha_2}}{\lambda^{\gamma_2\alpha_2}}.$$

Then in $\Omega \setminus \overline{\Omega_\delta}$, for $\lambda \gg 1$,

$$\begin{aligned} -\Delta_p\psi_1 &\leq (\lambda^{\gamma_1}\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1(p)\phi_1^{\frac{-p\alpha_1}{p-1+\alpha_1}+p} \\ &\leq \lambda\frac{A_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}}{(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}} \leq \lambda\frac{g_1(\psi_1,\psi_2)}{\psi_1^{\alpha_1}}. \end{aligned} \tag{5.6}$$

Similarly, we can show that in $\Omega \setminus \overline{\Omega_\delta}$, for $\lambda \gg 1$,

$$-\Delta_q\psi_2 \leq \lambda\frac{g_2(\psi_1,\psi_2)}{\psi_2^{\alpha_2}}. \tag{5.7}$$

Combining (5.4), (5.5), (5.6) and (5.7), we know that $(\psi_1,\psi_2) = \left(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}\right)$ is a sub-solution of (1.17) for $\lambda \gg 1$ in this case.

Next, we consider $(H16)$ when

$$g_1(s,t) \geq A_1 s^{\alpha_1} \text{ and } g_2(s,t) \geq A_2 s^{\alpha_2} \text{ for } s,t \gg 1. \tag{5.8}$$

Choose γ_1 such that

$$\frac{1}{(p-1)+\alpha_1} < \gamma_1 < \frac{1}{p-1}$$

and then choose γ_2 such that

$$\frac{1}{(q-1)+\alpha_2} < \gamma_2 < \frac{1+\gamma_1\alpha_2}{(q-1)+\alpha_2}.$$

Let $(\psi_1, \psi_2) = \left(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}\right)$, then in $\overline{\Omega_\delta}$, by the same argument as in the above case, we can see that (ψ_1, ψ_2) satisfies (5.4) and (5.5) in $\overline{\Omega_\delta}$ for $\lambda \gg 1$.

In $\Omega \setminus \overline{\Omega_\delta}$, from (5.8), we know that for $\lambda \gg 1$

$$g_1(\psi_1, \psi_2) \geq A_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}$$

and

$$g_2(\psi_1, \psi_2) \geq A_2(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_2}.$$

Since $\gamma_1(p-1) < 1$, if $\lambda \gg 1$,

$$(\lambda^{\gamma_1}\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1(p)\phi_1^p \leq \lambda\frac{A_1(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_1}}{\lambda^{\gamma_1\alpha_1}}$$

and since $\gamma_2(q-1) < 1+\gamma_1\alpha_2-\gamma_2\alpha_2$, if $\lambda \gg 1$,

$$(\lambda^{\gamma_2}\frac{q}{q-1+\alpha_2})^{q-1}\lambda_1(q)\phi_2^q \leq \lambda\frac{A_2(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_2}}{\lambda^{\gamma_2\alpha_2}}.$$

Then by the same argument as in the above case, we can show that in $\Omega \setminus \overline{\Omega_\delta}$, for $\lambda \gg 1$,

$$-\Delta_p\psi_1 \leq \lambda\frac{g_1(\psi_1, \psi_2)}{\psi_1^{\alpha_1}}, \text{ and } -\Delta_q\psi_2 \leq \lambda\frac{g_2(\psi_1, \psi_2)}{\psi_2^{\alpha_2}}.$$

Thus $(\psi_1, \psi_2) = \left(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}\right)$ is a sub-solution of (1.17) for $\lambda \gg 1$ in this case.

Now we consider $(H16)$ when

$$g_1(s,t) \geq A_1t^{\alpha_1} \text{ and } g_2(s,t) \geq A_2t^{\alpha_2} \text{ for } s,t \gg 1.$$

Choose γ_2 such that

$$\frac{1}{(q-1)+\alpha_2} < \gamma_2 < \frac{1}{q-1}$$

and then choose γ_1 such that

$$\frac{1}{(p-1)+\alpha_1} < \gamma_1 < \frac{1+\gamma_2\alpha_1}{(p-1)+\alpha_1}.$$

Then we can also show that $(\psi_1, \psi_2) = \left(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}\right)$ is a sub-solution of (1.17) for $\lambda \gg 1$ by a similar argument as in the previous case.

We finally consider $(H16)$ when

$$g_1(s,t) \geq A_1 t^{\alpha_1} \text{ and } g_2(s,t) \geq A_2 s^{\alpha_2} \text{ for } s,t \gg 1. \tag{5.9}$$

Choose γ_1 such that

$$\frac{1}{(p-1)+\alpha_1} < \gamma_1 < \frac{\alpha_1+\alpha_2+(q-1)}{(q-1)(p-1)+\alpha_1(q-1)+\alpha_2(p-1)}$$

and then choose γ_2 such that

$$\max\{\frac{\gamma_1((p-1)+\alpha_1)-1}{\alpha_1},\ \frac{1}{(q-1)+\alpha_2}\} < \gamma_2 < \frac{1+\gamma_1\alpha_2}{(q-1)+\alpha_2}.$$

Let $(\psi_1, \psi_2) = \left(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}}, \lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}}\right)$, then in $\overline{\Omega_\delta}$, by the same argument as in the first case, (ψ_1, ψ_2) satisfies (5.4) and (5.5) in $\overline{\Omega_\delta}$, for $\lambda \gg 1$.

In $\Omega \setminus \overline{\Omega_\delta}$, from (5.9), we know that for $\lambda \gg 1$,

$$g_1(\psi_1, \psi_2) \geq A_1(\lambda^{\gamma_2}\phi_2^{\frac{q}{q-1+\alpha_2}})^{\alpha_1}$$

and

$$g_2(\psi_1, \psi_2) \geq A_2(\lambda^{\gamma_1}\phi_1^{\frac{p}{p-1+\alpha_1}})^{\alpha_2}.$$

Since $\gamma_1(p-1) < 1+\gamma_2\alpha_1 - \gamma_1\alpha_1$ and $\gamma_2(q-1) < 1+\gamma_1\alpha_2 - \gamma_2\alpha_2$, if $\lambda \gg 1$,

$$(\lambda^{\gamma_1}\frac{p}{p-1+\alpha_1})^{p-1}\lambda_1(p)\phi_2^p \le \lambda\frac{A_1(\lambda^{\gamma_2}\phi_2^{\overline{\frac{q}{q-1+\alpha_2}}})^{\alpha_1}}{\lambda^{\gamma_1\alpha_1}}$$

and

$$(\lambda^{\gamma_2}\frac{q}{q-1+\alpha_2})^{q-1}\lambda_1(q)\phi_2^q \le \lambda\frac{A_2(\lambda^{\gamma_1}\phi_1^{\overline{\frac{p}{p-1+\alpha_1}}})^{\alpha_2}}{\lambda^{\gamma_2\alpha_2}}.$$

Then by the same argument as in the other cases, we can show that (ψ_1, ψ_2) is a sub-solution of (1.17) for $\lambda \gg 1$.

Now for all the above cases, we construct a super-solution $(Z_1, Z_2) \ge (\psi_1, \psi_2)$. Let $(Z_1, Z_2) = (m(\lambda)e_p, m(\lambda)e_q)$, where e_p and e_q are as defined in the proof of Theorem 9. Since $e_p > 0$, $e_q > 0$ in Ω and $\frac{\partial e_p}{\partial \nu} < 0$, $\frac{\partial e_q}{\partial \nu} < 0$ on $\partial\Omega$, there exist constants d_1, $d_2 > 0$ such that $e_p \le d_1 e_q$ and $e_q \le d_2 e_p$. Since $\alpha_1 < r_i < \alpha_1+(p-1)$ and $\alpha_2 < \theta_i < \alpha_2+(q-1)$, we have for $m(\lambda) \gg 1$,

$$(m(\lambda))^{(p-1)+\alpha_1} \ge \lambda(B_1 m(\lambda)^{r_1} e_p^{r_1-\alpha_1} + B_2 m(\lambda)^{r_2} d_2^{r_2} e_p^{r_2-\alpha_1})$$

and

$$(m(\lambda))^{(q-1)+\alpha_2} \ge \lambda(C_1 m(\lambda)^{\theta_1} d_1^{\theta_1} e_q^{\theta_1-\alpha_2} + C_2 m(\lambda)^{\theta_2} e_q^{\theta_2-\alpha_2}).$$

Hence from $(H17)$, for $m(\lambda) \gg 1$,

$$\begin{aligned} -\Delta_p Z_1 = (m(\lambda))^{p-1} &\ge \lambda\frac{B_1(m(\lambda)e_p)^{r_1} + B_2(m(\lambda)d_2 e_p)^{r_2}}{(m(\lambda)e_p)^{\alpha_1}} \\ &\ge \lambda\frac{B_1(m(\lambda)e_p)^{r_1} + B_2(m(\lambda)e_q)^{r_2}}{(m(\lambda)e_p)^{\alpha_1}} \ge \lambda\frac{g_1(Z_1, Z_2)}{Z_1^{\alpha_1}}. \end{aligned}$$

Similarly, we can show that for $m(\lambda) \gg 1$,

$$-\Delta_q Z_2 \ge \lambda\frac{g_2(Z_1, Z_2)}{Z_2^{\alpha_2}}.$$

Then (Z_1, Z_2) is a positive super-solution of (1.17). By choosing $m(\lambda)$ large enough, we will have $(Z_1, Z_2) \geq (\psi_1, \psi_2)$ in $\overline{\Omega}$. Therefore, problem (1.17) has a positive solution $(u, v) \in [(\psi_1, \psi_2), (Z_1, Z_2)]$. □

5.2.2 Proof of Theorem 11

Here, the construction of a sub-solution (ψ_1, ψ_2) is the same as in the proof of Theorem 10. For the construction of the super-solution, from [GST], we know that there exist functions $w_1 \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ and $w_2 \in W_0^{1,q}(\Omega) \cap C(\overline{\Omega})$ such that

$$\begin{cases} -\Delta_p w_1 = \frac{1}{w_1^{\alpha_1}} & \text{in } \Omega \\ \quad w_1 = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\Delta_q w_2 = \frac{1}{w_2^{\alpha_2}} & \text{in } \Omega \\ \quad w_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

and satisfying $w_1 \geq \varepsilon\phi_1$ and $w_2 \geq \varepsilon\phi_2$ for some $\varepsilon > 0$, where ϕ_1 and ϕ_2 are as defined in the proof of Theorem 9.

Let $(Z_1, Z_2) = (m(\lambda)w_1, (g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty))^{\frac{1}{q-1}} w_2)$. From $(H18)$, we know that for $m(\lambda) \gg 1$,

$$\frac{1}{\lambda} \geq \frac{g_1(m(\lambda)\|w_1\|_\infty, \|w_2\|_\infty (g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty))^{\frac{1}{q-1}})}{m(\lambda)^{\alpha_1+p-1}}.$$

Then for such $m(\lambda) \gg 1$,

$$\begin{aligned} -\Delta_p Z_1 &= \frac{m(\lambda)^{p-1}}{w_1^{\alpha_1}} \\ &\geq \lambda \frac{g_1(m(\lambda)\|w_1\|_\infty, (g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty)^{\frac{1}{q-1}} w_2)}{(m(\lambda)w_1)^{\alpha_1}} \\ &= \lambda \frac{g_1(Z_1, Z_2)}{Z_1^{\alpha_1}}. \end{aligned}$$

Also since $\lim_{s\to\infty} \frac{g_2(s,s)}{s^{q-1}} = 0$, there exists $m(\lambda) \gg 1$ such that

$$\frac{1}{\|w_2\|_\infty^{q-1}} > \frac{g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty)}{(m(\lambda)\|w_1\|_\infty)^{q-1}},$$

which implies

$$m(\lambda)\|w_1\|_\infty > (g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty))^{\frac{1}{q-1}}\|w_2\|_\infty.$$

Now from $(H16)$, we know that for $m(\lambda) \gg 1$,

$$g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty) > \lambda^{\frac{q-1}{\alpha_2}}.$$

Hence, for $m(\lambda) \gg 1$,

$$\begin{aligned}
-\Delta_q Z_2 &= \frac{g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty)}{w_2^{\alpha_2}} \\
&\geq \lambda \frac{g_2(m(\lambda)\|w_1\|_\infty, (g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty))^{\frac{1}{q-1}}\|w_2\|_\infty)}{(g_2(m(\lambda)\|w_1\|_\infty, m(\lambda)\|w_1\|_\infty))^{\frac{\alpha_2}{q-1}} w_2^{\alpha_2}} \\
&\geq \lambda \frac{g_2(Z_1, Z_2)}{Z_2^{\alpha_2}}.
\end{aligned}$$

Further, since $g_1(s,s) \to \infty$ and $g_2(s,s) \to \infty$ as $s \to \infty$, $m(\lambda)$ can be chosen large enough so that $(Z_1, Z_2) \geq (\psi_1, \psi_2)$ in $\overline{\Omega}$. Therefore, problem (1.17) has a positive solution $(u, v) \in [(\psi_1, \psi_2), (Z_1, Z_2)]$. $\square$

CHAPTER 6

INFINITE SEMIPOSITONE LAPLACIAN AND p-LAPLACIAN $n \times n$ SYSTEMS

6.1 Proof of Theorem 12

Let $\phi > 0$ be as defined in the proof of Lemma 2. From $(H19)$, there exists $\gamma \in \big(\frac{1}{1+\underline{\alpha}}, \frac{1}{1+(\overline{\alpha}-\sigma)}\big)$, let $\psi_i = \lambda^\gamma \phi^{\frac{2}{1+\alpha_i}}$, $i = 1, \cdots, n$. Then

$$-\Delta\psi_i = \big(\lambda^\gamma \frac{2}{1+\alpha_i}\big)\phi^{\frac{-2\alpha_i}{1+\alpha_i}}\big[\lambda_1\phi^2 - \big(\frac{1-\alpha_i}{1+\alpha_i}\big)|\nabla\phi|^2\big].$$

Let $\delta > 0$, $m > 0$ and $\mu > 0$ be such that

$$\big(\frac{1-\alpha_i}{1+\alpha_i}\big)|\nabla\phi|^2 - \lambda_1\phi^2 \geq m, \quad \text{in } \overline{\Omega}_\delta, \quad \forall i = 1, \cdots, n,$$

and $\phi \geq \mu > 0$ in $\Omega \setminus \overline{\Omega}_\delta$, where $\overline{\Omega}_\delta = \{x \in \Omega \mid d(x, \partial\Omega) \leq \delta\}$. This is possible since $|\nabla\phi| \neq 0$; $\partial\Omega$. Hence even though $g_i(0) < 0$, in $\overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$(\lambda^\gamma \frac{2}{1+\alpha_i}\big)\big[\lambda_1\phi^2 - \big(\frac{1-\alpha_i}{1+\alpha_i}\big)|\nabla\phi|^2\big] \leq \lambda\frac{g_i(0)}{(\lambda^\gamma)^{\alpha_i}},$$

since $1 - \gamma - \alpha_i\gamma < 0$. Therefore in $\overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$-\Delta\psi_i \leq \lambda\frac{g_i(0)}{(\lambda^\gamma\phi_i^{\frac{2}{1+\alpha_i}})^{\alpha_i}} \leq \lambda\frac{g_i(\psi_{i+1})}{\psi_i^{\alpha_i}}. \tag{6.1}$$

Next, in $\Omega \setminus \overline{\Omega}_\delta$, since $\phi_i \geq \mu > 0$, from $(H19)$, we know that for $\lambda \gg 1$,

$$g_i(\lambda^\gamma\phi^{\frac{2}{1+\alpha_{i+1}}}) \geq A(\lambda^\gamma\phi^{\frac{2}{1+\alpha_{i+1}}})^\sigma.$$

Also, since $0 < \mu \leq \phi < 1$ and $1 + (\sigma - \alpha_i)\gamma - \gamma > 0$, for $\lambda \gg 1$,

$$\big(\lambda^{\gamma} \frac{2}{1+\alpha_i}\big)\lambda_1 \phi^2 \leq \lambda \frac{A(\lambda^{\gamma} \phi^{\frac{2}{1+\alpha_{i+1}}})^{\sigma}}{\lambda^{\gamma\alpha_i}}.$$

Then in $\Omega \setminus \overline{\Omega}_\delta$, for $\lambda \gg 1$,

$$-\Delta\psi_i \leq \big(\lambda^{\gamma} \frac{2}{1+\alpha_i}\big)\lambda_1 \phi^{\frac{-2\alpha_i}{1+\alpha_i}+2} \leq \lambda \frac{g_i(\lambda^{\gamma} \phi^{\frac{2}{1+\alpha_{i+1}}})}{(\lambda^{\gamma} \phi^{\frac{2}{1+\alpha_i}})^{\alpha_i}} = \lambda \frac{g_i(\psi_{i+1})}{\psi_i^{\alpha_i}}. \tag{6.2}$$

Combining (6.1) and (6.2), we see that for $\lambda \gg 1$,

$$-\Delta\psi_i \leq \lambda \frac{g_i(\psi_{i+1})}{\psi_i^{\alpha_i}} \quad \text{in } \Omega.$$

Thus $(\psi_1, \cdots, \psi_n)$ is a positive sub-solution of (1.18).

Now, we construct a super-solution $(Z_1, \cdots, Z_n) \geq (\psi_1, \cdots, \psi_n)$. From [GST], we know that there exist $w_i \in C^1(\Omega) \cap C(\overline{\Omega})$, $i = 1, \cdots, n$. such that

$$\begin{cases} -\Delta w_i = \frac{1}{w_i^{\alpha_i}} & \text{in } \Omega \\ w_i = 0 & \text{on } \partial\Omega, \end{cases}$$

and satisfying $w_i \geq \varepsilon\phi$ for some $\varepsilon > 0$.

Let $\omega = \max_{i=1,2,\cdots,n} \|w_i\|$. Then, from $(H21)$, we can choose $C_\lambda \gg 1$ such that

$$\frac{g_1^{[\omega]} \circ g_2^{[\omega]} \circ \cdots \circ g_{n-1}^{[\omega]} \circ g_n(\|w_1\| C_\lambda)}{C_\lambda^{1+\alpha_1}} \leq \frac{1}{\lambda}. \tag{6.3}$$

Let

$$\begin{cases} \rho_1 = C_\lambda \\ \rho_n = g_n(\|w_1\|\rho_1) \\ \rho_j = g_j^{[\|w_{j+1}\|]}(\rho_{j+1}), \quad \text{for } j = n-1, n-2, \cdots, 3, 2, \end{cases}$$

and $(Z_1, Z_2, \cdots, Z_n) = (\rho_1 w_1, \rho_2 w_2, \cdots, \rho_n w_n)$. Then by (6.3),

$$
\begin{aligned}
-\Delta Z_1 = \frac{C_\lambda}{w_1^{\alpha_1}} &\geq \lambda \frac{g_1^{[\omega]} \circ g_2^{[\omega]} \circ \cdots \circ g_{n-1}^{[\omega]} \circ g_n(\|w_1\| C_\lambda)}{(C_\lambda)^{\alpha_1} w_1^{\alpha_1}} \\
&\geq \lambda \frac{g_1^{[\|w_2\|]} \circ g_2^{[\|w_3\|]} \circ \cdots \circ g_{n-1}^{[\|w_n\|]} \circ g_n(\|w_1\| C_\lambda)}{(C_\lambda w_1)^{\alpha_1}} \\
&\vdots \\
&= \lambda \frac{g_1^{[\|w_2\|]}(\rho_2)}{(C_\lambda w_1)^{\alpha_1}} = \lambda \frac{g_1(\rho_2 \|w_2\|)}{(C_\lambda w_1)^{\alpha_1}} \geq \lambda \frac{g_1(Z_2)}{(Z_1)^{\alpha_1}}.
\end{aligned}
$$

From $(H19)$, we know that $g_i(s) \to \infty$ as $s \to \infty$. Thus for $C_\lambda \gg 1$,

$$
\frac{\lambda}{\rho_i^{\alpha_i}} \leq 1 \qquad \forall i = 1, \cdots, n.
$$

Hence, for $j = 2, 3, \cdots, n-1$,

$$
\begin{aligned}
-\Delta Z_j = \frac{\rho_j}{w_j^{\alpha_j}} \geq \lambda \frac{\rho_j}{(\rho_j w_j)^{\alpha_j}} &= \lambda \frac{g_j^{[\|w_{j+1}\|]}(\rho_{j+1})}{(\rho_j w_j)^{\alpha_j}} \\
\geq \lambda \frac{g_j(\rho_{j+1} w_{j+1})}{(\rho_j w_j)^{\alpha_j}} &= \lambda \frac{g_j(Z_{j+1})}{(Z_j)^{\alpha_j}}
\end{aligned}
$$

and

$$
\begin{aligned}
-\Delta Z_n = \frac{\rho_n}{w_n^{\alpha_n}} \geq \lambda \frac{\rho_n}{(\rho_n w_n)^{\alpha_n}} &= \lambda \frac{g_n(\|w_1\| \rho_1)}{(\rho_n w_n)^{\alpha_n}} \\
\geq \lambda \frac{g_n(\rho_1 w_1)}{(\rho_n w_n)^{\alpha_n}} &= \lambda \frac{g_n(Z_1)}{(Z_n)^{\alpha_n}}.
\end{aligned}
$$

Thus $(Z_1, \cdots, Z_n)$ is a super-solution of (1.18). Further, C_λ can be chosen large enough so that$(Z_1, \cdots, Z_n) \geq (\psi_1, \cdots, \psi_n)$ in $\overline{\Omega}$. Therefore, problem (1.18) has a positive solution $(u_1, \cdots, u_n) \in [(\psi_1, \cdots, \psi_n), (Z_1, \cdots, Z_n)]$. □

6.2 Proof of Theorem 13

Let $\psi = \lambda^\gamma \phi^{\frac{2}{1+\alpha}}$, $\gamma \in \left(\frac{1}{1+\alpha}, \frac{1}{1+(\alpha-\sigma)}\right)$, and ϕ as before. Then by arguments similar to that in the proof of Theorem 12, we can show that $(\psi, \cdots, \psi)$ is a sub-solution. Now, we construct a super-solution $(Z_1, \cdots, Z_n) \geq (\psi, \cdots, \psi)$. By $(H22)$, there exists $C_\lambda \gg 1$ such that

$$\frac{\tilde{g}_1^{[\lambda\|e\|]} \circ \tilde{g}_2^{[\lambda\|e\|]} \circ \cdots \circ \tilde{g}_{n-1}^{[\lambda\|e\|]} \circ \tilde{g}_n(\|e\|C_\lambda)}{C_\lambda} \leq \frac{1}{\lambda}.$$

Let

$$\begin{cases} \rho_1 = C_\lambda \\ \rho_n = \lambda \tilde{g}_n(\|e\|\rho_1) \\ \rho_j = \lambda \tilde{g}_j^{[\|e\|]}(\rho_{j+1}), \quad \text{for } j = n-1, n-2, \cdots, 3, 2, \end{cases}$$

where e is as defined in the proof of Lemma 1.

Let $(Z_1, \cdots, Z_n) := (\rho_1 e, \rho_2 e, \cdots, \rho_n e)$. Then

$$\begin{aligned} -\Delta Z_1 = C_\lambda &\geq \lambda \tilde{g}_1^{[\lambda\|e\|]} \circ \tilde{g}_2^{[\lambda\|e\|]} \circ \cdots \circ \tilde{g}_{n-1}^{[\lambda\|e\|]} \circ \tilde{g}_n(\|e\|C_\lambda) \\ &= \lambda \tilde{g}_1^{[\lambda\|e\|]} \circ \tilde{g}_2^{[\lambda\|e\|]} \circ \cdots \circ \tilde{g}_{n-1}^{[\|e\|]}(\rho_n) \\ &\vdots \\ &= \lambda \tilde{g}_1^{[\|e\|]}(\rho_2) = \lambda \tilde{g}_1(\rho_2\|e\|) \geq \lambda \tilde{g}_1(Z_2). \end{aligned}$$

For $j = 2, \cdots, n-1$,

$$-\Delta Z_j = \rho_j = \lambda \tilde{g}_j^{[\|e\|]}(\rho_{j+1}) \geq \lambda \tilde{g}_j(\rho_{j+1} e) = \lambda \tilde{g}_j(Z_{j+1})$$

and

$$-\Delta Z_n = \rho_n = \lambda \tilde{g}_n(\|e\|\rho_1) \geq \lambda \tilde{g}_n(\rho_1 e) = \lambda \tilde{g}_n(Z_1).$$

Thus $(Z_1, \cdots, Z_n)$ is a super-solution of (1.19). Further, from $(H22)$, we know that $\tilde{g}_j(s) \to \infty$ as $s \to \infty$ for $j = 2, \cdots, n$. Thus C_λ can be chosen large enough so that$(Z_1, \cdots, Z_n) \geq (\psi, \cdots, \psi)$ in $\overline{\Omega}$. Therefore, problem (1.19) has a positive solution $(u_1, \cdots, u_n) \in [(\psi, \cdots, \psi), (Z_1, \cdots, Z_n)]$. □

6.3 Proof of Theorem 14

First, by an argument similar to the proof of Theorem 12, we can show that if $\psi_i := \lambda^\gamma \phi_1^{\frac{p}{p-1+\alpha_i}}$, for $\gamma \in (\frac{1}{p-1+\underline{\alpha}}, \frac{1}{p-1+(\overline{\alpha}-\sigma)})$, then $(\psi_1, \psi_2, \cdots, \psi_n)$ is a sub-solution of (1.20) for $\lambda \gg 1$. Here ϕ_1 is as defined in the proof of Theorem 4.

Also, by [GST], the problem

$$\begin{cases} -\Delta_p w_i = \frac{1}{w_i^{\alpha_i}} & \text{in } \Omega \\ w_i = 0 & \text{on } \partial\Omega, \end{cases}$$

has a solution $w_i \in W^{1,p}(\Omega) \times C(\overline{\Omega})$ such that $w_i \geq \varepsilon\phi_1$, for some $\varepsilon > 0$. Let $Z_i := \rho_i w_i$, where

$$\begin{cases} \rho_1 = C_\lambda \\ \rho_n = g_n(\|w_1\|\rho_1) \\ \rho_j = g_j^{[\|w_{j+1}\|]}(\rho_{j+1}), \quad \text{for } j = n-1, n-2, \cdots, 3, 2. \end{cases}$$

Then by the argument similar to that in the proof of Theorem 12. We can show that for $C_\lambda \gg 1$, $(Z_1, Z_2, \cdots, Z_n)$ is a super-solution of (1.20) and $(Z_1, Z_2, \cdots, Z_n) \geq (\psi_1, \psi_2, \cdots, \psi_n)$. Hence Theorem 14 holds. □

6.4 Proof of Theorem 15

To establish theorem 15, let $\psi := \lambda^{\gamma}\phi_1^{\frac{p}{p-1+\alpha}}$, for $\gamma \in (\frac{1}{p-1+\alpha}, \frac{1}{p-1+(\alpha-\sigma)})$, and $Z_i := \rho_i e_p$, where both ϕ_1 and e_p are as defined in the proof of Theorem 4 and

$$\begin{cases} \rho_1 = C_\lambda \\ \rho_n = \lambda^{\frac{1}{p-1}} g_n(\|w_1\|\rho_1) \\ \rho_j = \lambda^{\frac{1}{p-1}} g_j^{[\|w_{j+1}\|]}(\rho_{j+1}), \quad \text{for } j = n-1, n-2, \cdots, 3, 2. \end{cases}$$

Then by the argument similar to that in the proof of Theorem 13, $(\psi, \cdots, \psi)$ is a sub-solution of (1.21) for $\lambda \gg 1$ and for $C_\lambda \gg 1$, $(Z_1, Z_2, \cdots, Z_n)$ is a super-solution of (1.21) with $(Z_1, Z_2, \cdots, Z_n) \geq (\psi, \cdots, \psi)$. Hence Theorem 15 holds. $\square$

CHAPTER 7

INFINITE SEMIPOSITONE PROBLEMS WITH FALLING ZEROS

In this chapter proofs of Theorems 16-17 will be presented. In Section 7.1, proof of Theorem 16 will be presented and the proof of Theorem 17 will be presented in Section 7.2.

7.1 Proof of Theorem 16

From an anti-maximum principle (see Lemma 8), there exists $\sigma(\Omega) > 0$ such that the solution z_λ of

$$\begin{cases} -\Delta z - \lambda z = -1 & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases}$$

for $\lambda \in (\lambda_1, \lambda_1 + \sigma)$ is positive in Ω and is such that $\frac{\partial z}{\partial \nu} < 0$ on $\partial\Omega$, where ν is outward normal vector at $\partial\Omega$. Fix $\lambda^* \in (\lambda_1, \min(\lambda_1 + \sigma, \frac{1+\alpha}{2} a))$ and let

$$K := \min\Big\{\Big(\frac{2}{(1+\alpha)A\|z_{\lambda^*}\|_\infty^{\frac{2p-1+\alpha}{1+\alpha}}}\Big)^{\frac{1}{p-1}}, \Big(\frac{a - \frac{2}{1+\alpha}\lambda^*}{2A\|z_{\lambda^*}\|_\infty^{\frac{2p-2}{1+\alpha}}}\Big)^{\frac{1}{p-1}}\Big\}.$$

Define $\psi := K z_{\lambda^*}^{\frac{2}{1+\alpha}}$. Then

$$\nabla\psi = K\Big(\frac{2}{1+\alpha}\Big) z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \nabla z_{\lambda^*},$$

$$
\begin{aligned}
-\Delta\psi &= -K\big(\frac{2}{1+\alpha}\big)\big(\frac{1-\alpha}{1+\alpha}\big)z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}}|\nabla z_{\lambda^*}|^2 - K\big(\frac{2}{1+\alpha}\big)z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}}\Delta z_{\lambda^*} \\
&= -K\big(\frac{2}{1+\alpha}\big)\big(\frac{1-\alpha}{1+\alpha}\big)z_{\lambda^*}^{\frac{-2\alpha}{1+\alpha}}|\nabla z_{\lambda^*}|^2 - K\big(\frac{2}{1+\alpha}\big)z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}}(1-\lambda^* z_{\lambda^*}) \\
&= K\big(\frac{2}{1+\alpha}\big)\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - K\big(\frac{2}{1+\alpha}\big)z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} - K\big(\frac{2}{1+\alpha}\big)\big(\frac{1-\alpha}{1+\alpha}\big)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \qquad (7.1)
\end{aligned}
$$

and

$$
a\psi - g(\psi) - \frac{c}{\psi^\alpha} = aKz_{\lambda^*}^{\frac{2}{1+\alpha}} - g(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}) - \frac{c}{K^\alpha z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}. \qquad (7.2)
$$

Let $\delta > 0, \mu > 0, m > 0$ be such that

$$
|\nabla z_{\lambda^*}| \geq m \quad \text{in } \overline{\Omega_\delta}, \quad \text{and} \quad z_{\lambda^*} \geq \mu \quad \text{in} \quad \Omega \setminus \overline{\Omega_\delta},
$$

where $\overline{\Omega_\delta} := \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$. Let

$$
c^* := K^{1+\alpha}\min\big\{\frac{2}{1+\alpha}\frac{1-\alpha}{1+\alpha}m^2,\ \frac{1}{2}\mu^2\big(a - \frac{2}{1+\alpha}\lambda^*\big)\big\}.
$$

In $\overline{\Omega_\delta}$, we compare (7.1) and (7.2) term by term to see that for $c < c^*$,

$$
-\Delta\psi \leq a\psi - g(\psi) - \frac{c}{\psi^\alpha} \qquad \text{in } \overline{\Omega_\delta}.
$$

Since $\frac{2}{1+\alpha}\lambda^* \leq a$, we have

$$
K\big(\frac{2}{1+\alpha}\big)\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} \leq aKz_{\lambda^*}^{\frac{2}{1+\alpha}}, \qquad (7.3)
$$

and from the choice of K, we know that

$$
\frac{2}{1+\alpha} \geq AK^{p-1}\|z_{\lambda^*}\|^{\frac{2p-1+\alpha}{1+\alpha}}. \qquad (7.4)
$$

By (7.4) and $(H25)$, we have

$$
-K\frac{2}{1+\alpha}z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} \leq -AK^p z_{\lambda^*}^{\frac{2p}{1+\alpha}} \leq -g(Kz_{\lambda^*}^{\frac{2}{1+\alpha}}). \qquad (7.5)
$$

Next, we know that $K^{1+\alpha}\big(\frac{2}{1+\alpha}\big)\big(\frac{1-\alpha}{1+\alpha}\big)m^2 \geq c$ for $c < c^*$ and $|\nabla z_{\lambda^*}| \geq m$ in $\overline{\Omega_\delta}$. Thus

$$K\Big(\frac{2}{1+\alpha}\Big)\Big(\frac{1-\alpha}{1+\alpha}\Big)|\nabla z_{\lambda^*}|^2 \geq \frac{c}{K^\alpha}$$

and

$$-K\Big(\frac{2}{1+\alpha}\Big)\Big(\frac{1-\alpha}{1+\alpha}\Big)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \leq -\frac{c}{K^\alpha z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}. \tag{7.6}$$

Hence for $c < c^*$, combining (7.3), (7.5) and (7.6), we have $-\Delta\psi \leq a\psi - g(\psi) - \frac{c}{\psi^\alpha}$ in $\overline{\Omega_\delta}$.

Now, in $\Omega \setminus \overline{\Omega_\delta}$, from the fact that $c \leq \frac{1}{2}K^{1+\alpha}\mu^2(a - \frac{2}{1+\alpha}\lambda^*)$ for $c < c^*$ and $z_{\lambda^*} \geq \mu$, we have

$$\frac{c}{K^\alpha} \leq \frac{1}{2}K z_{\lambda^*}^2(a - \frac{2}{1+\alpha}\lambda^*). \tag{7.7}$$

Also from the choice of K, we have

$$AK^{p-1}z_{\lambda^*}^{\frac{2p-2}{1+\alpha}} \leq \frac{1}{2}(a - \frac{2}{1+\alpha}\lambda^*). \tag{7.8}$$

By using (7.7) and (7.8),

$$
\begin{aligned}
-\Delta\psi &= K\big(\frac{2}{1+\alpha}\big)\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} - K\big(\frac{2}{1+\alpha}\big) z_{\lambda^*}^{\frac{1-\alpha}{1+\alpha}} - K\big(\frac{2}{1+\alpha}\big)\big(\frac{1-\alpha}{1+\alpha}\big)\frac{|\nabla z_{\lambda^*}|^2}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \\
&\leq K\frac{2}{1+\alpha}\lambda^* z_{\lambda^*}^{\frac{2}{1+\alpha}} \\
&= \frac{1}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\big[\frac{1}{2}\frac{2}{1+\alpha}\lambda^* K z_{\lambda^*}^2 + \frac{1}{2}\frac{2}{1+\alpha}\lambda^* K z_{\lambda^*}^2\big] \\
&\leq \frac{1}{z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}}\big[\big(\frac{1}{2}K z_{\lambda^*}^2 a - \frac{c}{K^\alpha}\big) + K z_{\lambda^*}^2\big(\frac{1}{2}a - AK^{p-1} z_{\lambda^*}^{\frac{2p-2}{1+\alpha}}\big)\big] \\
&= aK z_{\lambda^*}^{\frac{2}{1+\alpha}} - AK^p z_{\lambda^*}^{\frac{2p}{1+\alpha}} - \frac{c}{K^\alpha} z_{\lambda^*}^{-\frac{2\alpha}{1+\alpha}} \\
&\leq aK z_{\lambda^*}^{\frac{2}{1+\alpha}} - g(K z_{\lambda^*}^{\frac{2}{1+\alpha}}) - \frac{c}{K^\alpha z_{\lambda^*}^{\frac{2\alpha}{1+\alpha}}} \\
&= a\psi - g(\psi) - \frac{c}{\psi^\alpha}.
\end{aligned}
$$

Hence ψ is a sub-solution of (1.22).

Next, we construct a super-solution. From $(H26)$, we know that there is a large $\overline{M} > 0$ such that $au - g(u) - \frac{c}{u^\alpha} \leq \overline{M}$ for all $u > 0$ and $\overline{M}e \geq \psi$ in Ω, where e is as defined in the proof of Lemma 1. Let $Z := \overline{M}e$. then

$$-\Delta Z = \overline{M} \geq aZ - g(Z) - \frac{c}{Z^\alpha} \qquad \text{in } \Omega.$$

Thus Z is a positive super-solution of (1.22) for $c < c^*$ satisfying $Z \geq \psi$ in $\overline{\Omega}$ and Theorem 16 is proven. □

7.2 Proof of Theorem 17

For fixed $\lambda^* \in (\lambda_1,\ \min(\lambda_1+\sigma,\ \frac{1+\alpha}{2}\overline{a}))$ where $\overline{a}=\min\{a_1,a_2\}$ and σ is as before from the anti-maximum principle in the previous section. Let

$$K := \min\Big\{\Big(\frac{2}{(1+\alpha)A\|z_{\lambda^*}\|_\infty^{\frac{2p-1+\alpha}{1+\alpha}}}\Big)^{\frac{1}{p-1}},\ \Big(\frac{\overline{a}-\frac{2}{1+\alpha}\lambda^*}{2A\|z_{\lambda^*}\|_\infty^{\frac{2p-2}{1+\alpha}}}\Big)^{\frac{1}{p-1}}\Big\}.$$

Define $(\psi_1,\psi_2) := (Kz_{\lambda^*}^{\frac{2}{1+\alpha}}, Kz_{\lambda^*}^{\frac{2}{1+\alpha}})$. Then by the same argument as in the proof of Theorem 16, we can show that (ψ_1,ψ_2) is a sub-solution of (1.23) for $\max\{c_1,c_2\}<c^*$ where

$$c^* := K^{1+\alpha}\min\Big\{\frac{2}{1+\alpha}\frac{1-\alpha}{1+\alpha}m^2,\ \ \frac{1}{2}\mu^2\big(\overline{a}-\frac{2}{1+\alpha}\lambda^*\big)\Big\}.$$

Also from $(H28)$ we know that there exists $\overline{M}>0$ large enough such that $(Z_1,Z_2) := (\overline{M}e,\overline{M}e)$ is a super-solution of (1.23) satisfying $(\psi_1,\psi_2)\le(Z_1,Z_2)$ in $\overline{\Omega}$. Thus (1.23) has a positive solution for $\max\{c_1,c_2\}<c^*$ and Theorem 17 is proven. $\square$

CHAPTER 8

CONCLUSIONS AND FUTURE DIRECTIONS

8.1 Conclusions

This thesis initiates an extensive study on the existence of positive solutions to classes of singular systems. In particular, it is focused on the challenging problem of constructing positive sub-solutions to infinite semipositone problems. Assuming combined sublinear conditions at ∞ on the nonlinearities, large enough super-solutions are constructed. Using these sub-super solutions, we obtain positive solutions for these systems.

8.2 Future directions

We plan to

(A) Extend this study of existence results to systems that satisfy other types of conditions at ∞ (e.g. combined superlinear condition at ∞),

(B) Investigate

(a) uniqueness results

(b) nonexistence results and

(c) multiplicity results

and

(C) Extend these studies when Ω is an unbounded domain (e.g. exterior domain, entire domain).

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