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Multiple positive solutions for classes of elliptic systems with combined nonlinear effects

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MULTIPLE POSITIVE SOLUTIONS FOR CLASSES OF ELLIPTIC SYSTEMS WITH COMBINED NONLINEAR EFFECTS

By

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We study positive solutions to nonlinear elliptic systems of the form:

$$-\Delta u = \lambda f(v) \text{ in } \Omega$$

$$-\Delta v = \lambda g(u) \text{ in } \Omega$$

$$u = 0 = v \text{ on } \partial\Omega$$

where Δu is the Laplacian of u, λ is a positive parameter and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. In particular, we will analyze the combined effects of the nonlinearities on the existence and multiplicity of positive solutions. We also study systems with multiparameters and stronger coupling. We extend our results to p-q-Laplacian systems and to $n \times n$ systems. We mainly use sub- and super-solutions to prove our results.

Key words: positone, ellitpic systems, multiple solutions, sub-super solutions, p-Laplacian

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LIST OF SYMBOLS, ABBREVIATIONS, AND NOMENCLATURE

Ω bounded domain of $\mathbb{R}^N$.

$\partial\Omega$ boundary of Ω.

Ω_δ $:= \{x \in \Omega \mid d(x, \partial\Omega) < \delta\}$.

$C((0,\infty))$ is the space of continuous real valued functions on $(0,\infty)$.

$C(\Omega)$ is the space of continuous real valued functions on Ω.

$C^m(\Omega)$ is the space of continuously m-times differentiable functions on Ω.

$C^\infty(\Omega)$ $= \bigcap_{k=0}^{\infty} C^k(\Omega)$.

$C_0^\infty(\Omega)$ is the space of functions in $C^\infty(\Omega)$ with compact support in Ω.

$W^{m,p}(\Omega)$ is the Sobolev space of order m for $1 \leq p \leq \infty$.

$W_0^{m,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Δu Laplacian of u, i.e., $\Delta u = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_Nx_N}$.

$\Delta_p u$ p-Laplacian of u, i.e., $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$.

CHAPTER 1

INTRODUCTION

The recent development of reaction diffusion systems in biology, ecology, chemistry and geology, and the traditional importance of these systems in physics, heat-mass transfer and engineering lead to extensive study of various aspects of nonlinear elliptic partial differential equations. It has also been discussed that reaction diffusion processes are an essential basis for processes connected to morphogenesis in biology (see [Hr]). Recently, reaction diffusion systems have been extensively studied in medical industry especially in medical imaging. A large amount of mathematically rich and physically interesting study has been available in the literature since the middle of the 1960s. This thesis is concerned with the study of multiple positive solutions for nonlinear elliptic systems. In particular, we initiate a careful analysis on the multiple positive solutions based on combined nonlinear effects. There has been considerable interest in the class of semilinear elliptic boundary value problems known as positone problems in the last three to four decades. The study of positone problems was initiated by Cohen and Keller in [CK]. The terminology *positone problems* was coined by Cohen and Keller when the nonlinearity involved in the problem was positive and monotone. Cohen and Keller were interested in problems arising from the theory of nonlinear heat generation; positone problems, however, also arise in many other situations. For an excellent review on positone problems see [Li], also see [Am],

[BIS], [CR], [FLN], [GNN], [JL], [KW], [La], [Ra] and [Sa] for important results on positone problems. In this thesis we study multiplicity results for various classes of positone systems.

There has been significant interest in the study of positive solutions to the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Δ is the Laplacian operator, λ is a positive parameter and Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. If $f : [0,\infty) \to (0,\infty)$ and $\lim_{x\to\infty} \frac{f(x)}{x} = 0$ then it is easy to prove the existence of positive solutions of (1.1) for every $\lambda > 0$. Further if we assume that $\frac{x}{f(x)}$ is nondecreasing, it is easy to prove the uniqueness of positive solutions of (1.1) for every $\lambda > 0$.

In [BIS], authors studied the multiplicity results for (1.1) when $\frac{x}{f(x)}$ is decreasing for certain range of x. Such problems arise in various applications.

Examples:

(a) Theory of combustion (see Parks [Pk], Sattinger [Sa], Parter [Pt] and Tam [Ta]) where $f(u) = e^{\frac{\alpha u}{\alpha+u}}$

(b) Chemical reactor theory (see Aris [Ar]) where $f(u) = (\sigma - u)e^{-\frac{k}{1+u}}$

(c) Immobilized enzyme systems (see Kernevez et al. [KJDBT]) where $f(u) = \dfrac{(c-u)}{1+(c-u)+K(c-u)^2}$.

In the rest of this chapter (§1.1 - §1.4) we will discuss all our main results and give examples that satisfy our theorems. In Chapter 2, some preliminary results will be discussed. In the Chapter 3, multiple positive solutions of (1.2) will be discussed. In Chapter 4, $n \times n$ systems will be discussed. Existence and multiplicity results of positive solutions for multiparameter p-q-Laplacian systems will be discussed in Chapter 5. In Chapter 6, existence and multiplicity results for strongly coupled systems will be discussed. Conclusion and future directions will be discussed in Chapter 7.

1.1 Multiple positive solutions for 2×2 systems

Our interest is to extend the study of multiple positive solutions in [BIS] to the following system:

$$\begin{cases} -\Delta u = \lambda f(v) & \text{in } \Omega \\ -\Delta v = \lambda g(u) & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We assume that f and g are $C^1([0,\infty))$ functions satisfying the following assumptions:

(H1) $f(0) \geq 0,\ g(0) \geq 0$ and f and g are strictly increasing

(H2) $\lim_{x\to\infty} \frac{f(Mg(x))}{x} = 0\ \forall M > 0$ (combined sublinear effect at ∞).

For such a system with $f(0)$ or $g(0)$ strictly positive the existence of a positive solution for every $\lambda > 0$ has been proven (see [HS1], [HS2] and [Da]). Further from [Da] it follows that if $\frac{x}{f(x)}$ and $\frac{x}{g(x)}$ are nondecreasing for all $x \geq 0$ then (1.2) will have at most one positive solution.

In this thesis, first, we will focus on the case when either $\frac{x}{f(x)}$ or $\frac{x}{g(x)}$ is decreasing for certain range of x. In particular, under certain combined effects of $\frac{x}{f(x)}$ and $\frac{x}{g(x)}$ we investigate the existence of multiple positive solutions to (1.2). To precisely state our results, for $a_i, b_i; i = 1, 2$ we define:

$$Q_1(a_1, a_2) := \min\Big\{\frac{a_1}{f(a_2)}, \frac{a_2}{g(a_1)}\Big\}$$
$$Q_2(b_1, b_2) := \max\Big\{\frac{b_1}{f(b_2)}, \frac{b_2}{g(b_1)}\Big\}.$$

Then we establish:

Theorem 1

Let B_R be the largest ball of radius R inscribed in Ω,

$$C_1(\Omega) := \inf_{\epsilon} \frac{N}{\epsilon^N} \cdot \frac{R^{N-1}}{(R-\epsilon)}\Big(= \frac{(N+1)^{N+1}}{N^{N-1}} \cdot \frac{1}{R^2}\Big)$$

and $C(\Omega) := C_1(\Omega)\|e\|_\infty$ where e is the unique solution of $-\Delta e = 1; \Omega, e = 0; \partial\Omega$. Assume (H1)-(H2) hold and $f(0)$ or $g(0)$ be strictly positive. Suppose $Q_1/Q_2 > C(\Omega)$ for some $a_i, b_i;\ i = 1, 2$ with $a_1 < b_1$ or $a_2 < b_2$, then (1.2) has at least three positive solutions for

$$\frac{C(\Omega)Q_2}{\|e\|_\infty} < \lambda < \frac{Q_1}{\|e\|_\infty}. \tag{1.3}$$

For the case when $f(0) = g(0) = 0$, we establish the following multiplicity result:

Theorem 2

Assume (H1)-(H2) hold and $f(0) = 0 = g(0)$. Suppose $Q_1/Q_2 > C(\Omega)$ for some $a_i, b_i;\ i = 1, 2$ with $a_1 < b_1$ or $a_2 < b_2$, then (1.2) has at least two positive solutions for

$$\frac{C(\Omega)Q_2}{\|e\|_\infty} < \lambda < \frac{Q_1}{\|e\|_\infty}.$$

In the above case, if we have more information on the derivatives of the nonlinearities at zero, namely, if $f'(0) = g'(0) = 0$ then, we prove:

Theorem 3

Assume (H1)-(H2) hold and $f(0) = g(0) = 0 = f'(0) = g'(0)$. For $\lambda > C_1(\Omega)Q$, (1.2) has at least two positive solutions where $Q := \inf_{r>0,s>0} \max\{\frac{r}{f(s)}, \frac{s}{g(r)}\}$.

Remark 1

In the application of Theorem 1, one may take $a_1 = a_2, b_1 = b_2$ i.e., look for a_1, b_1 such that $a_1 < b_1$ and

$$\left(\min\left\{\frac{a_1}{f(a_1)}, \frac{a_1}{g(a_1)}\right\}\Big/ \max\left\{\frac{b_1}{f(b_1)}, \frac{b_1}{g(b_1)}\right\}\right) > C(\Omega).$$

Remark 2

In the case of a single equation, $f \equiv g$ and hence our condition for multiplicity reads $\frac{a_1}{f(a_1)}/\frac{b_1}{f(b_1)} > C(\Omega)$ which closely resembles with the result in [BIS].

Remark 3

In the case of systems, our result shows that multiplicity occurs even when one of the function $\frac{x}{f(x)}$ or $\frac{x}{g(x)}$ is nondecreasing as illustrated in Subsection 1.1.2 of examples.

Remark 4

$\sigma(r) := \frac{1}{2N}[R^2 - r^2]$ solves $-\Delta\sigma = 1; B_R, \sigma = 0; \partial B_R$. By the maximum principle it is easy to see that $\|e\|_\infty \geq \|\sigma\|_\infty = \frac{R^2}{2N}$. Hence $C(\Omega) \geq \frac{(N+1)^{N+1}}{N^{N-1}} \cdot \frac{1}{2N} = \left(\frac{N+1}{N}\right)^N \left(\frac{N+1}{2}\right) > 1$. This is expected since in the single equation case of (1.5): $-\Delta u = \lambda f(u); \Omega,\ u = 0; \partial\Omega$, it is well known that if $\frac{u}{f(u)}$ is nondecreasing then there exists at most one positive solution. Clearly if $\frac{u}{f(u)}$ is nondecreasing then $\frac{Q_1}{Q_2} \leq 1$ for any choice of $a_1 = a_2 = a < b = b_1 = b_2$.

Remark 5

For a given domain Ω, $\|e\|_\infty$ is fixed. So by choosing the B_R to be the largest ball inscribed in Ω, we obtain the smallest possible value for

$$\frac{C(\Omega)}{\|e\|_\infty} = C_1(\Omega) = \frac{(N+1)^{N+1}}{N^{N-1}} \cdot \frac{1}{R^2}$$

giving us the best possible range for multiplicity.

In [BIS], a multiplicity result similar to Theorem 1 was established for the single equation case, and recently in [RS] it was extended to the p-Laplacian single equation case. In [BIS], the Greens function played a crucial role in the construction of a crucial subsolution, while in [RS] this was avoided in order to extend the result for the p-Laplacian case. We adopt and extend the ideas used in [RS] for our extensions to systems.

1.1.1 p-Laplacian systems

In this section we state the extensions of our results Theorems 1 - 3 to the following p-Laplacian system:

$$\begin{cases} -\Delta_p u = \lambda f(v) & \text{in } \Omega \\ -\Delta_p v = \lambda g(u) & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{1.4}$$

where $\Delta_p z := \text{div}(|\nabla z|^{p-2}\nabla z);\ p > 1$. We assume that f and g are $C^1([0,\infty))$ functions satisfying the following assumptions:

$(\widetilde{H}1)$ $f(0) \geq 0,\ g(0) \geq 0$ and f and g are strictly increasing

$(\widetilde{H}2)$ $\lim_{x\to\infty} \dfrac{f(M(g(x)^{\frac{1}{p-1}}))}{x^{p-1}} = 0 \ \forall M > 0$ (combined p-sublinear effect at ∞).

To precisely state our results, for positive constants $a_i, b_i; i = 1, 2$ we define:

$$\widetilde{Q}_1(a_1, a_2) := \min\Big\{\frac{a_1^{p-1}}{f(a_2)}, \frac{a_2^{p-1}}{g(a_1)}\Big\}$$

$$\widetilde{Q}_2(b_1, b_2) := \max\Big\{\frac{b_1^{p-1}}{f(b_2)}, \frac{b_2^{p-1}}{g(b_1)}\Big\}.$$

We now state the extensions of our Theorems 1-3 for the p-Laplacian system (1.4):

Theorem 4

Let B_R be the largest ball of radius R inscribed in Ω,

$$\widetilde{C}_1(\Omega) := \inf_{\epsilon} \frac{N}{\epsilon^N} \cdot \frac{R^{N-1}}{(R-\epsilon)^{p-1}} \Big(= \frac{(N+p-1)^{N+p-1}}{(p-1)^{p-1}N^{N-1}} \cdot \frac{1}{R^p} \Big)$$

and $\widetilde{C}(\Omega) := \widetilde{C}_1(\Omega)\|e_p\|_\infty^{p-1}$ where e_p is the unique solution of $-\Delta_p e_p = 1; \Omega$, $e_p = 0; \partial\Omega$. Assume $(\widetilde{H}1)$-$(\widetilde{H}2)$ hold and $f(0)$ or $g(0)$ be strictly positive. Suppose $\widetilde{Q}_1/\widetilde{Q}_2 > \widetilde{C}(\Omega)$ for some $a_i, b_i;\ i = 1, 2$ with $a_1 < b_1$ or $a_2 < b_2$, then (1.4) has at least three positive solutions for

$$\frac{\widetilde{C}(\Omega)\widetilde{Q}_2}{\|e_p\|_\infty^{p-1}} < \lambda < \frac{\widetilde{Q}_1}{\|e_p\|_\infty^{p-1}}.$$

Theorem 5

Assume $(\widetilde{H}1)$-$(\widetilde{H}2)$ hold and $f(0) = 0 = g(0)$. Suppose $\widetilde{Q}_1/\widetilde{Q}_2 > \widetilde{C}(\Omega)$ for some $a_i, b_i;\ i = 1, 2$ with $a_1 < b_1$ or $a_2 < b_2$, then (1.4) has at least two positive solutions for

$$\frac{\widetilde{C}(\Omega)\widetilde{Q}_2}{\|e_p\|_\infty^{p-1}} < \lambda < \frac{\widetilde{Q}_1}{\|e_p\|_\infty^{p-1}}.$$

Theorem 6

Assume ($\widetilde{H}1$)-($\widetilde{H}2$) hold and $f(0) = g(0) = 0 = f^{(k)}(0) = g^{(k)}(0)$ for $k = 1, 2 \cdots [p-1]$, where $[p-1]$ denotes the integer part of $p-1$. For $\lambda > \widetilde{C}_1(\Omega)Q$, (1.4) has at least two positive solutions where $Q := \inf_{r>0, s>0} \max\{\frac{r^{p-1}}{f(s)}, \frac{s^{p-1}}{g(r)}\}$.

Remark 6

It will be easy to see later (in Chapter 3) that the proofs of Theorems 1 - 3 for Laplacian systems naturally extend to p-Laplacian systems. As such we will omit the proofs of Theorems 4 - 6 in this thesis.

1.1.2 Examples

Here we discuss two examples for Laplacian systems. In particular we concentrate on the applications of Theorem 1.

Example 1

Let $f(x) = e^{\frac{\alpha x}{\alpha + x}}$ and $g(x) = e^x$. Clearly f and g satisfy (H1), and (H2) as

$$\lim_{x \to \infty} \frac{f(Mg(x))}{x} = \lim_{x \to \infty} \frac{e^{\frac{\alpha M e^x}{\alpha + M e^x}}}{x} = 0.$$

Choosing $a_1 = a_2 = 1,\ b_1 = b_2 = \alpha > 1$, we have $Q_1(1,1) := \min\left\{e^{-\frac{\alpha}{\alpha+1}}, e^{-1}\right\}$ and $Q_2(\alpha, \alpha) := \max\left\{\alpha e^{-\frac{\alpha}{2}}, \alpha e^{-\alpha}\right\}$, thus

$$\begin{aligned}(Q_1/Q_2) &= \min\left\{\frac{e^{\alpha - \frac{\alpha}{\alpha+1}}}{\alpha}, \frac{e^{\frac{\alpha}{2}-1}}{\alpha}, \frac{e^{\alpha-1}}{\alpha}, \frac{e^{\frac{\alpha}{2} - \frac{\alpha}{\alpha+1}}}{\alpha}\right\} \\ &\geq \min\left\{\frac{e^{\alpha-1}}{\alpha}, \frac{e^{\frac{\alpha}{2}-1}}{\alpha}, \frac{e^{\alpha-1}}{\alpha}, \frac{e^{\frac{\alpha}{2}-1}}{\alpha}\right\} \\ &\geq \frac{e^{\frac{\alpha}{2}-1}}{\alpha}.\end{aligned}$$

For any Ω we can choose α so large that $Q_1/Q_2 > C(\Omega)$. Hence Theorem 1 holds and there exist a range of λ for which there exists three positive solutions. See Figures 1.1 - 1.2 for the graph of $\frac{x}{f(x)}$ for $\alpha = 6$ and $\frac{x}{g(x)}$.

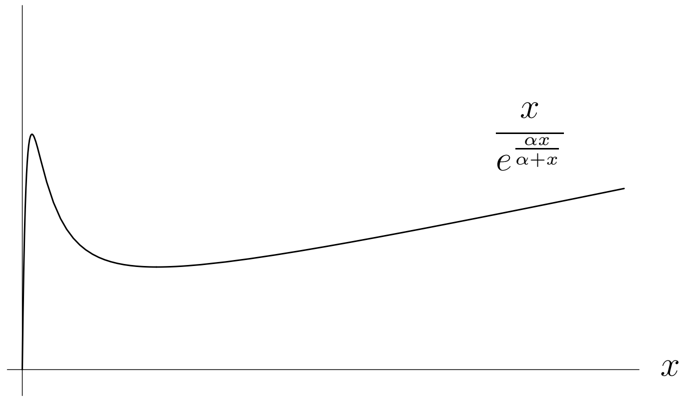

Figure 1.1

Graph of $\frac{x}{e^{\frac{\alpha x}{\alpha+x}}}$ for $\alpha = 6$

Example 2

Let $f(x) = e^{\frac{\alpha x}{\alpha+x}}$ and $g(x) = (x+1)^p, p > 0$. Clearly $f,\ g$ satisfies (H1) and (H2). Choosing $a_1 = a_2 = 1$, $b_1 = \alpha^{\frac{2}{p}} - 1,\ b_2 = \alpha$, we have $Q_1(1,1) := \min\left\{e^{-\frac{\alpha}{\alpha+1}}, 2^{-p}\right\}$ and $Q_2(\alpha^{\frac{2}{p}} - 1, \alpha) := \max\left\{(\alpha^{\frac{2}{p}} - 1)e^{-\frac{\alpha}{2}}, \alpha^{-1}\right\}$, thus

$$(Q_1/Q_2) = \min\left\{e^{-\frac{\alpha}{\alpha+1}}\alpha,\ \frac{e^{\frac{\alpha}{2}}}{2^p(\alpha^{\frac{2}{p}} - 1)},\ \frac{\alpha}{2^p},\ \frac{e^{\frac{\alpha}{2} - \frac{\alpha}{\alpha+1}}}{(\alpha^{\frac{2}{p}} - 1)}\right\}.$$

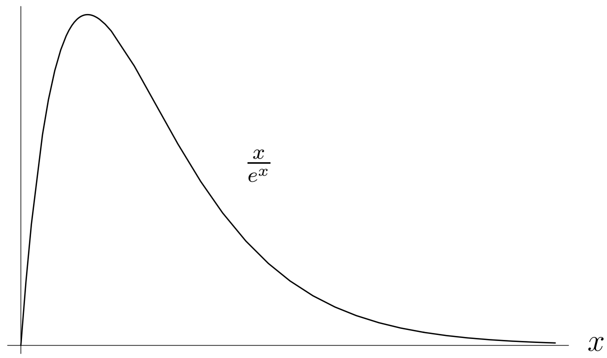

Figure 1.2

Graph of $\dfrac{x}{e^x}$

For any Ω we can choose α so large that $Q_1/Q_2 > C(\Omega)$. Hence Theorem 1 holds and there exist a range of λ for which three positive solutions exist. Note that for $p \leq 1$, $\frac{x}{g(x)}$ is nondecreasing. See Figures 1.3 - 1.4 for the graph of $\frac{x}{f(x)}$ for $\alpha = 5$ and $\frac{x}{g(x)}$ for $p = 0.8$.

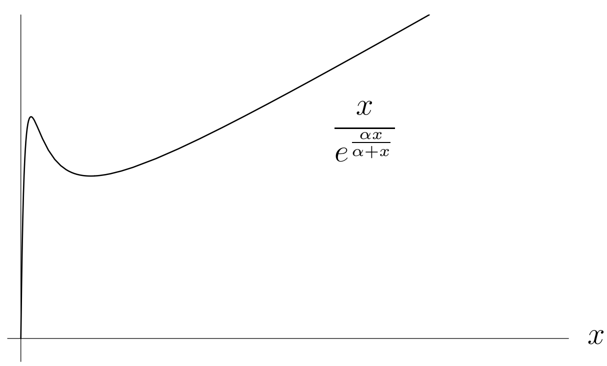

Figure 1.3

Graph of $\dfrac{x}{e^{\frac{\alpha x}{\alpha+x}}}$ for $\alpha = 5$

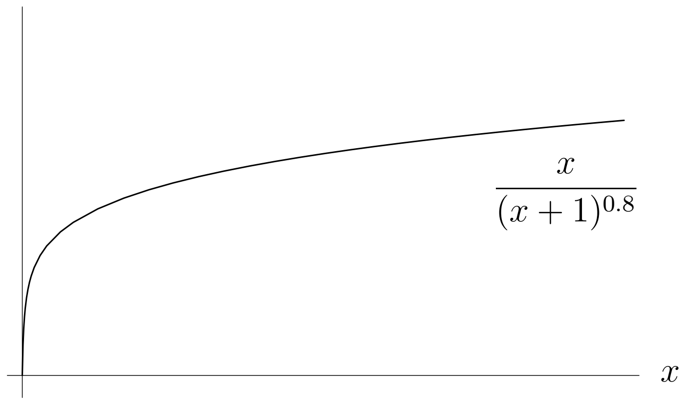

Figure 1.4

Graph of $\frac{x}{(x+1)^{0.8}}$

1.2 $n \times n$ systems

In this section we consider the boundary value problem

$$\begin{cases} -\Delta u_1 = \lambda f_1(u_2) & \text{in } \Omega \\ -\Delta u_2 = \lambda f_2(u_3) & \text{in } \Omega \\ \vdots = \vdots & \\ -\Delta u_{n-1} = \lambda f_{n-1}(u_n) & \text{in } \Omega \\ -\Delta u_n = \lambda f_n(u_1) & \text{in } \Omega \\ u_1 = u_2 = ... = u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Here $f_i \in C^1([0,\infty)),\ i \in \{1,2,\ldots,n\}$ satisfy the following assumptions:

(H3) $f_i, i \in \{1,\ldots,n\}$ are strictly increasing functions

(H4) $\lim_{x\to\infty} \dfrac{f_1^{[M]} \circ f_2^{[M]} \circ \cdots \circ f_{n-1}^{[M]} \circ f_n(x)}{x} = 0$ for every $M > 0$ where $f_k^{[M]}(x) := f_k(Mx);\ k = 1, 2, \ldots, n-1$.

In [Da], Dalmasso discussed the existence of positive solutions to such systems for $n = 2$ when $f_i(0)$'s are non-negative with at least one $f_i(0) > 0$ (positone problems). In Section 1.1, existence of multiple positive solutions to such positone problems for $n = 2$ was discussed. In particular, in the case when one of $\frac{x}{f_1(x)}$ or $\frac{x}{f_2(x)}$ decreases for some range of x, conditions for the existence of at least three positive solutions for a certain ranger of λ was discussed. In [HS1], Hai-Shivaji discussed the existence of positive solutions for $\lambda >> 1$ for the case when no sign conditions are assumed on $f_i(0),\ i \in \{1, 2\}$ (semipositone problems). We now extend these results for $n \times n$ systems. To state our results we first define:

$$Q_1(a_1, a_2, \ldots, a_n) := \min\left\{\frac{a_1}{f_1(a_2)}, \frac{a_2}{f_2(a_3)}, \ldots, \frac{a_k}{f_k(a_{k+1})}, \ldots, \frac{a_{n-1}}{f_{n-1}(a_n)}, \frac{a_n}{f_n(a_1)}\right\}$$

$$Q_2(b_1, b_2, \ldots, b_n) := \max\left\{\frac{b_1}{f_1(b_2)}, \frac{b_2}{f_2(b_3)}, \ldots, \frac{b_k}{f_k(b_{k+1})}, \ldots, \frac{b_{n-1}}{f_{n-1}(b_n)}, \frac{b_n}{f_n(b_1)}\right\}$$

for $a_i, b_i > 0,\ i \in \{1, 2, \ldots, n\}$. We establish the following results:

Theorem 7

Let (H3)-(H4) hold, $f_i(0) \geq 0, i \in \{1, \ldots, l-1, l+1, \ldots, n\}$ and $f_l(0) > 0$ for some $l \in \{1, \ldots, n\}$. Then (1.5) has a positive solution for every $\lambda > 0$.

Remark 7

(H3)-(H4) are satisfied by nonlinear f_i's which behave like x^{α_i} for x large where α_i's are positive constants such that $\prod_{i=1}^{n} \alpha_i < 1$. In particular, just one of the f_i's could be sublinear at infinity and rest of the f_i's be superlinear at infinity and still (H4) hold.

Theorem 8

Let B_R be the largest ball of radius R inscribed in Ω,

$$C_1(\Omega) := \inf_{\epsilon} \frac{N}{\epsilon^N} \cdot \frac{R^{N-1}}{(R-\epsilon)} \Big(= \frac{(N+1)^{N+1}}{N^{N-1}} \cdot \frac{1}{R^2} \Big)$$

and $C(\Omega) := C_1(\Omega)\|e\|_\infty$ where e is the unique solution of $-\Delta e = 1; \Omega, e = 0; \partial\Omega$. Assume (H3)-(H4) hold and $f_i(0) \geq 0, i \in \{1, \ldots, l-1, l+1, \ldots, n\}$ with $f_l(0) > 0$ for some $l \in \{1, \ldots, n\}$. Suppose $Q_1/Q_2 > C(\Omega)$ with $a_k < b_k$ for at least one of the $k \in \{1, 2, \ldots, n\}$ then (1.5) has at least three positive solutions for

$$\frac{C(\Omega)Q_2}{\|e\|_\infty} < \lambda < \frac{Q_1}{\|e\|_\infty}. \tag{1.6}$$

Remark 8

In the application of Theorem 8, one may take $a_i = a, b_i = b, i \in \{1, 2, \ldots, n\}$ i.e., look for a, b such that $a < b$ and

$$\Big(\min \Big\{ \frac{a}{f_1(a)}, \ldots, \frac{a}{f_n(a)} \Big\} \Big/ \max \Big\{ \frac{b}{f_1(b)}, \ldots, \frac{b}{f_n(b)} \Big\} \Big) > C(\Omega).$$

Note that in the case $f_1 = f_2 = \cdots = f_n = f,\ u_1 = u_2 = \cdots = u_n = u$ taking $a_1 = a_2 = \cdots = a_n = a, b_1 = b_2 = \cdots = b_n = b$ we get $Q_1/Q_2 = \frac{a}{f(a)} / \frac{b}{f(b)}$, and Theorem 8 closely resembles with the multiplicity result established for the single equation in [BIS].

Corollary 1

Let (H3)-(H4) hold and $f_i(0) = 0, i \in \{1, 2, \ldots, n\}$. Suppose $Q_1/Q_2 > C(\Omega)$ with $a_k < b_k$ for at least one of the $k \in \{1, 2, \ldots, n\}$ then (1.5) has at least two positive solutions for

$$\frac{C(\Omega)Q_2}{\|e\|_\infty} < \lambda < \frac{Q_1}{\|e\|_\infty}. \tag{1.7}$$

Corollary 2

Let (H3)-(H4) hold and $f_i(0) = 0 = f_i'(0), i \in \{1, 2, \ldots, n\}$. Suppose $\lambda > C_1(\Omega)Q$, then (1.5) has at least two positive solutions. Here

$$Q := \inf_{r_1, r_2, \ldots, r_n > 0} \max\Big\{\frac{r_1}{f_1(r_2)}, \frac{r_2}{f_2(r_3)}, \ldots, \frac{r_k}{f_k(r_{k+1})}, \ldots, \frac{r_n}{f_n(r_1)}\Big\}.$$

Theorem 9

Let (H3)-(H4) hold and $\lim_{x\to\infty} f_i(x) = \infty,\ i \in \{1, \ldots, n\}$. Then (1.5) has a positive solution $\underline{u} = (u_1, u_2, \ldots, u_n)$ provided λ is large. Further, $u_j(x) \to \infty\ \forall j \in \{1, 2, \ldots, n\}$ as $\lambda \to \infty\ \forall x \in \Omega$.

Remark 9

Theorem 9 establishes a positive solution for $\lambda >> 1$ without any sign condition assumed on the nonlinearities at the origin. In particular, $f_j(0) < 0$ for some $j \in \{1, 2, \ldots, n\}$ is allowed. Such problems are referred as *semipositone problems*. It has been well documented in the literature the fact that the study of positive solutions to semipositone problems are mathematically challenging (see [BCN] and [Li]). The difficulty is due to the fact that in the semipositone case, positive sub-solutions have to be superharmonic near the boundary and subharmonic in a large interior region of the domain. That is, a positive sub-solution in a single equation case, say ψ has to satisfy $-\Delta\psi < 0$ near boundary and $-\Delta\psi > 0$ in a large interior region. For more results on semipositone problems see [AnS], [AZ], [BCS], [BS], [CG], [CGS], [CHS], [CS], [Ha], [HS3], [HS4] and [Te].

Theorem 10

Let (H4)-(H3) hold, $\lim_{x\to\infty} f_i(x) = \infty,\ i \in \{1, \ldots, n\}$ and $f_i(0) = 0 = f_i'(0), i \in \{1, \ldots, n\}$. Then (1.5) has at least two positive solutions $\underline{u}^{(1)}$ and $\underline{u}^{(2)}$ provided λ is large. Further, $u_j^{(2)}(x) \to \infty\ \forall j \in \{1, 2, \ldots, n\}$ as $\lambda \to \infty\ \forall x \in \Omega$.

Remark 10

Theorem 10 is a result very similar to that of Corollary 1. However, in proving this Theorem 10 we will provide an alternate proof using our result in Theorem 9 for semipositone problems.

Since (1.5) is a cooperative system, it is well known that if there exist a sub-solution $(\psi_1, \psi_2, \ldots, \psi_n)$ and a super-solution $(z_1, z_2, \ldots, z_n)$ which satisfy $(\psi_1, \psi_2, \ldots, \psi_n) \leq (z_1, z_2, \ldots, z_n)$, then there exists a solution $(u_1, u_2, \ldots, u_n)$ such that

$$(\psi_1, \psi_2, \ldots, \psi_n) \leq (u_1, u_2, \ldots, u_n) \leq (z_1, z_2, \ldots, z_n)$$

(see [Am] and [Sa]).

The crucial part of establishing Theorem 8 depends on constructing a positive super-solution using the combined nonlinear effect at infinity, namely (H4). Here for the positone case, clearly the trivial function is a strict sub-solution. To prove Theorem 8, we use the multiplicity result in [Sh]. Namely, suppose there exist a sub-solution $(\psi_1, \psi_2, \ldots, \psi_n)$, a strict super-solution $(\zeta_1, \zeta_2, \ldots, \zeta_n)$, a strict sub-solution $(w_1, w_2, \ldots, w_n)$ and a super-solution $(z_1, z_2, \ldots, z_n)$ such that

$$(\psi_1, \psi_2, \ldots, \psi_n) \leq (\zeta_1, \zeta_2, \ldots, \zeta_n) \leq (z_1, z_2, \ldots, z_n),$$

$$(\psi_1, \psi_2, \ldots, \psi_n) \leq (w_1, w_2, \ldots, w_n) \leq (z_1, z_2, \ldots, z_n)$$

and

$$(w_1, w_2, \ldots, w_n) \not\leq (\zeta_1, \zeta_2, \ldots, \zeta_n).$$

Then there exists at least three distinct solutions $(u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)})$, $i = 1, 2, 3$ such that

$$(u_1^{(1)}, u_2^{(1)}, \ldots, u_n^{(1)}) \in [(\psi_1, \psi_2, \ldots, \psi_n), (\zeta_1, \zeta_2, \ldots, \zeta_n)],$$

$$(u_1^{(2)}, u_2^{(2)}, \dots, u_n^{(2)}) \in [(w_1, w_2, \dots, w_n), (z_1, z_2, \dots, z_n)]$$

and $(u_1^{(3)}, u_2^{(3)}, \dots, u_n^{(3)}) \in K$ where

$$K := [(\psi_1, \psi_2, \dots, \psi_n), (z_1, z_2, \dots, z_n)] \backslash \big([(\psi_1, \psi_2, \dots, \psi_n), (\zeta_1, \zeta_2, \dots, \zeta_n)] \cup [(w_1, w_2, \dots, w_n), (z_1, z_2, \dots, z_n)] \big).$$

The essential part of the proof of Theorem 8 relies on establishing these strict sub- and super-solutions in the given range of λ using the assumption $\frac{Q_1}{Q_2} > C(\Omega)$.

Next in establishing Theorem 9 for the semipositone case, the trivial solution is no longer a sub-solution. So one needs to also construct a positive sub-solution. We achieve this by using the square of the first eigenfunction φ of the negative Laplacian ($-\Delta$) with the Dirichlet boundary condition. In particular, we exploit the fact that $-\Delta(\varphi^2)$ is negative near the boundary $\partial\Omega$ (see [HS1]).

In the proof of Theorem 10, we will use Theorem 9 (semipositone case) to create a strict sub-solution and use the conditions of f_i's at zero to establish a strict super-solution.

1.2.1 p-Laplacian systems

In this section we state the extensions of our results Theorems 7 - 10 and Corollaries 1 - 2 for the following p-Laplacian $n \times n$ system:

$$
\begin{cases}
-\Delta_p u_1 = \lambda f_1(u_2) & \text{in } \Omega \\
-\Delta_p u_2 = \lambda f_2(u_3) & \text{in } \Omega \\
\quad \vdots \;\; = \;\; \vdots & \\
-\Delta_p u_{n-1} = \lambda f_{n-1}(u_n) & \text{in } \Omega \\
-\Delta_p u_n = \lambda f_n(u_1) & \text{in } \Omega \\
u_1 = u_2 = ... = u_n = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{1.8}
$$

Here $f_i \in C^1([0,\infty)),\ i \in \{1,2,\ldots,n\}$ satisfy the following assumptions:

$(\widetilde{H}3)$ $f_i, i \in \{1,\ldots,n\}$ are strictly increasing functions.

$(\widetilde{H}4)$ $\lim\limits_{x\to\infty} \dfrac{f_1^{[M]} \circ \left(f_2^{[M]}\right)^{\frac{1}{p-1}} \circ \cdots \circ \left(f_{n-1}^{[M]}\right)^{\frac{1}{p-1}} \circ \left(f_n(x)\right)^{\frac{1}{p-1}}}{x^{p-1}} = 0$ for every $M > 0$ where $f_k^{[M]}(x) := f_k(Mx);\ k = 1,2,\ldots,n-1.$

To state our results, we first define:

$$\widetilde{Q}_1(a_1,a_2,\ldots,a_n) := \min\left\{\frac{a_1^{p-1}}{f_1(a_2)}, \frac{a_2^{p-1}}{f_2(a_3)}, \ldots, \frac{a_k^{p-1}}{f_k(a_{k+1})}, \ldots, \frac{a_{n-1}^{p-1}}{f_{n-1}(a_n)}, \frac{a_n^{p-1}}{f_n(a_1)}\right\}$$

$$\widetilde{Q}_2(b_1,b_2,\ldots,b_n) := \max\left\{\frac{b_1^{p-1}}{f_1(b_2)}, \frac{b_2^{p-1}}{f_2(b_3)}, \ldots, \frac{b_k^{p-1}}{f_k(b_{k+1})}, \ldots, \frac{b_{n-1}^{p-1}}{f_{n-1}(b_n)}, \frac{b_n^{p-1}}{f_n(b_1)}\right\}$$

for $a_i, b_i > 0,\ i \in \{1,2,\ldots,n\}$. We now state the extensions of Theorems 7-10 and Corollaries 1-2.

Theorem 11

Let $(\widetilde{H}3)$-$(\widetilde{H}4)$ hold, $f_i(0) \geq 0, i \in \{1,\ldots,l-1,l+1,\ldots,n\}$ and $f_l(0) > 0$ for some $l \in \{1,\ldots,n\}$. Then (1.8) has a positive solution for every $\lambda > 0$.

Theorem 12

Let B_R be the largest ball of radius R inscribed in Ω,

$$\widetilde{C}_1(\Omega) := \inf_{\epsilon} \frac{N}{\epsilon^N} \cdot \frac{R^{N-1}}{(R-\epsilon)^{p-1}} \Big(= \frac{(N+p-1)^{N+p-1}}{(p-1)^{p-1}N^{N-1}} \cdot \frac{1}{R^p} \Big)$$

and $\widetilde{C}(\Omega) := \widetilde{C}_1(\Omega)\|e_p\|_\infty^{p-1}$ where e_p is the unique solution of $-\Delta_p e_p = 1$ in Ω, $e_p = 0$ on $\partial\Omega$. Assume ($\widetilde{H}3$)-($\widetilde{H}4$) hold and $f_i(0) \geq 0, i \in \{1, \dots, l-1, l+1, \dots, n\}$ with $f_l(0) > 0$ for some $l \in \{1, \dots, n\}$. Suppose $\widetilde{Q}_1/\widetilde{Q}_2 > \widetilde{C}(\Omega)$ with $a_k < b_k$ for at least one of the $k \in \{1, 2, \dots, n\}$ then (1.8) has at least three positive solutions for

$$\frac{\widetilde{C}(\Omega)\widetilde{Q}_2}{\|e\|_\infty} < \lambda < \frac{\widetilde{Q}_1}{\|e\|_\infty}. \tag{1.9}$$

Corollary 3

Let ($\widetilde{H}3$)-($\widetilde{H}4$) hold and $f_i(0) = 0, i \in \{1, 2, \dots, n\}$. Suppose $\widetilde{Q}_1/\widetilde{Q}_2 > C(\Omega)$ with $a_k < b_k$ for at least one of the $k \in \{1, 2, \dots, n\}$ then (1.8) has at least two positive solutions for

$$\frac{\widetilde{C}(\Omega)\widetilde{Q}_2}{\|e\|_\infty} < \lambda < \frac{\widetilde{Q}_1}{\|e\|_\infty}. \tag{1.10}$$

Corollary 4

Let ($\widetilde{H}3$)-($\widetilde{H}4$) hold and $f_i(0) = 0 = f_i^{(k)}(0), i \in \{1, 2, \dots, n\}$ for $k = 1, 2 \cdots [p-1]$, where $[p-1]$ denotes the integer part of $p-1$. Suppose $\lambda > \widetilde{C}_1(\Omega)\widetilde{Q}$, then (1.8) has at least two positive solutions. Here

$$\widetilde{Q} := \inf_{r_1, r_2, \dots, r_n > 0} \max\left\{ \frac{r_1^{p-1}}{f_1(r_2)}, \frac{r_2^{p-1}}{f_2(r_3)}, \dots, \frac{r_k^{p-1}}{f_k(r_{k+1})}, \dots, \frac{r_n^{p-1}}{f_n(r_1)} \right\}.$$

Theorem 13

Let ($\widetilde{H}3$)-($\widetilde{H}4$) hold and $\lim_{x\to\infty} f_i(x) = \infty,\ i \in \{1,\dots,n\}$. Then (1.8) has a positive solution $\underline{\mathrm{u}} = (u_1, u_2, \dots, u_n)$ provided λ is large. Further, $u_j(x) \to \infty\ \forall j \in \{1, 2, \dots, n\}$ as $\lambda \to \infty\ \forall x \in \Omega$.

Theorem 14

Let ($\widetilde{H}3$)-($\widetilde{H}4$) hold, $\lim_{x\to\infty} f_i(x) = \infty,\ i \in \{1,\dots,n\}$ and $f_i(0) = 0 = f_i^{(k)}(0), i \in \{1, 2, \dots, n\}$ for $k = 1, 2 \cdots [p-1]$, where $[p-1]$ denotes the integer part of $p-1$. Then (1.8) has at least two positive solutions $\underline{\mathrm{u}}^{(1)}$ and $\underline{\mathrm{u}}^{(2)}$ provided λ is large. Further, $u_j^{(2)}(x) \to \infty\ \forall j \in \{1, 2, \dots, n\}$ as $\lambda \to \infty\ \forall x \in \Omega$.

Remark 11

It will be easy to see later (in Chapter 4) that the proofs of Theorems 7 - 10 and Corollaries 1- 2 for Laplacian systems naturally extend to p-Laplacian systems. As such we will omit the proofs of Theorems 11 - 14 and Corollaries 3 - 4 in this thesis.

1.2.2 An example

In this section we discuss an example that satisfies Theorem 8.

Example 3

Let $f_1(x) = e^{\frac{\alpha x}{\alpha+x}}$, $f_i(x) = (x+1)^{p_{i-1}}, i \in \{2, 3, \dots, n\},\ p_j > 1, j \in \{1, 2, \dots, n-1\}$. Clearly $f_i, i \in \{1, 2, \dots, n\}$ satisfy (H3) and (H4). Choosing $a_i = 1, i \in \{1, 2, \dots, n\}, b_1 = \alpha^{\frac{2}{p_{n-1}}} - 1,\ b_2 = \alpha,\ b_i = \alpha^{\frac{2}{p_{i-2}}} - 1, i \in \{3, 4, \dots, n\}$ we have

$$Q_1(1, 1, \dots, 1) = \min\left\{e^{-\frac{\alpha}{\alpha+1}}, 2^{-p_1}, 2^{-p_2}, \dots, 2^{-p_{n-1}}\right\},$$
$$= \min\left\{e^{-\frac{\alpha}{\alpha+1}}, 2^{-p_k}\right\} \text{ where } p_k = \max\{p_1, p_2, \dots, p_{n-1}\} \text{ and}$$

$$Q_2(\alpha^{\frac{2}{p_{n-1}}} - 1, \alpha, \alpha^{\frac{2}{p_1}} - 1, \dots, \alpha^{\frac{2}{p_{n-2}}} - 1)$$
$$= \max\left\{(\alpha^{\frac{2}{p_{n-1}}} - 1)e^{-\frac{\alpha}{2}}, \alpha^{-1}, (\alpha^{\frac{2}{p_1}} - 1)\alpha^{-2}, \dots, (\alpha^{\frac{2}{p_{n-2}}} - 1)\alpha^{-2}\right\},$$
$$= \max\left\{(\alpha^{\frac{2}{p_{n-1}}} - 1)e^{-\frac{\alpha}{2}}, \alpha^{-1}, (\alpha^{\frac{2}{p_m}} - 1)\alpha^{-2}\right\} \text{ for } \alpha > 1$$

where $p_m = \min\{p_1, p_2, \dots, p_{n-2}\}$. Thus

$$(Q_1/Q_2) = \min\left\{\frac{e^{\frac{-\alpha}{\alpha+1}}e^{\frac{\alpha}{2}}}{(\alpha^{\frac{2}{p_{n-1}}} - 1)}, e^{-\frac{\alpha}{\alpha+1}}\alpha, \frac{e^{-\frac{\alpha}{\alpha+1}}\alpha^2}{(\alpha^{\frac{2}{p_m}} - 1)}, \frac{e^{\frac{\alpha}{2}}}{2^{p_k}(\alpha^{\frac{2}{p_{n-1}}} - 1)}, \frac{\alpha}{2^{p_k}}, \frac{\alpha^2}{2^{p_k}(\alpha^{\frac{2}{p_m}} - 1)}\right\}.$$

It is easy to see that $(Q_1/Q_2) \to \infty$ as $\alpha \to \infty$. Thus for any Ω we can choose α so large that $Q_1/Q_2 > C(\Omega)$. Hence Theorem 8 holds and there exist a range of λ for which three positive solutions exists.

1.3 Multiparameter p-q-Laplacian systems

In this section we investigate the existence and multiplicity of positive solutions to the following multiparameter p-q-Laplacian system:

$$\begin{cases} -\Delta_p u = \tau_1 f(v) + \mu_1 h(u) \text{ in } \Omega \\ -\Delta_q v = \tau_2 g(u) + \mu_2 \gamma(v) \text{ in } \Omega \\ u = 0 = v \text{ on } \partial\Omega \end{cases} \tag{1.11}$$

where τ_1, τ_2, μ_1 and μ_2 are nonnegative parameters with $\tau_1 + \mu_1$ and $\tau_2 + \mu_2$ are non zeros. Here we assume $f, g, h, \gamma \in C^1(0,\infty) \cap C[0,\infty)$ be strictly increasing functions. We prove the results under the following additional assumptions:

(H5) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \gamma(x) = \infty$

(H6) $\lim_{x\to\infty} \frac{f(M[g(x)]^{1/q-1})}{x^{p-1}} = 0$ for every $M > 0$

(H7) $\lim_{x\to\infty} \frac{h(x)}{x^{p-1}} = \lim_{x\to\infty} \frac{\gamma(x)}{x^{q-1}} = 0.$

We establish:

Theorem 15

Let (H5)-(H7) hold. There exists a large positive solution to (1.11) for $\tau_1 + \mu_1$ and $\tau_2 + \mu_2$ are large.

Remark 12

Theorem 15 holds without any sign condition on the nonlinearities at the origin (hence holds for the semipositone case).

Theorem 16

Let (H5)-(H7) hold. Let f, h, g and γ be sufficiently smooth functions in the neighborhood of zero with $f(0) = h(0) = g(0) = \gamma(0) = 0 = f^{(k)}(0) = h^{(k)}(0) = g^{(l)}(0) = \gamma^{(l)}(0)$ for $k = 1, 2, \cdots [p-1], l = 1, 2, \cdots [q-1]$ where $[s]$ denotes the integer part of s. Then (1.11) has at least two positive solutions provided $\tau_1 + \mu_1$ and $\tau_2 + \mu_2$ are large.

Next we study (1.11) with the hypotheses $(H6), (H7)$ and $(H8)$:

(H8) $f(0) > 0, g(0) > 0, h(0) \geq 0, \gamma(0) \geq 0$ and $\lim_{x\to\infty} g(x) = \infty.$

We prove the following multiplicity result. To precisely state our result we let $e_s, s = p, q$ be the unique positive solution of $-\Delta_s e_s = 1; \Omega$, $e_s = 0; \partial\Omega$ and for positive constants $a_i, b_i; i = 1, 2$ we define:

$$Q_1(a_1, a_2, b_1, b_2) := \frac{\left[a_1^{p-1}/f(b_1)\right]}{\left[a_2^{p-1}/f(b_2)\right]}$$

$$Q_2(a_1,a_2,b_1,b_2) := \frac{\left[b_1^{q-1}/g(a_1)\right]}{\left[b_2^{q-1}/g(a_2)\right]}.$$

Then we prove:

Theorem 17

Let R be the radius of the largest ball B_R inscribed in Ω, $0 < \sigma < R$ and let $C_1 := \frac{N}{\sigma^N}\frac{R^{N-1}}{(R-\sigma)^{p-1}}$ and $C_2 := \frac{N}{\sigma^N}\frac{R^{N-1}}{(R-\sigma)^{q-1}}$. If $Q_1 > C_1 d_1 \|e_p\|_\infty^{p-1}$ and $Q_2 > C_2 d_1 \|e_q\|_\infty^{q-1}$ for some $a_1 < a_2$ (or $b_1 < b_2$) and $d_1 > 1$ then (1.11) has at least three positive solutions for

$$(\tau_1,\tau_2,\mu_1,\mu_2) \in \Big(C_1\frac{a_2^{p-1}}{f(b_2)}, \frac{a_1^{p-1}}{d_1\|e_p\|_\infty^{p-1} f(b_1)}\Big) \times \Big(C_2\frac{b_2^{q-1}}{g(a_2)}, \frac{b_1^{q-1}}{d_1\|e_q\|_\infty^{q-1} g(a_1)}\Big)$$
$$\times \Big[0, \frac{a_1^{p-1}}{d_2\|e_p\|_\infty^{p-1} h(a_1)}\Big) \times \Big[0, \frac{b_1^{q-1}}{d_2\|e_q\|_\infty^{q-1} \gamma(b_1)}\Big).$$

Here $d_2 = \left[1 - \frac{1}{d_1}\right]^{-1}$.

Remark 13

When $p = q$, we can minimize the values of C_1 and C_2 (which is the same as C_1 in this case) by choosing $\sigma = \sigma_0 = \frac{N}{N+p-1}R$.

Theorem 18

If $\mu_1 = 0 = \mu_2$ and $Q_1 > C_1\|e_p\|_\infty^{p-1}$ and $Q_2 > C_2\|e_q\|_\infty^{q-1}$ for some $a_1 < a_2$ or $b_1 < b_2$ then (1.11) has at least three positive solutions for

$$(\tau_1,\tau_2) \in \Big(C_1\frac{a_2^{p-1}}{f(b_2)}, \frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1} f(b_1)}\Big) \times \Big(C_2\frac{b_2^{q-1}}{g(a_2)}, \frac{b_1^{q-1}}{\|e_q\|_\infty^{q-1} g(a_1)}\Big).$$

1.3.1 Examples

In this section we discuss some examples that satisfy Theorem 15, Theorem 16 and Theorem 17.

Example 4

Let $f(x) = \sum_{i=1}^{m} a_i x^{p_i} - c_1, g(x) = \sum_{j=1}^{n} b_j x^{q_j} - c_2, h(x) = \sum_{k=1}^{s} \alpha_k x^{r_k} - c_3$ and $\gamma(x) = \sum_{l=1}^{t} \beta_l x^{d_l} - c_4$ where $a_i, b_j, \alpha_k, \beta_l, p_i, q_j, r_k, d_l, c_1, c_2, c_3, c_4 \geq 0, p_i q_j < (p-1)(q-1), r_k < (p-1)$ and $d_l < (q-1)$. Then it is easy to see that f, g, h and γ satisfy the hypotheses of Theorem 15.

Example 5

Let $f(x) = \begin{cases} x^{p_1}; & x \leq 1 \\ \frac{p_1}{p_2} x^{p_2} + (1 - \frac{p_1}{p_2}); & x > 1, \end{cases}$ $h(x) = \begin{cases} x^{p_3}; & x \leq 1 \\ \frac{p_3}{p_4} x^{p_4} + (1 - \frac{p_3}{p_4}); & x > 1, \end{cases}$

$$g(x) = \begin{cases} x^{q_1}; & x \leq 1 \\ \frac{q_1}{q_2} x^{q_2} + (1 - \frac{q_1}{q_2}); & x > 1, \end{cases} \text{ and } \gamma(x) = \begin{cases} x^{q_3}; & x \leq 1 \\ \frac{q_3}{q_4} x^{q_4} + (1 - \frac{q_3}{q_4}); & x > 1. \end{cases}$$

Here we assume $p_1, p_3 > p - 1$ if p is an integer, $p_1, p_3 > [p]$ if p is not an integer, $q_1, q_3 > q - 1$ if q is an integer, $q_1, q_3 > [q]$ if q is not an integer, $p_2 q_2 < (p-1)(q-1)$, $p_4 < p - 1$ and $q_4 < q - 1$. Then it is easy to see that f, g, h and γ satisfy the hypotheses of Theorem 16.

Example 6

Let $f(x) = e^{\frac{\alpha x}{\alpha + x}}$, $g(x) = (x+1)^k$, $h(x) = x^r$ and $\gamma(x) = x^s$ where $k > q - 1, 0 < r < p - 1$ and $0 < s < q - 1$. Clearly f, g, h and γ satisfy (H5)-(H8).

Let $a_1 = 1 = b_1, a_2 = \alpha = b_2, \alpha > 1$. We note that

$$Q_1(1, \alpha, 1, \alpha) := \frac{e^{\frac{\alpha}{2}}}{\alpha^{p-1} e^{\frac{\alpha}{\alpha+1}}} \text{ and } Q_2(1, \alpha, 1, \alpha) := \frac{(\alpha+1)^k}{2^k \alpha^{q-1}}.$$

Since $Q_1, Q_2 \to \infty$ as $\alpha \to \infty$, for any Ω we can choose α so large that $Q_1 > C_1 d_1 \|e_p\|_\infty^{p-1}$ and $Q_2 > C_2 d_1 \|e_q\|_\infty^{q-1}$. Hence Theorem 17 holds and there exist a range of parameters for which three positive solutions exist.

1.4 Strongly coupled p-q-Laplacian systems

In this section we study the existence and multiplicity of positive solutions to the following strongly coupled system:

$$\begin{cases} -\Delta_p u = \lambda f(u,v) & \text{in } \Omega \\ -\Delta_q v = \lambda g(u,v) & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega. \end{cases} \tag{1.12}$$

Here we assume f and g satisfy the following:

(H9) $f, g \in C^1((0,\infty)\times(0,\infty)) \cap C([0,\infty)\times[0,\infty))$ be monotone functions such that $f_u, f_v, g_u, g_v \geq 0$ and $\lim_{u,v\to\infty} f(u,v) = \lim_{u,v\to\infty} g(u,v) = \infty$

(H10) $\lim_{x\to\infty} \frac{f(x, M[g(x,x)]^{1/q-1})}{x^{p-1}} = 0$ for every $M > 0$

(H11) $\lim_{x\to\infty} \frac{g(x,x)}{x^{q-1}} = 0.$

We establish the following existence and multiplicity results:

Theorem 19

Let (H9)-(H11) hold. There exists a large positive solution to (1.12) for λ large.

Remark 14

Theorem 15 holds without any sign condition on the nonlinearities at the origin (hence holds for the semipositone case).

Theorem 20

Let (H9)-(H11) hold and further let $F(s) = f(s, cs)$ and $G(s) = g(\tilde{c}s, s)$ for any $c, \tilde{c} > 0$. Assume that f and g be sufficiently smooth functions in the neighborhood of zero with $F(0) = G(0) = 0$, $F^{(k)}(0) = 0 = G^{(l)}(0)$ for $k = 1, 2, \ldots [p-1]$, $l = 1, 2, \ldots [q-1]$ where $[s]$ denotes the integer part of s. Then (1.12) has at least two positive solutions provided λ is large.

Remark 15

Let $f(0,0) \geq 0$ and $g(0,0) \geq 0$ with one of them strictly positive. Then (H9)-(H11) guarantees a positive solution of (1.12) for any $\lambda > 0$.

1.4.1 Examples

In this section we discuss some examples that satisfy Theorem 19 and Theorem 20.

Example 7

Consider the following problem

$$\begin{cases} -\Delta_p u = \lambda[v^\alpha + (uv)^\beta - 1] & \text{in } \Omega \\ -\Delta_q v = \lambda[u^\sigma + (uv)^{\gamma/2} - 1] & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{1.13}$$

where α, β, σ and γ are positive parameters. Then it is easy to see that (1.13) satisfies the hypotheses of Theorem 19 if $\max\{\sigma,\gamma\}\frac{\alpha}{q-1} < p-1$, $(\max\{\sigma,\gamma\}\frac{1}{q-1} + 1)\beta < p-1$ and $\max\{\sigma,\gamma\} < q-1$.

Example 8

Let h and g be defined as follows:

$$h(x) = \begin{cases} x^{\alpha}; & x \leq 1 \\ \frac{\alpha}{\sigma}x^{\sigma} + (1 - \frac{\alpha}{\sigma}); & x > 1 \end{cases} \quad \text{and } \gamma(x) = \begin{cases} x^{\mu}; & x \leq 1 \\ \frac{\mu}{\delta}x^{\delta} + (1 - \frac{\mu}{\delta}); & x > 1. \end{cases}$$

where α, σ, μ and δ are positive parameters. Here we assume $\alpha > p - 1$ if p is an integer, $\alpha > [p]$ if p is not an integer, $\mu > q - 1$ if q is an integer and $\mu > [q]$ if q is not an integer.

Consider the following problem

$$\begin{cases} -\Delta u = \lambda[1 + u^{\beta}]h(v) & \text{in } \Omega \\ -\Delta v = \lambda\gamma(u) & \text{in } \Omega \\ u = 0 = v & \text{on } \partial\Omega \end{cases} \tag{1.14}$$

where $0 \leq \beta < p-1$. Then it is easy to see that (1.14) satisfies the hypotheses of Theorem 20 if $\delta\sigma < [p - 1 - \beta](q - 1)$ and $\delta < q - 1$.

CHAPTER 2

PRELIMINARIES

In this chapter we will discuss some preliminary results that will be used to establish our results in the following chapters. In particular, we will discuss maximum principles, comparison principles and the method of sub- and super-solutions.

2.1 Linear elliptic boundary value problems

Consider the following Laplacian equation of the form:

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. Let $C^{m+r}(\Omega), 0 < r < 1$ be the space of m-times continuously differentiable functions whose m^{th} derivatives are Hölder continuous on Ω with Hölder exponent r. We shall consider classical solutions of (2.1), that is, $C^2(\Omega) \cap C^1(\overline{\Omega})$ functions satisfying (2.1) pointwise. Let $f \in C^{\alpha}(\overline{\Omega})$ with $\alpha = 0$ if $n = 1$ and $0 < \alpha < 1$ if $n \geq 2$. Then it is well known that (2.1) has a solution $u = Kf$ where $K : C^{\alpha}(\overline{\Omega}) \longrightarrow C^{2+\alpha}(\overline{\Omega})$ is a solution operator whose kernel is the Green's function $G(x, y)$ for (2.1), that is, $Kf(x) := \int_{\Omega} G(x, y)f(y)dy$.

2.2 Maximum and comparison principles

Here we recall the classical maximum principle, the Hopf maximum principle and two comparison principles.

Lemma 1 (Maximum principle, see [PW] and [GT])

Let $\Delta u \geq 0$ in Ω. If u attains its maximum at any interior point in Ω, then $u \equiv M$ in Ω.

Lemma 2 (Hopf Maximum Principle, see [PW] and [GT])

Let $\Delta u \geq 0$ in Ω. Suppose that $u \leq M$ in Ω and that $u = M$ at some $p \in \partial\Omega$. Then $\frac{\partial u}{\partial \eta} > 0$ at $p \in \partial\Omega$ unless $u \equiv M$. Here $\frac{\partial}{\partial \eta}$ denotes the outward normal derivative.

Lemma 3 (Weak comparison principle)

Assume that $\Delta u \geq \Delta v$ in Ω and $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in $\overline{\Omega}$.

Lemma 4 (Strong comparison principle)

Assume that $\Delta u > \Delta v$ in Ω and $u = v$ on $\partial\Omega$. Then $u < v$ in Ω and $\frac{\partial u}{\partial \eta} > \frac{\partial v}{\partial \eta}$ on $\partial\Omega$.

2.3 Degree theory

Consider a mapping $f \in C^1(\bar{D}, \mathbb{R}^N)$ where D is an open and bounded subset of $\mathbb{R}^N$. Suppose $p \in \mathbb{R}^N$ such that $f \neq p$ on ∂D. Assume that $D \cup f^{-1}(p)$ is finite and that the Jacobian matrix $f'(x)$ is non-singular at these points. Then the degree of f at p relative to D is defined by

$$d(f, D, p) = \sum_{x \in f^{-1}(b) \cap D} Sign|f'(x)|$$

where $|f'(x)|$ is the determinant of the Jacobian matrix.

The degree defined above can be extended to functions defined on a Banach Space X. Let $\chi(u) := u - T(u)$, where $T \in C(\bar{D}, E)$ with D bounded and open subset of E and T is completely continuous (compact). If $p \in E$ and $p \notin \chi(\partial D), d(\chi, D, p)$ can be defined by approximating χ with mappings over finite dimensional spaces, the degree so extended is called Leray-Schuder degree (see [Ld]). Our results in the rest of the preliminary sections can be proved using degree theory.

2.4 Sub- and super-solutions

In this section we discuss the method of sub- and super-solutions. Consider

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

A function $\psi \in C^2(\overline{\Omega})$ is called a sub-solution and $\phi \in C^2(\overline{\Omega})$ is called a super-solution of (2.2) if ψ and ϕ satisfy

$$\begin{cases} -\Delta \psi \leq f(\psi) & \text{in } \Omega \\ \psi \leq 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

and

$$\begin{cases} -\Delta \phi \geq f(\phi) & \text{in } \Omega \\ \phi \geq 0 & \text{on } \partial\Omega \end{cases} \tag{2.4}$$

respectively. A sub-solution which is not a solution is called a strict sub-solution and similarly a super-solution which is not a solution is called a strict super-solution. Now we state some theorems about the existence and multiplicity results using sub- and super-solutions.

Lemma 5 (See [Am] and [Sa])

Suppose there exists a sub-solution ψ and a super-solution ϕ for the problem (2.2) satisfying $\psi \leq \phi$, then there exists a solution u of (2.2) such that $\psi \leq u \leq \phi$.

Lemma 6 (See [Am])

Suppose there exists a sub-solution ψ_1, a strict super-solution ϕ_1, a strict sub-solution ψ_2 and a super-solution ϕ_2 for (2.2) such that $\psi_1 < \phi_1 < \psi_2 < \phi_2$. Then (2.2) has at least three distinct solutions u_1, u_2 and u_3 such that $\psi_1 \leq u_1 < u_2 < u_3 \leq \phi_2$.

However in the paper by Shivaji [Sh], the condition $\phi_1 < \psi_2$ was relaxed to $\psi_2 \nleq \phi_1$. The author proved the following result:

Lemma 7 (See [Sh])

Suppose there exists a sub-solution ψ_1, a strict super-solution ϕ_1, a strict sub-solution ψ_2 and a super-solution ϕ_2 for (2.2) such that $\psi_1 < \phi_1 < \phi_2$, $\psi_1 < \psi_2 < \phi_2$ and $\psi_2 \nleq \phi_1$. Then (2.2) has at least three distinct solutions u_1, u_2 and u_3 such that $u_1 \in [\psi_1, \phi_1], u_2 \in [\psi_2, \phi_2]$ and $u_3 \in [\psi_1, \phi_2] \setminus [\psi_1, \phi_1] \cup [\psi_2, \phi_2]$.

2.5 Laplacian systems

In this section, we discuss similar results for Laplacian systems of the form:

$$\begin{cases} -\Delta \underline{u} = \underline{F}(\underline{u}) & \text{in } \Omega \\ \underline{u} = \underline{0} & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

where $\underline{u} = (u_1, u_2, \ldots, u_n)$ and $\underline{F} = (F_1, F_2, \ldots, F_n) : \left[C^1([0,\infty)^n)\right]^n \longrightarrow \mathbb{R}^n$. Such a system (2.5) is called cooperative if $\frac{\partial F_i}{\partial u_j} \geq 0, i, j \in \{1, 2, \ldots, n\}, i \neq j$. We say $\underline{u} \leq \underline{v}$ if $u_i \leq v_i, i \in \{1, 2, \ldots, n\}$, $\underline{u} \lneq \underline{v}$ if $u_i \leq v_i, i \in \{1, 2, \ldots, n\}$ and $u_l < v_l$ for some $l \in \{1, 2, \ldots, n\}$.

For simplicity we present the results for 2×2 systems case. Consider the following cooperative system of the form:

$$\begin{cases} -\Delta u_1 = F_1(u_1, u_2) & \text{in } \Omega \\ -\Delta u_2 = F_2(u_1, u_2) & \text{in } \Omega \\ u_1 = 0 = u_2 & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

A function $(\psi_1, \psi_2) \in \left[C^2(\overline{\Omega})\right]^2$ is called a sub-solution and $(\phi_1, \phi_2) \in \left[C^2(\overline{\Omega})\right]^2$ is called super-solution of (2.6) if (ψ_1, ψ_2) and (ϕ_1, ϕ_2) satisfy

$$\begin{cases} -\Delta \psi_1 \leq F_1(\psi_1, \psi_2) & \text{in } \Omega \\ -\Delta \psi_2 \leq F_2(\psi_1, \psi_2) & \text{in } \Omega \\ (\psi_1, \psi_2) \leq (0, 0) & \text{on } \partial\Omega \end{cases} \tag{2.7}$$

and

$$\begin{cases} -\Delta \phi_1 \geq F_1(\phi_1, \phi_2) & \text{in } \Omega \\ -\Delta \phi_2 \geq F_2(\phi_1, \phi_2) & \text{in } \Omega \\ (\phi_1, \phi_2) \geq (0, 0) & \text{on } \partial\Omega \end{cases} \tag{2.8}$$

respectively.

Lemma 8 (See [Am] and [Sa])

Suppose there exists a sub-solution (ψ_1, ψ_2) and a super-solution (ϕ_1, ϕ_2) for the problem (2.6) satisfying $(\psi_1, \psi_2) \leq (\phi_1, \phi_2)$, then there exists a solution (u_1, u_2) of (2.6) such that $\psi_i \leq u_i \leq \phi_i, i = 1, 2$.

Lemma 9 (See [Sh])

Suppose there exist a sub-solution (ψ_1, ψ_2), a strict super-solution (w_1, w_2), a strict sub-solution (ζ, ζ_2) and a super-solution (ϕ_1, ϕ_1) for (2.6) such that $(\psi_1, \psi_2) \leq (w_1, w_2) \leq$

(ϕ_1, ϕ_2), $(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (\phi_1, \phi_2)$ and $(\zeta_1, \zeta_2) \nleq (w_1, w_2)$. Then (2.6) has at least three distinct solutions $(u_i, v_i),\ i = 1, 2, 3$ such that $(u_1, v_1) \in [(\psi_1, \psi_2), (w_1, w_2)]$, $(u_2, v_2) \in [(\zeta_1, \zeta_2), (\phi_1, \phi_2)]$ and $(u_3, v_3) \in [(\psi_1, \psi_1), (\phi_2, \phi_2)] \setminus \big([(\psi_1, \psi_2), (w_1, w_2)] \cup [(\zeta_1, \zeta_2), (\phi_1, \phi_2)]\big)$.

2.6 p-Laplacian equations

Consider the following p-Laplacian equation of the form:

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

By a weak solution (2.9), we mean a function $u \in W_0^{1,p}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \omega = \int_\Omega f(x)\omega,\ \forall\, \omega \in C_0^\infty(\Omega).$$

2.6.1 Regularity result

We here state the following regularity result:

Lemma 10 (see [DKT])

Let $u \in W_0^{1,p}(\Omega)$ be any weak solution of the Dirichlet problem (2.9). If $f(x) \in L^\infty(\Omega)$, then $u \in C^1(\overline{\Omega})$.

2.6.2 Comparison principles

Consider the following equations:

$$-\Delta_p u = f(x) \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega,$$

$$-\Delta_p v = g(x) \text{ in } \Omega; \quad v = 0 \text{ on } \partial\Omega.$$

We now state the weak and strong comparison principle for the above problem (see [DKT], [FT], [CT] and [PS].)

Lemma 11 (Weak comparison principle)

Let $f, g \in L^\infty(\Omega)$ satisfy $f \leq g$ in Ω. Then $u \leq v$ in Ω.

Lemma 12 (Strong comparison principle)

Let $f, g \in L^\infty(\Omega)$ satisfy $0 \leq f \leq g$ and $f \not\equiv g$ in Ω. Then $u < v$ in Ω and $0 \geq \dfrac{\partial u}{\partial \eta} > \dfrac{\partial v}{\partial \eta}$ on $\partial\Omega$.

2.6.3 Sub- and super-solutions

Consider the following p-Laplacian equation of the form:

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.10}$$

where $p > 1$, $f \in C^1$ and f is strictly increasing.

We say a function $\psi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is a sub-solution and $\phi \in W^{1,p}(\Omega) \cap C(\overline{\Omega})$ is a super-solution of (2.10) if ψ and ϕ satisfy $\psi = 0 = \phi$ on $\partial\Omega$ and

$$\int_\Omega |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \omega \leq \int_\Omega f(\psi)\omega, \ \forall\, \omega \in W$$

and

$$\int_\Omega |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla \omega \geq \int_\Omega f(\phi)\omega, \ \forall\, \omega \in W$$

respectively where $W := \{\xi \in C_0^\infty(\Omega) |\ \xi \geq 0 \text{ in } \Omega\}$.

Lemma 13 (See [DH])

Suppose there exists a sub-solution ψ and a super-solution ϕ for the problem (2.10) satisfying $\psi \leq \phi$, then there exists a solution u of (2.10) such that $\psi \leq u \leq \phi$.

Lemma 14 (see [RS] and [Sh])

Let f be nonnegative and suppose there exists a sub-solution ψ_1, a strict super-solution ϕ_1, a strict sub-solution ψ_2 and a super-solution ϕ_2 for (2.10) such that $\psi_1 < \phi_1 < \phi_2$, $\psi_1 < \psi_2 < \phi_2$ and $\psi_2 \nleq \phi_1$. Then (2.10) has at least three distinct solutions u_1, u_2 and u_3 such that $u_1 \in [\psi_1, \phi_1], u_2 \in [\psi_2, \phi_2]$ and $u_3 \in [\psi_1, \phi_2] \setminus [\psi_1, \phi_1] \cup [\psi_2, \phi_2]$.

2.7 p-Laplacian systems

In this section we present results similar to the previous sections for the following $\underline{p}$-Laplacian $n \times n$ systems of the form:

$$\begin{cases} -\Delta_{\underline{p}} \underline{u} = \underline{F}(\underline{u}) & \text{in } \Omega \\ \underline{u} = \underline{0} & \text{on } \partial\Omega. \end{cases} \tag{2.11}$$

Here $-\Delta_{\underline{p}} \underline{u} = (-\Delta_{p_1} u_1, -\Delta_{p_2} u_2, \ldots, -\Delta_{p_n} u_n); p_i > 1, i \in \{1, 2, \ldots, n\}$ and $\underline{F}(\underline{u}) = (F_1(\underline{u}), \ldots, F_n(\underline{u})) = (F_1(u_1, u_2, \ldots, u_n), \ldots, F_n(u_1, u_2, \ldots, u_n))$. We further assume that $\frac{\partial F_i}{\partial u_j} \geq 0, \forall j \in \{1, 2, \ldots, n\}$ and $\frac{\partial F_i}{\partial u_l} > 0$ for some $l \in \{1, 2, \ldots, n\}$ for all $i \in \{1, 2, \ldots, n\}$.

For simplicity we present the results for the p-q-Laplacian 2×2 system of the form:

$$\begin{cases} -\Delta_p u_1 = F_1(u_1, u_2) & \text{in } \Omega \\ -\Delta_q u_2 = F_2(u_1, u_2) & \text{in } \Omega \\ u_1 = 0 = u_2 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

A function $(\psi_1, \psi_2) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ is called a sub-solution and $(\phi_1, \phi_2) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ is called super-solution of (2.12) if (ψ_1, ψ_2) and (ϕ_1, ϕ_2) satisfy $(\psi_1, \psi_2) = 0 = (\phi_1, \phi_2)$ on $\partial\Omega$ and

$$\begin{cases} \int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla \omega \leq \int_\Omega F_1(\psi_1, \psi_2)\omega \\ \int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 \cdot \nabla \omega \leq \int_\Omega F_2(\psi_1, \psi_2)\omega \end{cases} \tag{2.13}$$

and

$$\begin{cases} \int_\Omega |\nabla \phi_1|^{p-2} \nabla \phi_1 \cdot \nabla \omega \leq \int_\Omega F_1(\phi_1, \phi_2)\omega \\ \int_\Omega |\nabla \phi_2|^{q-2} \nabla \phi_2 \cdot \nabla \omega \leq \int_\Omega F_2(\phi_1, \phi_2)\omega \end{cases} \tag{2.14}$$

respectively for all $\omega \in W$.

Lemma 15 (See [DKT])

Suppose there exists a sub-solution (ψ_1, ψ_2) and a super-solution (ϕ_1, ϕ_2) for the problem (2.12) satisfying $(\psi_1, \psi_2) \leq (\phi_1, \phi_2)$, then there exists a solution (u_1, u_2) of (2.12) such that $\psi_i \leq u_i \leq \phi_i, i = 1, 2$.

Lemma 16

Let $F_1, F_2 \geq 0$. Suppose there exist a sub-solution (ψ_1, ψ_2), a strict super-solution (w_1, w_2), a strict sub-solution (ζ, ζ_2) and a super-solution (ϕ_1, ϕ_1) for (2.12) such that $(\psi_1, \psi_2) \leq (w_1, w_2) \leq (\phi_1, \phi_2)$, $(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (\phi_1, \phi_2)$ and $(\zeta_1, \zeta_2) \nleq (w_1, w_2)$. Then (2.12) has at least three solutions $(u_i, v_i),\ i = 1, 2, 3$ such that $(u_1, v_1) \in [(\psi_1, \psi_2), (w_1, w_2)]$, $(u_2, v_2) \in [(\zeta_1, \zeta_2), (\phi_1, \phi_2)]$ and $(u_3, v_3) \in [(\psi_1, \psi_1), (\phi_2, \phi_2)] \setminus \big([(\psi_1, \psi_2), (w_1, w_2)] \cup [(\zeta_1, \zeta_2), (\phi_1, \phi_2)]\big)$.

Proof of Lemma 16 follows easily from [Sh].

CHAPTER 3

LAPLACIAN SYSTEMS

In this chapter proofs the Theorems 1 - 3 will be presented. Theorem 1 will be proved in Section 3.1 when Ω is a ball and in Section 3.2 when Ω is a bounded domain. In Section 3.3, proofs of Theorem 2 and 3 will be presented.

3.1 Proof of Theorem 1 (when Ω is a ball of radius R)

We will establish a pair of sub-solutions $(\psi_1, \overline{\psi}_1), (\psi_2, \overline{\psi}_2)$ and a pair of super-solutions $(\phi_1, \overline{\phi}_1), (\phi_2, \overline{\phi}_2)$ satisfying Lemma 9. Clearly $(\psi_1, \overline{\psi}_1) = (0, 0)$ is a sub-solution of (1.2) since $f(0) \geq 0$ and $g(0) \geq 0$.

Next, for $\lambda < \frac{1}{\|e\|_\infty} \min\{\frac{a_1}{f(a_2)}, \frac{a_2}{g(a_1)}\} = A$ (say), we construct a positive strict super-solution $(\phi_1, \overline{\phi}_1)$ of (1.2). Let $e \in C^2(\overline{\Omega})$ be the solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \Omega \\ e = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $(\phi_1, \overline{\phi}_1) = (a_1 \frac{e}{\|e\|_\infty}, a_2 \frac{e}{\|e\|_\infty})$. Since $\lambda < \frac{1}{\|e\|_\infty} \frac{a_1}{f(a_2)}$, we have $-\Delta \phi_1 = \frac{a_1}{\|e\|_\infty} > \lambda f(a_2) \geq \lambda f(\overline{\phi}_1)$. Similar argument shows that $\overline{\phi}_1$ satisfies $-\Delta \overline{\phi}_1 > \lambda g(\phi_1)$. This proves that $(\phi_1, \overline{\phi}_1)$ is a positive strict super-solution of (1.2). Note that $\|\phi_1\|_\infty = a_1$ and $\|\overline{\phi}_1\|_\infty = a_2$.

Now, for $\lambda > C_1(\Omega)\max\{\frac{b_2}{g(b_1)}, \frac{b_1}{f(b_2)}\} = B$ (say), we construct a positive sub-solution $(\psi_2, \overline{\psi}_2)$ of (1.2) where $C_1 := \inf_{\epsilon} \frac{N}{\epsilon^N}\frac{R^{N-1}}{R-\epsilon}$. For $0 < \epsilon < R, \alpha, \beta > 1$ define $\rho(r) : [0, R] \to [0, 1]$ by

$$\rho(r) = \begin{cases} 1, & 0 \le r \le \epsilon \\ 1 - (1 - (\frac{R-r}{R-\epsilon})^{\beta})^{\alpha} & \epsilon < r \le R. \end{cases}$$

Thus

$$\rho'(r) = \begin{cases} 0, & 0 \le r \le \epsilon \\ -\frac{\alpha\beta}{R-\epsilon}(1 - (\frac{R-r}{R-\epsilon})^{\beta})^{\alpha-1}(\frac{R-r}{R-\epsilon})^{\beta-1} & \epsilon < r \le R. \end{cases}$$

Note that $|\rho'(r)| \le \frac{\alpha\beta}{R-\epsilon}$.

Let $w(r) = b_1\rho(r)$ and $\overline{w}(r) = b_2\rho(r)$, define $\psi_2(r), \overline{\psi}_2(r)$ as the radially symmetric C^2 solutions of

$$\begin{cases} -\Delta\psi_2 = \lambda f(\overline{w}) & \text{in } B(0, R) \\ -\Delta\overline{\psi}_2 = \lambda g(w) & \text{in } B(0, R) \\ \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial B(0, R). \end{cases} \tag{3.1}$$

Then $\psi_2,\ \overline{\psi}_2$ satisfies

$$-\psi_2'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f(\overline{w}(s))ds$$
$$-\overline{\psi}_2'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}g(w(s))ds.$$

Since $|\rho'(r)| \le \frac{\alpha\beta}{R-\epsilon}$, we have

$$|w'(r)| \le \frac{b_1\alpha\beta}{R-\epsilon},\ \ |\overline{w}'(r)| \le \frac{b_2\alpha\beta}{R-\epsilon}. \tag{3.2}$$

Note that for $0 < r \leq \epsilon$, clearly $\psi_2'(r) < w'(r)$ and $\overline{\psi}_2'(r) < \overline{w}'(r)$. Now for $r > \epsilon$,

$$\begin{aligned} -\psi_2'(r) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f(\overline{w}(s))ds \\ &\geq \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1} f(\overline{w}(s))ds \\ &\geq \frac{\lambda}{R^{N-1}} f(b_2) \frac{\epsilon^N}{N}. \end{aligned} \tag{3.3}$$

Similar calculations shows that

$$-\overline{\psi_2}'(r) \geq \frac{\lambda}{R^{N-1}} g(b_1) \frac{\epsilon^N}{N}. \tag{3.4}$$

Note that the $\inf_\epsilon \frac{N}{\epsilon^N} \frac{R^{N-1}}{R-\epsilon}$ is attained at $\epsilon_0 = \frac{N}{N+1} R$ hence $C_1(\Omega) = \frac{N}{\epsilon_0^N} \frac{R^{N-1}}{R-\epsilon_0}$. Since $\lambda > C_1(\Omega) \max\{\frac{b_2}{g(b_1)}, \frac{b_1}{f(b_2)}\}$, choose $\alpha, \beta > 1$ so that

$\lambda > \alpha\beta C_1(\Omega) \max\{\frac{b_2}{g(b_1)}, \frac{b_1}{f(b_2)}\}$. Hence $\lambda > \alpha\beta \frac{N}{\epsilon_0^N} \frac{R^{n-1}}{R-\epsilon_0} \frac{b_2}{g(b_1)}$ and by (3.4),

$$-\overline{\psi_2}'(r) \geq \frac{\lambda}{R^{N-1}} g(b_1) \frac{\epsilon_0^N}{N} > \frac{\alpha\beta b_2}{R-\epsilon_0} \geq -\overline{w}'(r).$$

That is, we have $\overline{\psi_2}'(r) < \overline{w}'(r); 0 < r \leq R$. Similarly we can establish $\psi_2'(r) < w'(r); 0 < r \leq R$ by using (3.3) and (3.2). Using the continuity argument on λ these inequalities hold for $\lambda > B$. Since $\psi_2(R) = \overline{\psi}_2(R) = 0 = w(R) = \overline{w}(R)$, it is easy to see that

$$\psi_2(r) > w(r), \overline{\psi}_2(r) > \overline{w}(r) \text{ for } 0 \leq r < R. \tag{3.5}$$

Now since f and g are nondecreasing, using (3.1) and (3.5) we have

$$\begin{cases} -\Delta\psi_2 = \lambda f(\overline{w}) \leq \lambda f(\overline{\psi}_2) & \text{in } B(0,R) \\ -\Delta\overline{\psi}_2 = \lambda g(w) \leq \lambda g(\psi_2) & \text{in } B(0,R) \\ \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial B(0,R) \end{cases} \tag{3.6}$$

and hence $(\psi_2, \overline{\psi}_2)$ is a positive sub-solution of (1.2). We also note that $\|\psi_2\|_\infty \geq \|w\|_\infty = b_1$ and $\|\overline{\psi}_2\|_\infty \geq \|\overline{w}\|_\infty = b_2$. Since $b_1 > a_1$ or $b_2 > a_2$, we have $(\psi_2, \overline{\psi}_2) \nleq (\phi_1, \overline{\phi}_1)$.

Finally, using $(H2)$, we will construct a large positive super-solution $(\phi_2, \overline{\phi}_2)$. If both f and g are bounded, let $(\phi_2, \overline{\phi}_2) = (\lambda C_\lambda \frac{e}{\|e\|_\infty}, \lambda C_\lambda \frac{e}{\|e\|_\infty})$ and choose C_λ so large that $\frac{C_\lambda}{\|e\|_\infty} > \max\{\|f\|_\infty, \|g\|_\infty\}$. Then it is easy to see that $(\phi_2, \overline{\phi}_2)$ is a positive supersolution of (1.2). Suppose $g(x) \to \infty$ as $x \to \infty$, let $(\phi_2, \overline{\phi}_2) = (C_\lambda e, \lambda g(C_\lambda \|e\|_\infty) e)$. Then by $(H2)$, choosing C_λ large we have

$$\frac{f\Big(\lambda \|e\|_\infty g\big(C_\lambda \|e\|_\infty\big)\Big)}{C_\lambda \|e\|_\infty} \leq \frac{1}{\lambda \|e\|_\infty}.$$

Thus we have $-\Delta \phi_2 = C_\lambda \geq \lambda f\Big(\lambda \|e\|_\infty g\big(C_\lambda \|e\|_\infty\big)\Big) \geq \lambda f\Big(\lambda g\big(C_\lambda \|e\|_\infty\big) e\Big) = \lambda f(\overline{\phi}_2)$. Also we have $-\Delta \overline{\phi}_2 = \lambda g\big(C_\lambda \|e\|_\infty\big) \geq \lambda g(\phi_2)$, showing that $(\phi_2, \overline{\phi}_2)$ is a supersolution of (1.2). (If $g(x)$ is bounded and $f(x) \to \infty$ as $x \to \infty$, then $\lim_{x \to \infty} \frac{g(Mf(x))}{x} = 0 \ \forall M > 0$ and we can prove that $(\phi_2, \overline{\phi}_2) = (\lambda f\big(C_\lambda \|e\|_\infty\big) e, C_\lambda e)$ is a supersolution.) Also since $e > 0$ in Ω and $\frac{\partial e}{\partial \eta} < 0$ on $\partial\Omega$, for C_λ large enough, in all the above cases we have $(\phi_2, \overline{\phi}_2) \geq (\phi_1, \overline{\phi}_1)$ and $(\phi_2, \overline{\phi}_2) \geq (\psi_2, \overline{\psi}_2)$. Now by Lemma 9, if $C_1 Q_2 < \frac{Q_1}{\|e\|_\infty}$ that is $Q_1/Q_2 > C(\Omega) := C_1 \|e\|_\infty$, then for all λ satisfying (1.3), (1.2) has a solution on each of the following components $[(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$, $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$. Clearly the solutions in $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$ are positive. Since $f(0)$ or $g(0)$ is strictly positive, the solutions in $[(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$ is also positive.

3.2 Proof of Theorem 1 (when Ω is a general bounded domain)

In this section we will prove the Theorem 1 when Ω is a general bounded domain. First we construct a positive sub-solution $(z, \overline{z})$ of (1.2) in Ω with $\|z\|_\infty \geq b_1$ and $\|\overline{z}\|_\infty \geq b_2$. Let B_R be the largest inscribed ball or radius R in Ω. Assume $\lambda \geq C_1(\Omega)Q_2$ and let $(\psi_2(r), \overline{\psi}_2(r))$ be the sub-solution of (1.2) constructed in B_R of the previous theorem.

Now define

$$z(x) = \begin{cases} \psi_2(|x|) \; ; & x \in B_R \\ 0 \; ; & x \in \Omega - B_R \end{cases}$$

and

$$\overline{z}(x) = \begin{cases} \overline{\psi}_2(|x|) \; ; & x \in B_R \\ 0 \; ; & x \in \Omega - B_R. \end{cases}$$

Then $z, \overline{z} \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ and $z = 0 = \overline{z}$ on $\partial\Omega$. Further, on B_R we have

$$\begin{cases} -\Delta z = -\Delta\psi_2 \leq \lambda f(\overline{\psi}_2) = \lambda f(\overline{z}) \\ -\Delta\overline{z} = -\Delta\overline{\psi}_2 \leq \lambda g(\psi_2) = \lambda g(z) \end{cases}$$

while outside B_R we have

$$\begin{cases} -\Delta z = 0 < \lambda f(0) = \lambda f(\overline{z}) \\ -\Delta\overline{z} = 0 < \lambda g(0) = \lambda g(z). \end{cases}$$

Hence $(z, \overline{z})$ is a sub-solution of (1.2) in Ω for $\lambda \geq C_1(\Omega)Q_2$ with $\|z\|_\infty \geq b_1$ and $\|\overline{z}\|_\infty \geq b_2$. The rest of the proof is identical to the previous case except that here for the second sub-solution we will use $(z, \overline{z})$ described above.

3.3 Proofs of Theorem 2 and 3

3.3.1 Proof of Theorem 2

Since $f(0) = g(0) = 0$, it is obvious to see that $(\psi_1, \overline{\psi}_1) = (0, 0)$ is a solution to (1.2), and constructing sub- and super-solutions as in the proof of Theorem 1, it is easy to see that (1.2) has at least two positive solutions in $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$.

3.3.2 Proof of Theorem 3

Since $f(0) = g(0) = 0$, we have the first sub-solution (solution) $(\psi_1, \overline{\psi}_1) = (0, 0)$ for (1.2) for any $\lambda > 0$. Now let $\lambda > C_1(\Omega) \inf_{r>0, s>0} \max\{\frac{r}{f(s)}, \frac{s}{g(r)}\}$ be fixed. Then there exists $b_1 > 0, b_2 > 0$ such that $\lambda > C_1(\Omega) Q_2(b_1, b_2)$. Hence we have the second positive sub-solution $(\psi_2, \overline{\psi}_2)$ of (1.2) (as discussed in the proof of Theorem 1) with $\|\psi_2\|_\infty \geq b_1$, $\|\overline{\psi}_2\|_\infty \geq b_2$. Since $f(0) = g(0) = 0 = f'(0) = g'(0)$, we have $\frac{x}{f(x)}$ and $\frac{x}{g(x)} \to \infty$ as $x \to 0$. Thus there exists $a < b_1$ (or b_2) such that $\big(\min\{\frac{a}{f(a)}, \frac{a}{g(a)}\}\big)/\|e\|_\infty > \lambda$. Let $(\phi_1, \overline{\phi}_1) = (a\frac{e}{\|e\|_\infty}, a\frac{e}{\|e\|_\infty})$. Then it is easy to see that $(\phi_1, \overline{\phi}_1)$ is a super-solution of (1.2). Note that $(\psi_2, \overline{\psi}_2) \nleq (\phi_1, \overline{\phi}_1)$ since $a < b_1$. Also using $(H2)$ as in the proof of Theorem 1 there exists a large positive super-solution $(\phi_2, \overline{\phi}_2)$ of (1.2) such that $(\phi_1, \overline{\phi}_1) \leq (\phi_2, \overline{\phi}_2)$ and $(\psi_2, \overline{\psi}_2) \leq (\phi_2, \overline{\phi}_2)$. Now by Lemma 9, (1.2) has at least two positive solutions in $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$.

CHAPTER 4

$n \times n$ SYSTEMS

In this chapter we will prove Theorems 7 - 10 and Corollaries 1 - 2. We prove Theorem 7 in Section 4.1, Theorem 8 and Corollaries 1 - 2 in Section 4.2. We establish Theorem 9 in Section 4.3 and Theorem 10 in section 4.4.

To prove our results we first define $f_i(x) = f(0), i \in \{1, \ldots, n\}$ for $x < 0$. Here sub-solution $(\psi_1, \psi_2, \ldots, \psi_n) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^n$ of (1.5) satisfies

$$\begin{cases} -\Delta\psi_1 \leq \lambda f_1(\psi_2) & \text{in } \Omega \\ -\Delta\psi_2 \leq \lambda f_2(\psi_3) & \text{in } \Omega \\ \quad\vdots \quad \leq \quad \vdots & \\ -\Delta\psi_{n-1} \leq \lambda f_{n-1}(\psi_n) & \text{in } \Omega \\ -\Delta\psi_n \leq \lambda f_n(\psi_1) & \text{in } \Omega \\ \psi_1 = \psi_2 = \cdots = \psi_n = 0 & \text{on } \partial\Omega \end{cases}$$

and super-solution $(z_1, z_2, \ldots, z_n) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^n$ of (1.5) satisfies

$$\begin{cases} -\Delta z_1 \geq \lambda f_1(z_2) & \text{in } \Omega \\ -\Delta z_2 \geq \lambda f_2(z_3) & \text{in } \Omega \\ \quad \vdots \quad \geq \quad \vdots & \\ -\Delta z_{n-1} \geq \lambda f_{n-1}(z_n) & \text{in } \Omega \\ -\Delta z_n \geq \lambda f_n(z_1) & \text{in } \Omega \\ z_1 = z_2 = \cdots = z_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Since (1.5) is a cooperative system, it is well known that if there exist a sub-solution $(\psi_1, \psi_2, \ldots, \psi_n)$ and a super-solution $(z_1, z_2, \ldots, z_n)$ which satisfy $(\psi_1, \psi_2, \ldots, \psi_n) \leq (z_1, z_2, \ldots, z_n)$, then there exists a solution $(u_1, u_2, \ldots, u_n)$ such that $(\psi_1, \psi_2, \ldots, \psi_n) \leq (u_1, u_2, \ldots, u_n) \leq (z_1, z_2, \ldots, z_n)$ (see [Am] and [Sa]).

Here we also state the $n \times n$ version of Lemma 9, which will be used in our proofs.

Lemma 17 (See [Am] and [Sh])

Let $f_i, i = 1, 2, \ldots, n$ be nonnegative and strictly increasing. Suppose there exist a sub-solution $(\psi_1, \psi_2, \ldots, \psi_n)$, a strict super-solution $(\zeta_1, \zeta_2, \ldots, \zeta_n)$, a strict sub-solution $(w_1, w_2, \ldots, w_n)$ and a super-solution $(z_1, z_2, \ldots, z_n)$ for (1.5) such that

$$(\psi_1, \psi_2, \ldots, \psi_n) \leq (\zeta_1, \zeta_2, \ldots, \zeta_n) \leq (z_1, z_2, \ldots, z_n),$$

$$(\psi_1, \psi_2, \ldots, \psi_n) \leq (w_1, w_2, \ldots, w_n) \leq (z_1, z_2, \ldots, z_n)$$

and $(w_1, w_2, \ldots, w_n) \nleq (\zeta_1, \zeta_2, \ldots, \zeta_n)$. Then (1.5) has at least three distinct solutions $(u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)}), i = 1, 2, 3$ such that

$$(u_1^{(1)}, u_2^{(1)}, \ldots, u_n^{(1)}) \in [(\psi_1, \psi_2, \ldots, \psi_n), (\zeta_1, \zeta_2, \ldots, \zeta_n)],$$

$$(u_1^{(2)}, u_2^{(2)}, \ldots, u_n^{(2)}) \in [(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)]$$

and $(u_1^{(3)}, u_2^{(3)}, \ldots, u_n^{(3)}) \in K$ where

$$K := [(\psi_1, \psi_2, \ldots, \psi_n), (z_1, z_2, \ldots, z_n)] \setminus \Big([(\psi_1, \psi_2, \ldots, \psi_n), (\zeta_1, \zeta_2, \ldots, \zeta_n)] \cup [(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)] \Big).$$

4.1 Proof of Theorem 7

In this section we prove Theorem 7. It is easy to see that $(\psi_1, \psi_2, \ldots, \psi_n) = (0, 0, \ldots, 0)$ is a strict sub-solution of (1.5). We now construct the super-solution $(z_1, z_2, \ldots, z_n)$.

Let $\alpha_k,\ k \in \{2, 3, \ldots, n\}$ be defined by the following recursive relation:

$$\begin{cases} \alpha_n = f_n(c_\lambda \|e\|_\infty) \\ \alpha_{j-1} = f_{j-1}^{[\lambda\|e\|_\infty]}(\alpha_j),\ j = 3, 4, \ldots, n. \end{cases}$$

Let $(z_1, z_2, \cdots, z_k, \cdots, z_n) := \Big(c_\lambda, \lambda\alpha_2, \lambda\alpha_3, \ldots, \lambda\alpha_n\Big)e.$ We note that by choosing c_λ large and using (H4) we get

$$\begin{aligned} -\Delta z_1 = c_\lambda &\geq \lambda f_1^{[\lambda\|e\|_\infty]} \circ f_2^{[\lambda\|e\|_\infty]} \circ \cdots f_{n-1}^{[\lambda\|e\|_\infty]} \circ f_n(c_\lambda\|e\|_\infty) \\ &= \lambda f_1(\lambda\|e\|_\infty \alpha_2) \\ &\geq \lambda f_1(z_2). \end{aligned}$$

We also observe that for $k = 2, \ldots, n-1$ we have

$$
\begin{aligned}
-\Delta z_k &= \lambda \alpha_k \\
&= \lambda f_k^{[\lambda \|e\|_\infty]}(\alpha_{k+1}) \\
&= \lambda f_k(\lambda \|e\|_\infty \alpha_{k+1}) \\
&\geq \lambda f_k(z_{k+1}).
\end{aligned}
$$

Thus we see that $-\Delta z_k \geq \lambda f_k(z_{k+1})$ for $k = 1, \ldots, n-1$. Finally

$$
\begin{aligned}
-\Delta z_n &= \lambda \alpha_n \\
&= \lambda f_n(c_\lambda \|e\|_\infty) \\
&\geq \lambda f_n(z_1).
\end{aligned}
$$

Hence $(z_1, \cdots, z_n)$ is a super-solution of (1.5). Thus, there exists a solution $(u_1, u_2, \ldots, u_n)$ of (1.5) with $\psi_i \leq u_i \leq z_i, i \in \{1, \ldots, n\}$. Since $(\psi_1, \psi_2, \ldots, \psi_n) = (0, 0, \ldots, 0)$ is a strict sub-solution, it is easy to see that $(u_1, u_2, \ldots, u_n)$ is a positive solution. This completes the proof of Theorem 7.

4.2 Proofs of Theorem 8 and Corollaries 1 - 2

In this section we prove Theorem 8 and Corollaries 1 - 2. We prove Theorem 8 when Ω is a ball in Section 4.2.1. Here the crucial part of the proof relies on establishing the strict sub-solution $(w_1, w_2, \ldots, w_n)$. We produce such a $(w_1, w_2, \ldots, w_n)$ with the help of radially symmetric properties of the solutions. We use the results in Section 4.2.1 to prove the Theorem 8 for the general bounded Ω in Section 4.2.2. In Section 4.2.3, we prove

Corollaries 1 - 2. Corollary 1 easily follows from Theorem 8. We show the Corollary 2 by using Theorem 8 and the conditions on f_i's and f_i''s at zero.

4.2.1 Proof of Theorem 8 (when Ω is a ball of radius R)

We will establish a pair of sub-solutions $(\psi_1, \psi_2, \ldots, \psi_n), (w_1, w_2, \ldots, w_n)$ and a pair of super-solutions $(\zeta_1, \zeta_2, \ldots, \zeta_n), (z_1, z_2, \ldots, z_n)$ satisfying $n \times n$ version of Lemma 9, namely Lemma 17. Clearly $(\psi_1, \psi_2, \ldots, \psi_n) = (0, 0, \ldots, 0)$ is a strict sub-solution of (1.5) since $f_i(0) \geq 0, \ \forall i \in \{1, 2, \ldots, l-1, l+1, \ldots, n\}$ and $f_l(0) > 0$ for some $l \in \{1, 2, \ldots, n\}$.

We next construct a strict positive super-solution $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ of (1.5) when $\lambda < \frac{1}{\|e\|_\infty} \min\{\frac{a_1}{f_1(a_2)}, \ \frac{a_2}{f_2(a_3)}, \ldots, \frac{a_k}{f_k(a_{k+1})}, \ldots, \frac{a_{n-1}}{f_{n-1}(a_n)}, \frac{a_n}{f_n(a_1)}\} = A$ (say). Let $(\zeta_1, \zeta_2, \ldots, \zeta_n) = (a_1 \frac{e}{\|e\|_\infty}, a_2 \frac{e}{\|e\|_\infty}, \ldots, a_n \frac{e}{\|e\|_\infty})$. Since $\lambda < \frac{1}{\|e\|_\infty} \frac{a_1}{f(a_2)}$, we have $-\Delta\zeta_1 = \frac{a_1}{\|e\|_\infty} > \lambda f(a_2) \geq \lambda f(\zeta_2)$. Similar argument shows that ζ_k satisfies $-\Delta\zeta_k > \lambda f_k(\zeta_{k+1}), i \in \{2, 3, \ldots, n-1\}$ and $-\Delta\zeta_n > \lambda f_n(\zeta_1)$. This proves that $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is a positive strict super-solution of (1.5). Note that $\|\zeta_i\|_\infty = a_i, i \in \{1, 2, \ldots, n\}$.

Now we construct a positive strict sub-solution $(w_1, w_2, \ldots, w_n)$ of (1.5) when $\lambda > C_1(\Omega) \max\{\frac{b_1}{f_1(b_2)}, \ \frac{b_2}{f_2(b_3)}, \ldots, \frac{b_k}{f_k(b_{k+1})}, \ldots, \frac{b_{n-1}}{f_{n-1}(b_n)}, \frac{b_n}{f_n(b_1)}\} = B$ (say). For $0 < \epsilon < R$, $\alpha, \beta > 1$, define $\rho(r) : [0, R] \rightarrow [0, 1]$ by

$$\rho(r) = \begin{cases} 1, & 0 \leq r \leq \epsilon \\ 1 - (1 - (\frac{R-r}{R-\epsilon})^\beta)^\alpha & \epsilon < r \leq R. \end{cases}$$

Then

$$\rho'(r) = \begin{cases} 0, & 0 \le r \le \epsilon \\ -\frac{\alpha\beta}{R-\epsilon}(1-(\frac{R-r}{R-\epsilon})^{\beta})^{\alpha-1}(\frac{R-r}{R-\epsilon})^{\beta-1} & \epsilon < r \le R \end{cases}$$

and hence $|\rho'(r)| \le \frac{\alpha\beta}{R-\epsilon}$.

Let $\xi_i(r) = b_i\rho(r), i \in \{1, 2, \ldots, n\}$. Note that

$$|\xi_i'(r)| \le \frac{b_i\alpha\beta}{R-\epsilon}, \quad i \in \{1, 2, \ldots, n\}. \tag{4.1}$$

Define $w_1(r), w_2(r), \ldots, w_n(r)$ as the radially symmetric C^2 solutions of

$$\begin{cases} -\Delta w_1 = \lambda f_1(\xi_2) & \text{in } B(0,R) \\ -\Delta w_2 = \lambda f_2(\xi_3) & \text{in } B(0,R) \\ \vdots = \vdots & \\ -\Delta w_{n-1} = \lambda f_{n-1}(\xi_n) & \text{in } B(0,R) \\ -\Delta w_n = \lambda f_1(\xi_1) & \text{in } B(0,R) \\ w_1 = w_2 = ... = w_n = 0 & \text{on } \partial B(0,R). \end{cases} \tag{4.2}$$

Then $w_1(r), w_2(r), \ldots, w_n(r)$ satisfy

$$-w_1'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f_1(\xi_2)ds$$
$$-w_2'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f_2(\xi_3)ds$$
$$\vdots = \vdots$$
$$-w_{n-1}'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f_{n-1}(\xi_n)ds$$
$$-w_n'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1}f_n(\xi_1)ds.$$

Note that for $0 < r \le \epsilon$, clearly $w_i'(r) < \xi_i'(r), i \in \{1, 2, \ldots, n\}$. Now for $r > \epsilon$,

$$\begin{aligned} -w_1'(r) &= \frac{\lambda}{r^{N-1}} \int_0^r s^{N-1} f_1(\xi_2(s))ds \\ &\ge \frac{\lambda}{r^{N-1}} \int_0^\epsilon s^{N-1} f_1(\xi_2(s))ds \\ &\ge \frac{\lambda}{R^{N-1}} f_1(b_2) \frac{\epsilon^N}{N}. \end{aligned} \tag{4.3}$$

Similar calculations shows that

$$-w_2'(r) \ge \frac{\lambda}{R^{N-1}} f_2(b_3) \frac{\epsilon^N}{N} \tag{4.4}$$

$$\vdots \quad \ge \quad \vdots \tag{4.5}$$

$$-w_{n-1}'(r) \ge \frac{\lambda}{R^{N-1}} f_{n-1}(b_n) \frac{\epsilon^N}{N} \tag{4.6}$$

$$-w_n'(r) \ge \frac{\lambda}{R^{N-1}} f_n(b_1) \frac{\epsilon^N}{N}. \tag{4.7}$$

Since $\lambda > C_1(\Omega) \max\{\frac{b_1}{f_1(b_2)}, \frac{b_2}{f_2(b_3)}, \ldots, \frac{b_k}{f_k(b_{k+1})}, \ldots, \frac{b_{n-1}}{f_{n-1}(b_n)}, \frac{b_n}{f_n(b_1)}\}$, choose $\alpha, \beta > 1$ so that $\lambda > \alpha\beta C_1(\Omega) \max\{\frac{b_1}{f_1(b_2)}, \frac{b_2}{f_2(b_3)}, \ldots, \frac{b_{n-1}}{f_{n-1}(b_n)}, \frac{b_n}{f_n(b_1)}\}$. Note that $C_1(\Omega) = \frac{N}{\epsilon_0^N} \frac{R^{N-1}}{R-\epsilon_0}$ where $\epsilon_0 = \frac{N}{N+1}R$. Choosing $\epsilon = \epsilon_0$, $\lambda > \alpha\beta \frac{N}{\epsilon_0^N} \frac{R^{N-1}}{R-\epsilon_0} \frac{b_1}{f_1(b_2)}$ and by (4.1) and (4.3) we have

$$-w_1'(r) \ge \frac{\lambda}{R^{N-1}} f_1(b_2) \frac{\epsilon_0^N}{N} > \frac{\alpha\beta b_1}{R - \epsilon_0} \ge -\xi_1'(r).$$

That is, we have $w_1'(r) < \xi_1'(r); 0 < r \le R$. Similarly we can establish $w_2'(r) < \xi_2'(r), \ldots, w_n'(r) < \xi_n'(r); 0 < r \le R$ by using (4.4)-(4.7) and (4.1). Since $w_1(R) = w_2(R) = \cdots = w_n(R) = 0 = \xi_1(R) = \xi_2(R) = \cdots = \xi_n(R)$, it is easy to see that

$$w_i > \xi_i, \ i \in \{1, 2, \ldots, n\} \text{ for } 0 \le r < R. \tag{4.8}$$

Now since $f_i, i \in \{1, 2, \ldots, n\}$ are increasing, using (4.2) and (4.8) we have

$$
\begin{cases}
-\Delta w_1 = \lambda f_1(\zeta_2) < \lambda f_1(w_2) & \text{in } B(0,R) \\
-\Delta w_2 = \lambda f_2(\zeta_3) < \lambda f_2(w_3) & \text{in } B(0,R) \\
\quad \vdots \quad = \quad \vdots & \\
-\Delta w_{n-1} = \lambda f_{n-1}(\zeta_n) < \lambda f_{n-1}(w_n) & \text{in } B(0,R) \\
-\Delta w_n = \lambda f_n(\zeta_1) < \lambda f_n(w_1) & \text{in } B(0,R) \\
w_1 = w_2 = \cdots = w_n = 0 & \text{on } \partial B(0,R),
\end{cases}
$$

and hence $(w_1, w_2, \ldots, w_n)$ is a strict positive sub-solution of (1.5). We also note that $\|w_i\|_\infty \geq \|\xi_i\|_\infty = b_i, i \in \{1, 2, \ldots, n\}$. Since at least one of the $b_k > a_k$ we have $(w_1, w_2, \ldots, w_n) \nleq (\zeta_1, \zeta_2, \ldots, \zeta_n)$.

Let $(z_1, z_2, \cdots, z_k, \cdots, z_n)$ be the super solution as in the proof of Theorem 7. Further $z_i \geq w_i, z_i \geq \zeta_i, i \in \{1, 2, \ldots, n\}$ for c_λ large (assuming $f_i(x) \to \infty$ as $x \to \infty, i \in \{2, \ldots, n\}$, if any of the functions f_i's are bounded, then we consider systems with $\tilde{f}_i$'s such that $\tilde{f}_i \geq f_i$, $\tilde{f}_i \to \infty$ and satisfy (H4) to create the super-solution). Hence there exist positive solutions $(u_1^{(i)}, u_2^{(i)}, \ldots, u_n^{(i)})$, $i = 1, 2, 3$ such that

$$(u_1^{(1)}, u_2^{(1)}, \ldots, u_n^{(1)}) \in [(\psi_1, \psi_2, \ldots, \psi_n), (\zeta_1, \zeta_2, \ldots, \zeta_n)],$$

$$(u_1^{(2)}, u_2^{(2)}, \ldots, u_n^{(2)}) \in [(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)]$$

and $(u_1^{(3)}, u_2^{(3)}, \ldots, u_n^{(3)}) \in K$ where

$$K := [(\psi_1, \psi_2, \ldots, \psi_n), (z_1, z_2, \ldots, z_n)] \setminus \big([(\psi_1, \psi_2, \ldots, \psi_n), (\zeta_1, \zeta_2, \ldots, \zeta_n)] \cup [(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)]\big).$$

4.2.2 Proof of Theorem 8 (when Ω is a general bounded domain)

In this section we will prove the Theorem 8 when Ω is a general bounded domain. First we construct a strict positive sub-solution $(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)$ of (1.5) in Ω with $\|\overline{w}_i\|_\infty \geq b_i, i \in \{1, 2, \ldots, n\}$. Let B_R be the largest inscribed ball of radius R in Ω. Assume $\lambda > C_1(\Omega)Q_2$ and let $(w_1(r), w_2(r), \ldots, w_n(r))$ be the strict positive sub-solution of (1.5) constructed in B_R of the previous section. Now define

$$\overline{w}_1(x) = \begin{cases} w_1(|x|)\,; & x \in B_R \\ 0 \qquad ; & x \in \Omega - B_R \end{cases}$$

$$\overline{w}_2(x) = \begin{cases} w_2(|x|)\,; & x \in B_R \\ 0 \qquad ; & x \in \Omega - B_R \end{cases}$$

$$\vdots$$

$$\overline{w}_n(x) = \begin{cases} w_n(|x|)\,; & x \in B_R \\ 0 \qquad ; & x \in \Omega - B_R. \end{cases}$$

Then $\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ and $\overline{w}_1 = \overline{w}_2 = \cdots = \overline{w}_n = 0$ on $\partial\Omega$. Further, on B_R we have

$$\begin{cases} -\Delta\overline{w}_1 = -\Delta w_1 < \lambda f_1(w_2) = \lambda f_1(\overline{w}_2) \\ -\Delta\overline{w}_2 = -\Delta w_2 < \lambda f_2(w_3) = \lambda f_2(\overline{w}_3) \\ \qquad \vdots \\ -\Delta\overline{w}_{n-1} = -\Delta w_3 < \lambda f_{n-1}(w_n) = \lambda f_{n-1}(\overline{w}_n) \\ -\Delta\overline{w}_n = -\Delta w_n < \lambda f_n(w_1) = \lambda f_n(\overline{w}_1) \end{cases}$$

while outside B_R we have

$$\begin{cases} -\Delta\overline{w}_1 = 0 \leq \lambda f_1(0) = \lambda f_1(\overline{w}_2) \\ -\Delta\overline{w}_2 = 0 \leq \lambda f_2(0) = \lambda f_2(\overline{w}_3) \\ \vdots \\ -\Delta\overline{w}_l = 0 < \lambda f_l(0) = \lambda f_l(\overline{w}_{l+1}) \\ \vdots \\ -\Delta\overline{w}_{n-1} = 0 \leq \lambda f_{n-1}(0) = \lambda f_{n-1}(\overline{w}_n) \\ -\Delta\overline{w}_n = 0 \leq \lambda f_n(0) = \lambda f_n(\overline{w}_1). \end{cases}$$

Hence $(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)$ is a strict positive sub-solution of (1.5) in Ω for $\lambda > C_1(\Omega)Q_2$ with $\|\overline{w}_i\|_\infty \geq b_i, i \in \{1, 2, \ldots, n\}$. The rest of the proof is identical to the previous case except that here for the second sub-solution we will use $(\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_n)$ described above.

4.2.3 Proofs of Corollaries 1 - 2

4.2.3.1 Proof of Corollary 1

Since $f_i(0) = 0, i \in \{1, 2, \ldots, n\}$, it is obvious to see that $(\psi_1, \psi_2, \ldots, \psi_n) = (0, 0, \ldots, 0)$ is a solution to (1.5), and constructing sub and super-solutions as in Theorem 8, it is easy to see that (1.5) has at least two positive solutions in the following components $[(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)]$ and

$$[(\psi_1, \psi_2, \ldots, \psi_n), (z_1, z_2, \ldots, z_n)] \setminus \big([(\psi_1, \psi_2, \ldots, \psi_n), (\zeta_1, \zeta_2, \ldots, \zeta_n)] \\ \cup [(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)]\big).$$

4.2.3.2 Proof of Corollary 2

Since $f_i(0) = 0$, we have the first sub-solution (solution) $(\psi_1, \psi_2, \ldots, \psi_n) = (0, 0, \ldots, 0)$ for (1.5) for any $\lambda > 0$.

Now let $\lambda > C_1(\Omega) \inf_{r_1, r_2, \ldots, r_n > 0} \max\{\frac{r_1}{f_1(r_2)}, \frac{r_2}{f_2(r_3)}, \ldots, \frac{r_k}{f_k(r_{k+1})}, \ldots, \frac{r_n}{f_n(r_1)}\}$ be fixed. Then there exists $b_1 > 0, b_2 > 0, \ldots, b_n > 0$ such that $\lambda > C_1(\Omega) Q_2(b_1, \ldots, b_n)$. Hence we have the second positive sub-solution $(w_1, w_2, \ldots, w_n)$ of (1.5) (as discussed in the proof of Theorem 8) with $\|w_i\|_\infty \geq b_i, i \in \{1, 2, \ldots, n\}$. Since $f_i(0) = 0 = f_i'(0)$, we have $\frac{x}{f_i(x)} \to \infty$ as $x \to 0$ for every $i \in \{1, 2, \ldots, n\}$. Thus there exists $a < b_1$ (or one of b_i's $i \in \{2, \ldots, n\}$) such that $\big(\min\{\frac{a}{f_1(a)}, \ldots, \frac{a}{f_n(a)}\}\big)/\|e\|_\infty > \lambda$. Let $(\zeta_1, \zeta_2, \ldots, \zeta_n) = (a\frac{e}{\|e\|_\infty}, a\frac{e}{\|e\|_\infty}, \ldots, a\frac{e}{\|e\|_\infty})$. Then it is easy to see that $(\zeta_1, \zeta_2, \ldots, \zeta_n)$ is a super-solution of (1.5). Note that $(w_1, w_2, \ldots, w_n) \not\leq (\zeta_1, \zeta_2, \ldots, \zeta_n)$ since $a < b_1$. Also using $(H4)$ as in the proof of Theorem 8 there exists a large positive super-solution $(z_1, z_2, \ldots, z_n)$ of (1.5) such that $(\zeta_1, \zeta_2, \ldots, \zeta_n) \leq (z_1, z_2, \ldots, z_n)$ and $(w_1, w_2, \ldots, w_n) \leq (z_1, z_2, \ldots, z_n)$. Now by Lemma 17, (1.5) has at least two positive solutions in the following components $[(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)]$ and

$$[(\psi_1, \psi_2, \ldots, \psi_n), (z_1, z_2, \ldots, z_n)] \backslash \big([(\psi_1, \psi_2, \ldots, \psi_n), (\zeta_1, \zeta_2, \ldots, \zeta_n)] \cup [(w_1, w_2, \ldots, w_n), (z_1, z_2, \ldots, z_n)]\big).$$

4.3 Proof of Theorem 9

Let $\lambda_1 > 0$ be the principal eigenvalue and $\varphi > 0$ be with $\|\varphi\|_\infty = 1$ the corresponding eigenfunction of $-\Delta$ with the Dirichlet boundary conditions. It is well known that $\frac{\partial \varphi}{\partial \eta} < 0$ on $\partial\Omega$ where η is the unit outward normal. Hence there exists $\delta, m, \mu > 0$ such that

$$|\nabla \varphi|^2 - \lambda_1 \varphi^2 \geq m \text{ on } \overline{\Omega}_\delta \tag{4.9}$$

$$\varphi \geq \mu \text{ on } \Omega - \overline{\Omega}_\delta \tag{4.10}$$

where $\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) < \delta\}$. Let $k_0 > 0$ be such that $f_i(x) \geq -k_0$ for all $x \geq 0$ and for all $i \in \{1, \cdots, n\}$.

We shall verify that

$$(\psi_1, \psi_2, \cdots, \psi_n) := \Big(\Big[\frac{\lambda k_0}{2m}\Big]\varphi^2, \Big[\frac{\lambda k_0}{2m}\Big]\varphi^2, \cdots, \Big[\frac{\lambda k_0}{2m}\Big]\varphi^2 \Big)$$

is a sub-solution of (1.5) for λ large.

Let $\psi = \Big[\frac{\lambda k_0}{2m}\Big]\varphi^2$. Now we observe that

$$\begin{aligned} -\Delta\psi &= -\nabla \cdot \nabla \Big(\Big[\frac{\lambda k_0}{2m}\Big]\varphi^2 \Big) \\ &= -\nabla \cdot \Big(\Big[\frac{\lambda k_0}{m}\Big]\varphi \nabla \varphi \Big) \\ &= -\Big[\frac{\lambda k_0}{m}\Big](\nabla\varphi \cdot \nabla\varphi + \varphi\Delta\varphi) \\ &= \Big[\frac{\lambda k_0}{m}\Big](\lambda_1\varphi^2 - |\nabla\varphi|^2). \end{aligned} \tag{4.11}$$

Note that on $\overline{\Omega}_\delta$ we have $|\nabla\varphi|^2 - \lambda_1\varphi^2 \geq m$. Therefore on $\overline{\Omega}_\delta$ we have

$$\begin{aligned} -\Delta\psi &= \Big[\frac{\lambda k_0}{m}\Big](\lambda_1\varphi^2 - |\nabla\varphi|^2) \\ &\leq -\lambda k_0 \leq \lambda f_i(\psi) \quad \text{for any } i \in \{1, 2, \cdots, n\}. \end{aligned}$$

Also on $\Omega - \overline{\Omega}_\delta$ since $\phi \geq \mu$ for some $0 < \mu < 1$, for λ large, we have $f_i(\psi) \geq \frac{k_0 \lambda_1}{m}, i \in \{1, \cdots, n\}$ since $\lim_{x \to \infty} f_i(x) = \infty, i \in \{1, 2, \ldots, n\}$. Hence

$$\begin{aligned}
-\Delta\psi &= \Big[\frac{\lambda k_0}{m}\Big](\lambda_1 \varphi^2 - |\nabla\varphi|^2) \\
&\leq \Big[\frac{\lambda k_0}{m}\Big]\lambda_1 \varphi^2 \\
&< \lambda\Big[\frac{k_0 \lambda_1}{m}\Big] \\
&\leq \lambda f_i(\psi) \quad \text{for any } i \in \{1, \cdots, n\},
\end{aligned}$$

i.e. $-\Delta\psi \leq \lambda f_i(\psi)$ on Ω for $i \in \{1, \cdots, n\}$. Therefore clearly that $(\psi_1, \psi_2, \cdots, \psi_n) := \left(\Big[\frac{\lambda k_0}{2m}\Big]\varphi^2, \Big[\frac{\lambda k_0}{2m}\Big]\varphi^2, \cdots, \Big[\frac{\lambda k_0}{2m}\Big]\varphi^2\right)$ is a sub-solution of (1.5).

We choose the super-solution $(z_1, z_2, \ldots, z_n)$ as in proof of Theorem 7. Since $e > 0$ in Ω and $\frac{\partial e}{\partial \eta} < 0$ on $\partial\Omega$, by choosing c_λ large we have $(z_1, z_2, \ldots, z_n) \geq (\psi_1, \psi_2, \cdots, \psi_n)$. Note that this is possible since $\lim_{x \to \infty} f_i(x) = \infty, \forall i \in \{2, \ldots, n\}$. Thus, there exists a positive solution $(u_1, u_2, \ldots, u_n)$ of (1.5) with $\psi_i \leq u_i \leq z_i, i \in \{1, \ldots, n\}$. Since $u_i(x) \geq \frac{\lambda k_0}{2m}[\varphi(x)]^2, u_i(x) \to \infty,\ \forall i \in \{1, 2, \ldots, n\}$ as $\lambda \to \infty$ $\forall x \in \Omega$. Hence Theorem 9 is proven.

4.4 Proof of Theorem 10

We first note that $(\psi_1, \psi_2, \ldots, \psi_n) = (0, 0, \ldots, 0)$ is a sub-solution (indeed a solution). As in Section 4.1, we can always construct a large super-solution $(z_1, z_2, \ldots, z_n)$. We next consider

$$
\begin{cases}
-\Delta w_1 = \lambda \tilde{f}_1(w_2) & \text{in } \Omega \\
-\Delta w_2 = \lambda \tilde{f}_2(w_3) & \text{in } \Omega \\
\vdots = \vdots & \\
-\Delta w_{n-1} = \lambda \tilde{f}_{n-1}(w_n) & \text{in } \Omega \\
-\Delta w_n = \lambda \tilde{f}_n(w_1) & \text{in } \Omega \\
w_1 = w_2 = ... = w_n = 0 & \text{on } \partial\Omega
\end{cases}
\tag{4.12}
$$

where $\tilde{f}_i(s) = f_i(s) - 1, i \in \{1, \dots, n\}$. Then by Theorem 9, (4.12) has a positive solution $(w_1, w_2, \dots, w_n)$ when λ is large. Clearly this $(w_1, w_2, \dots, w_n)$ is a strict sub-solution of (1.5).

Finally we construct the strict super-solution $(\zeta_1, \zeta_2, \dots, \zeta_n)$. Let φ be as described in Section 4. Let $(\zeta_1, \zeta_2, \dots, \zeta_n) = (\epsilon\varphi, \epsilon\varphi, \dots, \epsilon\varphi)$ where $\epsilon > 0$ and $H_i(x) := \lambda_1 x - \lambda f_i(x), i \in \{1, \dots, n\}$. Observe that $H_i(0) = 0, i \in \{1, \dots, n\}$. Since $H_i'(0) = \lambda_1 > 0$, there exists θ such that $H_i(x) > 0$ for $x \in (0, \theta], i \in \{1, 2, \dots, n\}$. Hence for $0 < \epsilon \leq \theta$, we have $\lambda_1(\epsilon\varphi) > \lambda f_i(\epsilon\varphi), x \in \Omega$. Therefore

$$
-\Delta(\zeta_i) = \lambda_1(\epsilon\varphi) > \lambda f_i(\zeta_{i+1}), i \in \{1, \dots, n-1\}, x \in \Omega \tag{4.13}
$$

$$
-\Delta(\zeta_n) = \lambda_1(\epsilon\varphi) > \lambda f_n(\zeta_1), x \in \Omega \tag{4.14}
$$

Thus $(\zeta_1, \zeta_2, \dots, \zeta_n)$ is a strict super-solution. Here we can choose ϵ small so that $(w_1, w_2, \dots, w_3) \not\leq (\zeta_1, \zeta_2, \dots, \zeta_n)$. Hence by Lemma 17, there exists three distinct solutions $(u_1^{(i)}, u_2^{(i)}, \dots, u_n^{(i)})$, $i = 1, 2, 3$ such that

$$
(u_1^{(1)}, u_2^{(1)}, \dots, u_n^{(1)}) \in [(\psi_1, \psi_2, \dots, \psi_n), (\zeta_1, \zeta_2, \dots, \zeta_n)],
$$

$$(u_1^{(2)}, u_2^{(2)}, \dots, u_n^{(2)}) \in [(w_1, w_2, \dots, w_n), (z_1, z_2, \dots, z_n)]$$

and $(u_1^{(3)}, u_2^{(3)}, \dots, u_n^{(3)}) \in K$ where

$$K := [(\psi_1, \psi_2, \dots, \psi_n), (z_1, z_2, \dots, z_n)] \setminus \big([(\psi_1, \psi_2, \dots, \psi_n), (\zeta_1, \zeta_2, \dots, \zeta_n)] \\ \cup [(w_1, w_2, \dots, w_n), (z_1, z_2, \dots, z_n)] \big).$$

Since $(\psi_1, \dots, \psi_n) \equiv (0, \dots, 0)$ is a solution it may turn out that $(u_1^{(1)}, u_2^{(1)}, \dots, u_n^{(1)}) \equiv (\psi_1, \psi_2, \dots, \psi_n) \equiv (0, 0, \dots, 0)$. In any case we have two solutions $(u_1^{(2)}, u_2^{(2)}, \dots, u_n^{(2)})$ and $(u_1^{(3)}, u_2^{(3)}, \dots, u_n^{(3)})$ that are positive. We note that $u_j^{(2)}(x_0) \to \infty$ as $\lambda \to \infty$ $\forall j = 1, 2, \dots, n,\ \forall x_0 \in \Omega$ since $w_j(x_0) \to \infty$ as $\lambda \to \infty$ $\forall j = 1, 2, \dots, n,\ \forall x_0 \in \Omega$. Hence Theorem 10 holds.

CHAPTER 5

MULTIPARAMETER p-q-LAPLACIAN SYSTEMS

In this chapter proofs of Theorems 15 - Theorem 17 will be presented. In Section 5.1, proof of Theorem 15 will be discussed, in Section 5.2, Theorem 16 will be discussed. Proof of Theorem 17 for the case when Ω is a ball will be presented in Section 5.3 and the proof of Theorem 17 for the case when Ω is a general bounded domain will follow in Section 5.4

To establish our results, we first define $f(x) = f(0), g(x) = g(0), h(x) = h(0)$ and $\gamma(x) = \gamma(0)$ for $x < 0$. We shall establish Theorem 15 by constructing a positive weak sub-solution $(\psi, \overline{\psi}) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ and a weak super-solution $(\phi, \overline{\phi}) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ of (1.11) such that $\psi \leq \phi$ and $\overline{\psi} \leq \overline{\phi}$. Here our weak sub-solution $(\psi, \overline{\psi})$ and weak super-solution $(\phi, \overline{\phi})$ satisfy $(\psi, \overline{\psi}) = (0, 0) = (\phi, \overline{\phi})$ on $\partial\Omega$ and

$$\begin{cases} \int_\Omega |\nabla \psi|^{p-2} \nabla \psi \cdot \nabla \omega dx \leq \tau_1 \int_\Omega f(\overline{\psi}) \omega dx + \mu_1 \int_\Omega h(\psi) \omega dx \\ \int_\Omega |\nabla \overline{\psi}|^{p-2} \nabla \overline{\psi} \cdot \nabla \omega dx \leq \tau_2 \int_\Omega g(\psi) \omega dx + \mu_2 \int_\Omega \gamma(\overline{\psi}) \omega dx \end{cases}$$

$$\begin{cases} \int_\Omega |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla \omega dx \geq \tau_1 \int_\Omega f(\overline{\phi}) \omega dx + \mu_1 \int_\Omega h(\phi) \omega dx \\ \int_\Omega |\nabla \overline{\phi}|^{p-2} \nabla \overline{\phi} \cdot \nabla \omega dx \geq \tau_2 \int_\Omega g(\phi) \omega dx + \mu_2 \int_\Omega \gamma(\overline{\phi}) \omega dx \end{cases}$$

for all $\omega \in W := \{\eta \in C_0^\infty(\Omega) : \eta \geq 0 \text{ in } \Omega\}$.

Since (1.11) is a cooperative system, suppose there exists a sub-solution $(\psi, \overline{\psi})$ and a super-solution $(\phi, \overline{\phi})$ for (1.11) satisfying $(\psi, \overline{\psi}) \leq (\phi, \overline{\phi})$, then (1.11) has a solution (u, v) such that $\psi \leq u \leq \phi$ and $\overline{\psi} \leq u \leq \overline{\phi}$.

5.1 Proof of Theorem 15

Let σ_p, σ_q be the respective first eigenvalues of $-\Delta_p, -\Delta_q$ with Dirichlet boundary conditions and φ_p, φ_q be the corresponding eigenfunctions with $\varphi_p, \varphi_q > 0$ on Ω and $\|\varphi_p\|_\infty = 1 = \|\varphi_q\|_\infty$. Let $k_0, m, \delta > 0$ be such that $f(x), g(x), h(x), \gamma(x) \geq -k_0$ for all $x \geq 0$ and $|\nabla\varphi_p|^p - \sigma_p\varphi_p^p \geq m$, $|\nabla\varphi_q|^q - \sigma_q\varphi_q^q \geq m$ on $\overline{\Omega}_\delta = \{x \in \Omega | d(x, \partial\Omega) \leq \delta\}$. (This is possible since $|\nabla\varphi_s| \neq 0$ on $\partial\Omega$ while $\varphi_s = 0$ on $\partial\Omega$ for $s = p, q$). We shall verify that

$$(\psi, \overline{\psi}) := \Big(\big[\frac{(\tau_1 + \mu_1)}{m} k_0\big]^{1/p-1} \big(\frac{p-1}{p}\big) \varphi_p^{p/p-1}, \big[\frac{(\tau_2 + \mu_2)}{m} k_0\big]^{1/q-1} \big(\frac{q-1}{q}\big) \varphi_q^{q/q-1} \Big)$$

is a sub-solution of (1.11) for $\tau_1 + \mu_1$ and $\tau_2 + \mu_2$ large. Let $\omega \in W$, calculation shows that if $\psi = \big(\frac{Ak_0}{m}\big)^{1/s-1} \big(\frac{s-1}{s}\big) \varphi_s^{s/s-1}$ where A is a positive constant then

$$\begin{aligned} \int_\Omega |\nabla\psi|^{s-2} \nabla\psi \cdot \nabla\omega dx &= \Big(\frac{Ak_0}{m}\Big) \int_\Omega \varphi_s |\nabla\varphi_s|^{s-2} \nabla\varphi_s \cdot \nabla\omega dx \\ &= \Big(\frac{Ak_0}{m}\Big) \Big\{ \int_\Omega |\nabla\varphi_s|^{s-2} \nabla\varphi_s \cdot \nabla(\varphi_s\omega) dx - \int_\Omega |\nabla\varphi_s|^s \omega dx \Big\} \\ &= \Big(\frac{Ak_0}{m}\Big) \Big\{ \int_\Omega [\sigma_s\varphi_s^s - |\nabla\varphi_s|^s] \omega dx \Big\} \text{ for } s = p, q. \end{aligned}$$

Note that on $\overline{\Omega}_\delta$ we have $|\nabla\varphi_s|^s - \sigma_s\varphi_s^s \geq m$ for $s = p, q$. Also on $\Omega - \overline{\Omega}_\delta$ since $\varphi_p \geq \mu_p, \varphi_q \geq \mu_q$ for some $0 < \mu_p, \mu_q < 1$, if $\tau_1 + \mu_1$ and $\tau_2 + \mu_2$ are large then by (H5) $f(\overline{\psi}), h(\psi), g(\psi), \gamma(\overline{\psi}) \geq \frac{k_0}{m}\max\{\sigma_p, \sigma_q\}$. Hence

$$\begin{aligned}
\int_\Omega |\nabla\psi|^{p-2}\nabla\psi\cdot\nabla\omega dx &= \frac{(\tau_1+\mu_1)k_0}{m}\Big\{\int_\Omega[\sigma_p\varphi_p^p - |\nabla\varphi_p|^p]\omega dx\Big\} \\
&= \frac{(\tau_1+\mu_1)k_0}{m}\int_{\Omega_\delta}[\sigma_p\varphi_p^p - |\nabla\varphi_p|^p]\omega dx \\
&\qquad + \frac{(\tau_1+\mu_1)k_0}{m}\int_{\Omega-\overline{\Omega}_\delta}[\sigma_p\varphi_p^p - |\nabla\varphi_p|^p]\omega dx \\
&\leq -(\tau_1+\mu_1)k_0\int_{\Omega_\delta}\omega dx + \frac{(\tau_1+\mu_1)k_0}{m}\int_{\Omega-\overline{\Omega}_\delta}\sigma_p\omega dx \\
&\leq \int_{\Omega_\delta}[\tau_1 f(\overline{\psi}) + \mu_1 h(\psi)]\omega dx + \int_{\Omega-\overline{\Omega}_\delta}[\tau_1 f(\overline{\psi}) + \mu_1 h(\psi)]\omega dx \\
&= \int_\Omega[\tau_1 f(\overline{\psi}) + \mu_1 h(\psi)]\omega dx.
\end{aligned}$$

Similarly

$$\int_\Omega |\nabla\overline{\psi}|^{q-2}\nabla\overline{\psi}\cdot\nabla\omega dx \leq \int_\Omega[\tau_2 g(\psi) + \mu_2\gamma(\overline{\psi})]\omega dx,$$

i.e. $(\psi, \overline{\psi})$ is a sub-solution of (1.11).

Next, let e_p, e_q be the solution of $-\Delta_s e_s = 1; \Omega$, $e_s = 0; \partial\Omega$ for $s = p, q$. Let $(\phi, \overline{\phi}) := \Big(Ce_p, (\tau_2+\mu_2)^{1/q-1}[g(C\|e_p\|_\infty)]^{1/q-1}e_q\Big)$. Then

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla\omega dx = C^{p-1}\int_\Omega |\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\omega dx = C^{p-1}\int_\Omega \omega dx.$$

By (H6)-(H7) we can choose C large enough so that

$$\begin{aligned}
C^{p-1} &\geq \tau_1 f\big([\tau_2+\mu_2]^{1/q-1}\|e_q\|_\infty[g(C\|e_p\|_\infty)]^{1/q-1}\big) + \mu_1 h(C\|e_p\|_\infty) \\
&\geq \tau_1 f(\overline{\phi}) + \mu_1 h(\phi).
\end{aligned}$$

Hence

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi\cdot\nabla\omega dx \geq \int_\Omega \tau_1 f(\overline{\phi})\omega dx + \int_\Omega \mu_1 h(\phi)\omega dx.$$

Also

$$\begin{aligned}\int_\Omega |\nabla\overline{\phi}|^{q-2}\nabla\overline{\phi}\cdot\nabla\omega dx &= (\tau_2+\mu_2)\int_\Omega g(C\|e_p\|_\infty)\omega dx\\ &\geq \int_\Omega \tau_2 g(\phi)\omega dx + \int_\Omega \mu_2 g(C\|e_p\|_\infty)\omega dx.\end{aligned}$$

Again by (H6) for C large enough we have

$$\begin{aligned}g(C\|e_p\|_\infty) &\geq \gamma\Big([\tau_2+\mu_2]^{1/q-1}\big(g(C\|e_p\|_\infty)\big)^{1/q-1}\|e_q\|_\infty\Big)\\ &\geq \gamma(\overline{\phi}).\end{aligned}$$

Hence

$$\int_\Omega |\nabla\overline{\phi}|^{q-2}\nabla\overline{\phi}\cdot\nabla\omega dx \geq \int_\Omega \tau_2 g(\phi)\omega dx + \int_\Omega \mu_2\gamma(\overline{\phi})\omega dx,$$

i.e. $(\phi,\overline{\phi})$ is a super-solution of (1.11). Further $\phi\geq\psi$ and $\overline{\phi}\geq\overline{\psi}$ for C large. Thus there exists a solution (u,v) of (1.11) with $\psi\leq u\leq\phi$ and $\overline{\psi}\leq v\leq\overline{\phi}$. This completes the proof of Theorem 15.

5.2 Proof of Theorem 16

To prove Theorem 16, we will construct a sub-solution $(\psi_1,\overline{\psi}_1)$, a strict super-solution $(\phi_1,\overline{\phi}_1)$, a strict sub-solution $(\psi_2,\overline{\psi}_2)$ and a super-solution $(\phi_2,\overline{\phi}_2)$ for (1.11) such that $(\psi_1,\overline{\psi}_1)\leq(\phi_1,\overline{\phi}_1)\leq(\phi_2,\overline{\phi}_2)$, $(\psi_1,\overline{\psi}_1)\leq(\psi_2,\overline{\psi}_2)\leq(\phi_2,\overline{\phi}_2)$ and $(\psi_2,\overline{\psi}_2)\not\leq(\phi_1,\overline{\phi}_1)$. Then (1.11) has at least three distinct nonnegative solutions $(u_i,v_i),\ i=1,2,3$ such that $(u_1,v_1)\in[(\psi_1,\overline{\psi}_1),(\phi_1,\overline{\phi}_1)]$, $(u_2,v_2)\in[(\psi_2,\overline{\psi}_2),(\phi_2,\overline{\phi}_2)]$ and

$$(u_3,v_3)\in[(\psi_1,\overline{\psi}_1),(\phi_2,\overline{\phi}_2)]\setminus\big([(\psi_1,\overline{\psi}_1),(\phi_1,\overline{\phi}_1)]\cup[(\psi_2,\overline{\psi}_2),(\phi_2,\overline{\phi}_2)]\big).$$

We first note that $(\psi_1, \overline{\psi}_1) = (0,0)$ is a solution (hence a sub-solution). As in Section 5.1, we can always construct a large super-solution $(\phi_2, \overline{\phi}_2)$. We next consider

$$\begin{cases} -\Delta_p \psi_2 = \tau_1 \tilde{f}(\overline{\psi}_2) + \mu_1 \tilde{h}(\psi_2) & \text{in } \Omega \\ -\Delta_q \overline{\psi}_2 = \tau_2 \tilde{g}(\psi_2) + \mu_2 \tilde{\gamma}(\overline{\psi}_2) & \text{in } \Omega \\ \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial\Omega \end{cases} \tag{5.1}$$

where $\tilde{f}(s) = f(s) - 1, \tilde{h}(s) = h(s) - 1, \tilde{g}(s) = g(s) - 1$ and $\tilde{\gamma}(s) = \gamma(s) - 1$. Then by Theorem 11, (5.1) has a positive solution $(\psi_2, \overline{\psi}_2)$ when $(\tau_1 + \mu_1)$ and $(\tau_2 + \mu_2)$ is large. Clearly this $(\psi_2, \overline{\psi}_2)$ is a strict sub-solution of (1.11). Finally we construct the strict super-solution $(\phi_1, \overline{\phi}_1)$. Let φ_p, φ_q be as described in Section 5.1. We first note that there exists positive constants c_1 and c_2 such that

$$\varphi_p \leq c_1 \varphi_q \text{ and } \varphi_q \leq c_2 \varphi_p. \tag{5.2}$$

Let $(\phi_1, \overline{\phi}_1) = (\epsilon\varphi_p, \epsilon\varphi_q)$ where $\epsilon > 0$. Let $H_p(x) := \sigma_p x^{p-1} - \tau_1 f(c_2 x) - \mu_1 h(x)$ and $H_q(x) := \sigma_q x^{q-1} - \tau_2 g(c_1 x) - \mu_2 \gamma(x)$. Observe that $H_p(0) = H_q(0) = 0$, $H_p^{(k)}(0) = H_q^{(l)}(0) = 0$ for $k = 1, 2, \cdots [p-2]$ and $l = 1, 2, \cdots [q-2]$. $H_p^{(p-1)}(0) > 0$ and $H_q^{(q-1)}(0) > 0$ if p, q are integers, while $\lim_{r\to 0} H^{([p])}(r) = +\infty = \lim_{r\to 0} H^{([q])}(r)$ if p, q are not integers. Thus there exists θ such that $H_p(x) > 0$ and $H_q(x) > 0$ for $x \in (0, \theta]$. Hence on Ω, for $0 < \epsilon \leq \theta$ we have

$$\begin{aligned} \sigma_p(\phi_1)^{p-1} = \sigma_p(\epsilon\varphi_p)^{p-1} &> \tau_1 f(c_2 \epsilon\varphi_p) + \mu_1 h(\epsilon\varphi_p) \\ &\geq \tau_1 f(\epsilon\varphi_q) + \mu_1 h(\epsilon\varphi_p) \\ &= \tau_1 f(\overline{\phi}_1) + \mu_1 h(\phi_1) \end{aligned} \tag{5.3}$$

and similarly on Ω, we get

$$\begin{aligned}\sigma_q(\overline{\phi}_1)^{q-1} = \sigma_q(\epsilon\varphi_q)^{q-1} &> \tau_2 g(c_1\epsilon\varphi_q) + \mu_2\gamma(\epsilon\varphi_q) \\ &\geq \tau_2 g(\epsilon\varphi_p) + \mu_2\gamma(\epsilon\varphi_q) \\ &= \tau_2 g(\phi_1) + \mu_2\gamma(\overline{\phi}_1).\end{aligned} \tag{5.4}$$

Using the inequalities (5.3) and (5.4) we have

$$\begin{aligned}\int_\Omega |\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla\omega dx &= \epsilon^{p-1}\int_\Omega |\nabla\varphi_p|^{p-2}\nabla\varphi_p\cdot\nabla\omega \\ &= \int_\Omega \sigma_p(\epsilon\varphi_p)^{p-1}\omega dx \\ &> \tau_1\int_\Omega f(\overline{\phi}_1)\omega dx + \mu_1\int_\Omega h(\phi_1)\omega dx.\end{aligned}$$

Similarly we have

$$\int_\Omega |\nabla\overline{\phi}_1|^{q-2}\nabla\overline{\phi}_1\cdot\nabla\omega dx > \tau_2\int_\Omega g(\phi_1)\omega dx + \mu_2\int_\Omega \gamma(\overline{\phi}_1)\omega dx.$$

Thus $(\phi_1, \overline{\phi}_1)$ is a strict super-solution. Here we can choose ϵ small so that $(\psi_2, \overline{\psi}_2) \not\leq (\phi_1, \overline{\phi}_1)$. Hence there exists three distinct nonnegative solutions $(u_i, v_i), i = 1, 2, 3$ such that $(u_1, v_1) \in [(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$, $(u_2, v_2) \in [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and

$$(u_3, v_3) \in [(\psi, \overline{\psi}), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big).$$

Since $(\psi_1, \overline{\psi}_1) \equiv (0, 0)$ is a solution it may turn out that $(u_1, v_1) \equiv (\psi_1, \overline{\psi}_1) \equiv (0, 0)$. In any case we have two positive solutions (u_2, v_2) and (u_3, v_3). Hence Theorem 16 holds.

5.3 Proof of Theorem 17 (when Ω is a ball of radius R)

In this section we will prove Theorem 17 for the case when Ω is a ball. Let Ω be a ball of radius R, say B_R. We will establish a pair of sub-solutions $(\psi_1, \overline{\psi}_1), (\psi_2, \overline{\psi}_2)$ and a pair of super-solutions $(\phi_1, \overline{\phi}_1), (\phi_2, \overline{\phi}_2)$ for the system (1.11) satisfying Lemma 16. Clearly $(\psi_1, \overline{\psi}_1) = (0, 0)$ is a strict sub-solution of (1.11) since $f(0) > 0, g(0) > 0, h(0) \geq 0$ and $\gamma(0) \geq 0$.

We next construct a strict positive super-solution $(\phi_1, \overline{\phi}_1)$ of (1.11) when

$$\tau_1 < \frac{a_1^{p-1}}{d_1 \|e_p\|_\infty^{p-1} f(b_1)},\ \tau_2 < \frac{b_1^{q-1}}{d_1 \|e_q\|_\infty^{q-1} g(a_1)},\ \mu_1 < \frac{a_1^{p-1}}{d_2 \|e_p\|_\infty^{p-1} h(a_1)}$$

and $\mu_2 < \dfrac{b_1^{q-1}}{d_2 \|e_q\|_\infty^{q-1} \gamma(b_1)}$.

Let $(\phi_1, \overline{\phi}_1) = (a_1 \dfrac{e_p}{\|e_p\|_\infty}, b_1 \dfrac{e_q}{\|e_q\|_\infty})$. Then

$$\begin{aligned}
\frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1}} &= \frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1}} (\frac{1}{d_1} + \frac{1}{d_2}) \\
&= \frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1} d_1} + \frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1} d_2} \\
&> \tau_1 f(b_1) + \mu_1 h(a_1) \\
&\geq \tau_1 f(\overline{\phi}_1) + \mu_1 h(\phi_1).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\frac{b_1^{q-1}}{\|e_q\|_\infty^{q-1}} &= \frac{b_1^{q-1}}{\|e_q\|_\infty^{q-1}} (\frac{1}{d_1} + \frac{1}{d_2}) \\
&= \frac{b_1^{q-1}}{\|e_p\|_\infty^{q-1} d_1} + \frac{b_1^{q-1}}{\|e_q\|_\infty^{q-1} d_2} \\
&> \tau_2 g(a_1) + \mu_2 \gamma(b_1) \\
&\geq \tau_2 g(\phi_1) + \mu_2 \gamma(\overline{\phi_1}).
\end{aligned}$$

Let $\omega \in W$. We note that

$$\begin{aligned}\int_\Omega |\nabla \phi_1|^{p-2}\nabla\phi_1 \cdot \nabla\omega dx &= \frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1}}\int_\Omega |\nabla e_p|^{p-2}\nabla e_p \cdot \nabla\omega dx \\ &= \int_\Omega \Big[\frac{a_1^{p-1}}{\|e_p\|_\infty^{p-1}}\Big]\omega dx \\ &> \tau_1 \int_\Omega f(\overline{\phi}_1)\omega dx + \mu_1 \int_\Omega h(\phi_1)\omega dx.\end{aligned}$$

Similarly we have

$$\int_\Omega |\nabla \overline{\phi}_1|^{q-2}\nabla\overline{\phi}_1 \cdot \nabla\omega dx > \tau_2 \int_\Omega g(\phi_1)\omega dx + \mu_2 \int_\Omega \gamma(\overline{\phi_1})\omega dx.$$

This proves that $(\phi_1, \overline{\phi}_1)$ is a positive super-solution of (1.11). Note that $\|\phi_1\|_\infty = a_1$ and $\|\overline{\phi}_1\|_\infty = b_1$.

Now we construct a positive sub-solution $(\psi_2, \overline{\psi}_2)$ of (1.11) when $\tau_1 > C_1 \frac{a_2^{p-1}}{f(b_2)}$ and $\tau_2 > C_2 \frac{b_2^{q-1}}{g(a_2)}$. To do so, we first construct a positive sub-solution $(\psi_2, \overline{\psi}_2)$ for the following problem

$$\begin{cases} -\Delta_p u = \tau_1 f(v) \text{ in } \Omega \\ -\Delta_q v = \tau_2 g(u) \text{ in } \Omega \\ \quad u = 0 = v \text{ on } \partial\Omega. \end{cases} \tag{5.5}$$

We observe here that since $h(x) \geq 0,\ \gamma(x) \geq 0,\ \forall x \geq 0$, we have $(\psi_2, \overline{\psi}_2)$ satisfies

$$\begin{cases} -\Delta_p \psi_2 \leq \tau_1 f(\overline{\psi}_2) \leq \tau_1 f(\overline{\psi}_2) + \mu_1 h(\psi_2) & \text{in } \Omega \\ -\Delta_q \overline{\psi}_2 \leq \tau_2 g(\psi_2) \leq \tau_2 g(\psi_2) + \mu_2 \gamma(\overline{\psi}_2) & \text{in } \Omega \\ \quad \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial\Omega. \end{cases} \tag{5.6}$$

Hence $(\psi_2, \overline{\psi}_2)$ is a sub-solution of (1.11).

Construction of $(\psi_2, \overline{\psi}_2)$ is as follows. For $\alpha, \beta > 1$, define $\rho(r) : [0, R] \to [0, 1]$ by

$$\rho(r) = \begin{cases} 1, & 0 \le r \le \sigma \\ 1 - (1 - (\frac{R-r}{R-\sigma})^{\beta})^{\alpha} & \sigma < r \le R. \end{cases}$$

Then

$$\rho'(r) = \begin{cases} 0, & 0 \le r \le \sigma \\ -\frac{\alpha\beta}{R-\sigma}(1 - (\frac{R-r}{R-\sigma})^{\beta})^{\alpha-1}(\frac{R-r}{R-\sigma})^{\beta-1} & \sigma < r \le R \end{cases}$$

and hence $|\rho'(r)| \le \frac{\alpha\beta}{R-\sigma}$.

Let $w(r) = a_2\rho(r)$ and $\overline{w}(r) = b_2\rho(r)$. Note that

$$|w'(r)| \le \frac{a_2\alpha\beta}{R-\sigma}, \quad |\overline{w}'(r)| \le \frac{b_2\alpha\beta}{R-\sigma}. \tag{5.7}$$

Define $\psi_2(r), \overline{\psi}_2(r)$ as the radially symmetric solutions of

$$\begin{cases} -\Delta_p \psi_2 = \lambda_1 f(\overline{w}) & \text{in } B(0, R) \\ -\Delta_q \overline{\psi}_2 = \lambda_2 g(w) & \text{in } B(0, R) \\ \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial B(0, R). \end{cases} \tag{5.8}$$

Then $\psi_2, \ \overline{\psi}_2$ satisfies

$$-[r^{N-1}G_p(\psi_2'(r))]' = \tau_1 r^{N-1} f(\overline{w}(s))$$

$$-[r^{N-1}G_q(\overline{\psi}_2'(r))]' = \tau_2 r^{N-1} g(w(s))$$

$$\psi_2'(0) = 0; \psi_2(R) = 0, \quad \overline{\psi}_2'(0) = 0; \overline{\psi}_2(R) = 0$$

where $G_s(t) = |t|^{s-2}t$ for $s = p, q$. Integrating for $0 < r < R$, we have

$$-G_p(\psi_2'(r)) = \frac{\lambda_1}{r^{N-1}} \int_0^r s^{N-1} f(\overline{w}(s))ds$$
$$-G_q(\overline{\psi}_2'(r)) = \frac{\lambda_2}{r^{N-1}} \int_0^r s^{N-1} g(w(s))ds.$$

Hence

$$-\psi_2'(r) = G_p^{-1}\Big(\frac{\lambda_1}{r^{N-1}} \int_0^r s^{N-1} f(\overline{w}(s))ds\Big)$$
$$-\overline{\psi}_2'(r) = G_q^{-1}\Big(\frac{\lambda_2}{r^{N-1}} \int_0^r s^{N-1} g(w(s))ds\Big).$$

Note that for $0 < r \leq \sigma$, clearly $\psi_2'(r) < w'(r)$ and $\overline{\psi}_2'(r) < \overline{w}'(r)$. Now for $r > \sigma$,

$$\begin{aligned} -\psi_2'(r) &= G_p^{-1}\Big(\frac{\lambda_1}{r^{N-1}} \int_0^r s^{N-1} f(\overline{w}(s))ds\Big) \\ &\geq G_p^{-1}\Big(\frac{\lambda_1}{r^{N-1}} \int_0^\sigma s^{N-1} f(\overline{w}(s))ds\Big) \\ &\geq G_p^{-1}\Big(\frac{\lambda_1}{R^{N-1}} f(b_2)\frac{\sigma^N}{N}\Big). \end{aligned} \tag{5.9}$$

Similar calculations shows that

$$-\overline{\psi_2}'(r) \geq G_q^{-1}\Big(\frac{\tau_2}{R^{N-1}} g(a_2)\frac{\sigma^N}{N}\Big). \tag{5.10}$$

Since $\tau_1 > C_1 \frac{a_2^{p-1}}{f(b_2)}$ and $\tau_2 > C_2 \frac{b_2^{q-1}}{g(a_2)}$, choose $\alpha, \beta > 1$ so that $\tau_1 > \alpha^{p-1}\beta^{p-1}C_1\frac{a_2^{p-1}}{f(b_2)}$ and $\tau_2 > \alpha^{q-1}\beta^{q-1}C_2\frac{b_2^{q-1}}{g(a_2)}$. By the definition of C_1 we have $\tau_1 > \alpha^{p-1}\beta^{p-1}\frac{N}{\sigma^N}\frac{R^{N-1}}{(R-\sigma)^{p-1}}\frac{a_2^{p-1}}{f(b_2)}$. Thus

$$\frac{\lambda_1}{R^{N-1}} f(b_2)\frac{\sigma^N}{N} > (\frac{\alpha\beta a_2}{R-\sigma})^{p-1} = G_p(\frac{\alpha\beta a_2}{R-\sigma}).$$

By the monotonicity of G_p and G_p^{-1}, we have

$$G_p^{-1}\Big(\frac{\lambda_1}{R^{N-1}} f(b_2)\frac{\sigma^N}{N}\Big) > \frac{\alpha\beta a_2}{R-\sigma}.$$

Using (5.9), we get

$$-\psi_2'(r) \geq G_p^{-1}\Big(\frac{\lambda_1}{R^{N-1}}f(b_2)\frac{\sigma^N}{N}\Big) > \frac{\alpha\beta a_2}{R-\sigma} \geq -w'(r).$$

That is, we have $\psi_2'(r) < w'(r); 0 < r \leq R$. Similarly using (5.10) and (5.7) we can establish $\overline{\psi}_2'(r) < \overline{w}'(r); 0 < r \leq R$ for $\tau_2 > C_2\frac{b_2^{q-1}}{g(a_2)}$. Since $\psi_2(R) = \overline{\psi}_2(R) = 0 = w(R) = \overline{w}(R)$, it is easy to see that

$$\psi_2(r) > w(r), \overline{\psi}_2(r) > \overline{w}(r) \text{ for } 0 \leq r < R. \tag{5.11}$$

Now since f and g are strictly increasing, using (5.8) and (5.11), we have

$$\begin{cases} -\Delta_p\psi_2 = \lambda_1 f(\overline{w}) < \lambda_1 f(\overline{\psi}_2) & \text{in } B(0,R) \\ -\Delta_q\overline{\psi}_2 = \lambda_2 g(w) < \lambda_2 g(\psi_2) & \text{in } B(0,R) \\ \quad \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial B(0,R) \end{cases}$$

and hence $(\psi_2, \overline{\psi}_2)$ is a strict positive sub-solution of (5.5) so is for (1.11). We also note that $\|\psi_2\|_\infty > \|w\|_\infty = a_2$ and $\|\overline{\psi}_2\|_\infty > \|\overline{w}\|_\infty = b_2$. Since $a_1 < a_2$ or $b_1 < b_2$ we have $(\psi_2, \overline{\psi}_2) \nleq (\phi_1, \overline{\phi}_1)$.

Next, let e_p, e_q be the solution of $-\Delta_s e_s = 1; \Omega$, $e_s = 0; \partial\Omega$ for $s = p, q$. Let $(\phi_2, \overline{\phi}_2) := \Big(Ce_p, \ (\tau_2 + \mu_2)^{1/q-1}[g(C\|e_p\|_\infty)]^{1/q-1}e_q\Big)$ for $C > 0$. Then

$$\int_\Omega |\nabla\phi_2|^{p-2}\nabla\phi_2 \cdot \nabla\omega dx = C^{p-1}\int_\Omega |\nabla e_p|^{p-2}\nabla e_p \cdot \nabla\omega dx = C^{p-1}\int_\Omega \omega dx.$$

By (H6)-(H8) we can choose C large enough so that

$$\begin{aligned} C^{p-1} &\geq \tau_1 f\big([\tau_2 + \mu_2]^{1/q-1}\|e_q\|_\infty[g(C\|e_p\|_\infty)]^{1/q-1}\big) + \mu_1 h(C\|e_p\|_\infty) \\ &\geq \tau_1 f(\overline{\phi}_2) + \mu_1 h(\phi_2). \end{aligned}$$

Hence

$$\int_\Omega |\nabla \phi_2|^{p-2}\nabla \phi_2 \cdot \nabla \omega dx \geq \int_\Omega \tau_1 f(\overline{\phi}_2)\omega dx + \int_\Omega \mu_1 h(\phi_2)\omega dx.$$

Also

$$\begin{aligned}\int_\Omega |\nabla \overline{\phi}_2|^{q-2}\nabla \overline{\phi}_2 \cdot \nabla \omega dx &= (\tau_2+\mu_2)\int_\Omega g(C\|e_p\|_\infty)\omega dx \\ &\geq \int_\Omega \tau_2 g(\phi_2)\omega dx + \int_\Omega \mu_2 g(C\|e_p\|_\infty)\omega dx.\end{aligned}$$

Again by (H6) for C large enough we have

$$\begin{aligned}g(C\|e_p\|_\infty) &\geq \gamma\Big([\tau_2+\mu_2]^{1/q-1}\big(g(C\|e_p\|_\infty)\big)^{1/q-1}\|e_q\|_\infty\Big) \\ &\geq \gamma(\overline{\phi}_2).\end{aligned}$$

Hence

$$\int_\Omega |\nabla \overline{\phi}_2|^{q-2}\nabla \overline{\phi}_2 \cdot \nabla \omega dx \geq \int_\Omega \tau_2 g(\phi_2)\omega dx + \int_\Omega \mu_2 \gamma(\overline{\phi}_2)\omega dx$$

i.e. $(\phi_2, \overline{\phi}_2)$ is a super-solution of (1.11). Further $\phi_2 \geq \psi_2$ and $\overline{\phi}_2 \geq \overline{\psi}_2$ for C large. Hence by Lemma 5, (1.11) has a solution on each of the following components $[(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$, $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$. Clearly the solutions in $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$ are positive. Since $f(0)$ or $g(0)$ is strictly positive, the solutions in $[(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$ is also positive. This completes the proof of Theorem 17 when Ω is a Ball.

5.4 Proof of Theorem 17 (when Ω is a general bounded domain)

In this section we will prove the Theorem 17 for any bounded domain Ω. First we construct a positive strict sub-solution $(z, \overline{z})$ of (5.1) in Ω with $\|z\|_\infty \geq a_2$ and $\|\overline{z}\|_\infty \geq b_2$. Let B_R

be the largest inscribed ball in Ω with radius R. Assume $\tau_1 > C_1 \frac{a_2^{p-1}}{f(b_2)}$ and $\tau_2 > C_2 \frac{b_2^{q-1}}{g(a_2)}$ and let $(\psi_2(r), \overline{\psi}_2(r))$ be the sub-solution of (1.11) constructed in B_R as in the previous case. Now define

$$z(x) = \begin{cases} \psi_2(|x|)\ ; & x \in B_R \\ 0 \quad ; & x \in \Omega - B_R \end{cases}$$

and

$$\overline{z}(x) = \begin{cases} \overline{\psi}_2(|x|)\ ; & x \in B_R \\ 0 \quad ; & x \in \Omega - B_R. \end{cases}$$

Then $(z, \overline{z}) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ and $z = 0 = \overline{z}$ on $\partial\Omega$. Further, on B_R we have

$$\begin{cases} -\Delta_p z = -\Delta \psi_2 \leq \lambda f(\overline{\psi}_2) = \lambda f(\overline{z}) \\ -\Delta_q \overline{z} = -\Delta \overline{\psi}_2 \leq \lambda g(\psi_2) = \lambda g(z) \end{cases}$$

while outside B_R we have

$$\begin{cases} -\Delta_p z = 0 < \lambda f(0) = \lambda f(\overline{z}) \\ -\Delta_p \overline{z} = 0 < \lambda g(0) = \lambda g(z) \end{cases}$$

Hence $(z, \overline{z})$ is a strict sub-solution of (1.11) in Ω for $\tau_1 > C_1 \frac{a_2^{p-1}}{f(b_2)}$ and $\tau_2 > C_2 \frac{b_2^{q-1}}{g(a_2)}$ with $\|z\|_\infty \geq a_2$ and $\|\overline{z}\|_\infty \geq b_2$. The rest of the proof is identical to the previous case except that here for the second sub-solution we will use $(z, \overline{z})$ described above.

Proof of Theorem 17 follows easily from the proof of Theorem 17 assuming $\mu_1 = 0 = \mu_2$.

CHAPTER 6

STRONGLY COUPLED p-q-LAPLACIAN SYSTEMS

In this chapter proofs of Theorems 19 - 20 will be proved. In Section 6.1, proof of Theorem 19 will be presented and the proof of Theorem 20 will be presented in Section 6.2.

6.1 Proof of Theorem 19

In this section, proof of Theorem 19 will be presented. We first extend $f(u,v)$ and $g(u,v)$ for all $(u,v) \in \mathbb{R}^2$ smoothly such that there exists constant $k_0 > 0$ such that $f(u,v), g(u,v) \geq -k_0$ for all $(u,v) \in \mathbb{R}^2$. We shall establish Theorem 19 by constructing a positive weak sub-solution $(\psi_1, \psi_2) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ and a super-solution $(z_1, z_2) \in W^{1,p}(\Omega) \cap C(\overline{\Omega}) \times W^{1,q}(\Omega) \cap C(\overline{\Omega})$ of (1.12) such that $\psi_i \leq z_i$ for $i = 1, 2$. That is, ψ_i, z_i satisfies $(\psi_1, \psi_2) = (0, 0) = (z_1, z_2)$ on $\partial\Omega$ and

$$\begin{cases} \int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla \xi dx \leq \lambda \int_\Omega f(\psi_1, \psi_2) \xi dx \\ \int_\Omega |\nabla \psi_2|^{p-2} \nabla \psi_2 \cdot \nabla \xi dx \leq \lambda \int_\Omega g(\psi_1, \psi_2) \xi dx \end{cases}$$

$$\begin{cases} \int_\Omega |\nabla z_1|^{p-2} \nabla z_1 \cdot \nabla \xi dx \geq \lambda \int_\Omega f(z_1, z_2) \xi dx \\ \int_\Omega |\nabla z_2|^{p-2} \nabla z_2 \cdot \nabla \xi dx \geq \lambda \int_\Omega g(z_1, z_2) \xi dx \end{cases}$$

for all $\xi \in W := \left\{ \eta \in C_0^\infty(\Omega) : \eta \geq 0 \text{ in } \Omega \right\}$.

Let $\lambda_1^{(r)}$ the first eigenvalue of $-\Delta_r$ with Dirichlet boundary conditions and φ_r the corresponding eigenfunction with $\varphi_r > 0$ on Ω and $\|\varphi_r\|_\infty = 1$ for $r = p, q$. Let $m, \delta > 0$ be such that $|\nabla\varphi_r|^r - \lambda_1^{(r)}\varphi_r^r \geq m$ on $\overline{\Omega}_\delta = \{x \in \Omega | d(x, \partial\Omega) \leq \delta\}$ for $r = p, q$. (This is possible since $|\nabla\varphi_r| \neq 0$ on $\partial\Omega$ while $\varphi_r = 0$ on $\partial\Omega$ for $r = p, q$). We shall verify that

$$(\psi_1, \psi_2) := \Big([\frac{\lambda k_0}{m}]^{1/p-1} (\frac{p-1}{p}) \varphi_p^{p/p-1}, [\frac{\lambda k_0}{m}]^{1/q-1} (\frac{q-1}{q}) \varphi_q^{q/q-1} \Big),$$

is a sub-solution of (1.12) for λ large. Let $\xi \in W$. Then

$$\begin{aligned}\int_\Omega |\nabla\psi_1|^{p-2}\nabla\psi_1 \cdot \nabla\xi dx &= \Big(\frac{\lambda k_0}{m}\Big) \int_\Omega \varphi_p |\nabla\varphi_p|^{p-2}\nabla\varphi_p \cdot \nabla\xi dx \\ &= \Big(\frac{\lambda k_0}{m}\Big) \Big\{ \int_\Omega |\nabla\varphi_p|^{p-2}\nabla\varphi_p \cdot \nabla(\varphi_p\xi) dx - \int_\Omega |\nabla\varphi_p|^p \xi dx \Big\} \\ &= \Big(\frac{\lambda k_0}{m}\Big) \Big\{ \int_\Omega [\lambda_1^{(p)}\varphi_p^p - |\nabla\varphi_p|^p]\xi dx \Big\}.\end{aligned}$$

Similarly

$$\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2 \cdot \nabla\xi dx = \Big(\frac{\lambda k_0}{m}\Big) \Big\{ \int_\Omega [\lambda_1^{(q)}\varphi_q^q - |\nabla\varphi_q|^q]\xi dx \Big\}.$$

Now on $\overline{\Omega}_\delta$ we have $|\nabla\varphi_r|^r - \lambda_1^{(s)}\varphi_r^r \geq m$ for $r = p, q$. Which implies that

$$\begin{aligned}&\frac{k_0}{m}\Big(\lambda_1^{(p)}\varphi_p^p - |\nabla\varphi_p|^p\Big) - f(\psi_1, \psi_2) \leq 0 \quad \text{and} \\ &\frac{k_0}{m}\Big(\lambda_1^{(q)}\varphi_q^q - |\nabla\varphi_q|^q\Big) - g(\psi_1, \psi_2) \leq 0.\end{aligned}$$

Next on $\Omega - \overline{\Omega}_\delta$ we have $\varphi_p \geq \mu$, $\varphi_q \geq \mu$ for some $\mu > 0$, and therefore for λ large

$$f(\psi_1, \psi_2) \geq \frac{k_0}{m}\lambda_1^{(p)} \geq \frac{k_0}{m}\lambda_1^{(p)}\varphi_p^p - |\nabla\varphi_p|^p \quad \text{and}$$

$$g(\psi_1, \psi_2) \geq \frac{k_0}{m}\lambda_1^{(q)} \geq \frac{k_0}{m}\lambda_1^{(q)}\varphi_q^q - |\nabla\varphi_q|^q.$$

Hence

$$\int_\Omega |\nabla \psi_1|^{p-2}\nabla\psi_1\cdot\nabla\xi dx \le \lambda\int_\Omega f(\psi_1,\psi_2)\xi dx$$

and

$$\int_\Omega |\nabla \psi_2|^{q-2}\nabla\psi_2\cdot\nabla\xi dx \le \lambda\int_\Omega g(\psi_1,\psi_2)\xi dx$$

i.e. (ψ_1,ψ_2) is a sub-solution of (1.12) for λ large.

Next let e_r be the solution of $-\Delta_r e_r = 1$ in Ω, $e_r = 0$ on $\partial\Omega$ for $r = p, q$. Let $(z_1, z_2) := \left(\frac{c}{\mu_p}\lambda^{1/p-1}e_p, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}e_q\right)$ where $\mu_r = \|e_r\|_\infty;\ r = p, q$. Then

$$\begin{aligned}\int_\Omega |\nabla z_1|^{p-2}\nabla z_1\cdot\nabla\xi dx &= \lambda\big(\frac{c}{\mu_p}\big)^{p-1}\int_\Omega |\nabla e_p|^{p-2}\nabla e_p\cdot\nabla\xi dx\\ &= \frac{1}{(\mu_p)^{p-1}}(c\lambda^{1/p-1})^{p-1}\int_\Omega \xi dx.\end{aligned}$$

By (H10) we can choose c large enough so that

$$\begin{aligned}\frac{1}{(\mu_p)^{p-1}}(c\lambda^{1/p-1})^{p-1}\int_\Omega \xi dx &\ge \lambda\int_\Omega f(c\lambda^{1/p-1}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}\mu_q)\xi dx\\ &\ge \lambda\int_\Omega f(c\lambda^{1/p-1}\frac{e_p}{\mu_p}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}e_q)\xi dx\\ &= \lambda\int_\Omega f(z_1, z_2)\xi dx.\end{aligned}$$

Next

$$\begin{aligned}\int_\Omega |\nabla z_2|^{q-2}\nabla z_2\cdot\nabla\xi dx &= \lambda[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]\int_\Omega |\nabla e_q|^{q-2}\nabla e_q\cdot\nabla\xi dx\\ &= \lambda[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]\int_\Omega \xi dx.\end{aligned}$$

By (H11) choose c large so that $\frac{1}{\lambda^{1/q-1}}\mu_q \geq \frac{[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}}{c\lambda^{1/p-1}}$, then

$$\begin{aligned}
\lambda[g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]\int_\Omega \xi dx &\geq \lambda\int_\Omega g\big(c\lambda^{1/p-1}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}\mu_q\big)\xi dx \\
&\geq \lambda\int_\Omega g\big(c\lambda^{1/p-1}\frac{e_p}{\mu_p}, [g(c\lambda^{1/p-1}, c\lambda^{1/p-1})]^{1/q-1}\lambda^{1/q-1}e_q\big)\xi dx \\
&= \lambda\int_\Omega g(z_1, z_2)\xi dx,
\end{aligned}$$

i.e., (z_1, z_2) is a super-solution of (1.12) with $z_i \geq \psi_i$ for c large, $i = 1, 2$. (Note $|\nabla e_r| \neq 0$ on $\partial\Omega$ for $r = p, q$).

Thus there exists a solution (u, v) of (1.12) with $\psi_1 \leq u \leq z_1, \psi_2 \leq v \leq z_2$. This completes the proof of Theorem 19.

6.2 Proof of Theorem 20

To prove Theorem 20, we will construct a sub–solution (ψ_1, ψ_2), a strict super-solution (ζ_1, ζ_2), a strict sub-solution (w_1, w_2) and a super-solution (z_1, z_2) for (1.12) such that $(\psi_1, \psi_2) \leq (\zeta_1, \zeta_2) \leq (z_1, z_2)$, $(\psi_1, \psi_2) \leq (w_1, w_2) \leq (z_1, z_2)$, and $(w_1, w_2) \nleq (\zeta_1, \zeta_2)$. Then (1.12) has at least three distinct solutions $(u_i, v_i),\ i = 1, 2, 3$ such that $(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)]$, $(u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$ and

$$(u_3, v_3) \in [(\psi_1, \psi_2), (z_1, z_2)] \setminus \big([(\psi_1, \psi_2), (\zeta_1, \zeta_2)] \cup [(w_1, w_2), (z_1, z_2)]\big).$$

We first note that $(\psi_1, \psi_2) = (0, 0)$ is a solution (hence a sub-solution). As in Section 6.1, we can always construct a large super-solution (z_1, z_2). We next consider

$$\begin{cases} -\Delta_p w_1 = \lambda \tilde{f}(w_1, w_2) & \text{in } \Omega \\ -\Delta_q w_2 = \lambda \tilde{g}(w_1, w_2) & \text{in } \Omega \\ w_1 = 0 = w_2 & \text{on } \partial\Omega \end{cases} \tag{6.1}$$

where $\tilde{f}(u, v) = f(u, v) - 1$ and $\tilde{g}(u, v) = g(u, v) - 1$. Then by Theorem 19, (6.1) has a positive solution (w_1, w_2) when λ is large. Clearly this (w_1, w_2) is a strict sub-solution of (1.12). Finally we construct the strict super-solution (ζ_1, ζ_2).

To do so, we let φ_p, φ_q as described in Section 6.1. We note that there exists positive constants c_1 and c_2 such that

$$\varphi_p \le c_1 \varphi_q \text{ and } \varphi_q \le c_2 \varphi_p. \tag{6.2}$$

Let $(\zeta_1, \zeta_2) = (\epsilon\varphi_p, \epsilon\varphi_q)$ where $\epsilon > 0$. Let $H_p(s) := \lambda_1^{(p)} s^{p-1} - \lambda f(s, c_2 s)$ and $H_q(s) := \lambda_1^{(q)} s^{q-1} - \lambda g(c_1 s, s)$. Observe that $H_p(0) = H_q(0) = 0$, $H_p^{(k)}(0) = 0 = H_q^{(l)}(0)$ for $k = 1, 2, \cdots [p-2]$ and $l = 1, 2, \cdots [q-2]$. $H_p^{(p-1)}(0) > 0$ and $H_q^{(q-1)}(0) > 0$ if p, q are integers, while $\lim_{r \to 0} H^{([p])}(r) = +\infty = \lim_{r \to 0} H^{([q])}(r)$ if p, q are not integers. Thus there exists θ such that $H_p(s) > 0$ and $H_q(s) > 0$ for $s \in (0, \theta]$. Hence on Ω for $0 < \epsilon \le \theta$ we have

$$\begin{aligned} \lambda_1^{(p)} (\zeta_1)^{p-1} = \lambda_1^{(p)} (\epsilon\varphi_p)^{p-1} &> \lambda f(\epsilon\varphi_p, c_2 \epsilon\varphi_p) \\ &\ge \lambda f(\epsilon\varphi_p, \epsilon\varphi_q) \\ &= \lambda f(\zeta_1, \zeta_2) \end{aligned} \tag{6.3}$$

and similarly on Ω we get

$$\begin{aligned}\lambda_1^{(q)}(\zeta_2)^{q-1} = \lambda_1^{(q)}(\epsilon\varphi_q)^{q-1} &> \lambda g(c_1\epsilon\varphi_q, \epsilon\varphi_q)\\ &\geq \lambda g(\epsilon\varphi_p, \epsilon\varphi_q)\\ &= \lambda g(\zeta_1, \zeta_2) \end{aligned} \tag{6.4}$$

Using the inequalities (6.3) and (6.4) we have

$$\begin{aligned}\int_\Omega |\nabla\zeta_1|^{p-2}\nabla\zeta_1 \cdot \nabla\xi dx = \epsilon^{p-1}&\int_\Omega |\nabla\varphi_p|^{p-2}\nabla\varphi_p \cdot \nabla\xi\\ &= \int_\Omega \lambda_1^{(p)}(\epsilon\varphi_p)^{p-1}\xi dx\\ &> \lambda\int_\Omega f(\zeta_1, \zeta_2)\xi dx.\end{aligned}$$

Similarly we have

$$\int_\Omega |\nabla\zeta_2|^{q-2}\nabla\zeta_2 \cdot \nabla\xi dx > \lambda\int_\Omega g(\zeta_1, \zeta_2)\xi dx.$$

Thus (ζ_1, ζ_2) is a strict super-solution. Here we can choose ϵ small so that $(w_1, w_2) \nleq (\zeta_1, \zeta_2)$. Hence there exists solutions three nonnegative solutions $(u_i, v_i), i = 1, 2, 3$ such that $(u_1, v_1) \in [(\psi_1, \psi_2), (\zeta_1, \zeta_2)]$, $(u_2, v_2) \in [(w_1, w_2), (z_1, z_2)]$ and

$$(u_3, v_3) \in [(\psi_1, \psi_2), (z_1, z_2)] \setminus \big([(\psi_1, \psi_2), (\zeta_1, \zeta_2)] \cup [(w_1, w_2), (z_1, z_2)]\big).$$

Since $(\psi_1, \psi_2) \equiv (0, 0)$ is a solution it may turn out that $(u_1, v_1) \equiv (\psi_1, \psi_2) \equiv (0, 0)$. In any case we have two positive solutions (u_2, v_2) and (u_3, v_3). Hence Theorem 20 holds.

Remark 16

Note that in the construction of the super-solution (ζ_1, ζ_2) we require the conditions at zero on F and G only for the constants $c = c_2$ and $\tilde{c} = c_1$.

CHAPTER 7

CONCLUSIONS AND FUTURE DIRECTIONS

7.1 Conclusions

This thesis initiates an extensive study on the existence of multiple positive solutions for classes of Laplacian and p-Laplacian systems. It is well known that if $\frac{u}{f_i(u)}$ is nondecreasing for all $i \in \{1, 2, \ldots, n\}$, then the systems will have at most one positive solution (see [Da]). We question whether $\frac{u}{f_i(u)}$ needs to have decreasing regions for each $i \in \{1, 2, \ldots, n\}$ for multiplicity to occur. Our results conclude that this is not necessary. In fact, we show that multiplicity can occur even when only one of these $\frac{u}{f_i(u)}$ has a decreasing region. In our study we also do not require that each one of the nonlinearities to be sublinear at ∞. We only require that the nonlinearities satisfy the combined sublinear condition.

7.2 Future directions

We plan to investigate these models in

(A) exterior domains and

(B) unbounded domains.

Also, we will investigate these models for

(A) uniqueness results and

(B) exact multiplicity results.

Further, models with stronger coupling will be investigated. Also combined linear and superlinear effects will be analyzed in the study of

(A) existence results

(B) non existence results

(C) uniqueness results and

(D) exact multiplicity results.

REFERENCES

[Am] Amann H. *Fixed point equations and nonlinear eigenvalue problems in ordered banach spaces*. SIAM Rev. **18** (1976), 620-709.

[AnS] Anuradha V., and Shivaji R. *Sign-changin solutions for a class of multiparameter semipositone problems.* Nonlin. Anal. TMA, **24(11)** (1995), 1581-1596.

[AZ] Arcoya D., and Zertiti A. *Existence and nonexistence of radially symmetric non-negative solutions for a class of semipositone problems in an annulus*. Rendiconti di Mathematica, Series 8. Vol. 14 (1994), 625-646.

[Ar] Aris R. *On stability criteria of chemical reaction engineering*. Chem. engng. Sci. **24**(1969), 149-169.

[BCN] Berestycki H., Caffarelli L.A., and Nirenberg L. *Inequalities for second order elliptic equations with applications to unbounded domains*. A Celebration of John F. Nash Jr., Duke Math. J. **81** (1996), 467-494.

[BCS] Brown K.J., Castro A., and Shivaji R. *Nonexistence of radially symmetric solutions for a class of semipositone problems*. Diff. and Int. Eqns., **2(4)** (1989), 541-545.

[BIS] Brown K.J., Ibrahim M.M.A., and Shivaji R. S-Shaped bifurcation curves. *Nonlin. Anal. TMA,* **5(5)** (1981), 475-486.

[BS] Brown K.J., and Shivaji R. *Instability of nonnegative solutions for a class of semipositone problems*. Proc. AMS, **112(1)** (1991), 121-124.

[CG] Castro A., and Gadam S. *Uniquesness of stable and unstable positive solutions for semipositone problems*. Nonlin. Anal. TMA, **22** (1994), 425-429.

[CGS] Castro A., Garner J.B., and Shivaji R. *Existence results for classes of sublinear semipositone problems*. Results in Mathematics, **23** (1993), 214-220.

[CHS] Castro A., Hassanpour M., and Shivaji R. *Uniqueness of nonnegative solutions for a semipositone problem with concave nonlinearity*. Comm. Part. Diff. Eqns. **20(11-12)** (1995), 1927-1936.

[CK] Cohen D.S., and Keller H.B. *Some positone problems suggested by nonlinear heat generation*. J. Math. Mech. **16**(1967), 1361-1376.

[CMS] Castro A., Maya C., and Shivaji R. *Nonlinear eigenvalue problems with semipositone structure*. Elec. Jour. of Diff. Eqns, Conf **05** (2000), 33-49.

[CS] Castro A., and Shivaji R. *Semipositone problems*. Semigroups of linear and nonlinear Operators and Applications. Kluwer Academic Publishers, New York, Eds. J. A. Goldstein and G. Goldstein, (1993), 109-119. (Invited review paper).

[CR] Crandall M.G., and Rabinowitz P. *Some continuation and variational methods for positive solutions of nonlinear ellitptic boundary value problems*. Arch. Rat. Mech. Anal, **52** (1973), 161-180.

[CT] Cuesta M., and Takác P. *A Strong Comparison Principle for Positive Solutions of Degenerate Elliptic Equations*. Diff. Int. Eqns. **13** (2000), no. 4-6, 721-746

[Da] Dalmasso R. *Existence and uniqueness of positive solutions of semilinear elliptic systems*. Nonlin. Anal. **39**(2000), 559-568.

[DH] Drábek P., and Hernandez J. *Existence and uniqueness of positive solutions for some quasilinear elliptic problem*. Nonlin. Anal. **44** (2001), 189-204.

[DKT] Drábek P., Krejčí P., and Takáč P. *Nonlinear Differential Equations,* Chapman & Hall/CRC, 1999.

[FLN] D.G. de Figueiredo, Lions P.L., and Nussbaum R.D. *A priori estimates and existence of positive solutions os semilinear elliptic equations*. Jour. Math. Pures et Appl. **61** (1982), 41-63.

[GNN] Gidas B., Ni W., and Nirenberg L. *Symmetry and related properties via the Maximum principle*. Comm. Math. Phys. **68** (1979), 209-243.

[GT] Gilbert D., and Trudinger N. *Elliptic Partial Differential Equations of Second Order*. Springer, New York, 1983.

[FT] Fleckinger J., and Takáč P. *Uniqueness of positive solutions for nonlinear cooperative systems with the p-Laplacian,* Indiana Univ. Math. J **43** (1994), no. 4, 1227-1253.

[Ha] Hai D.D. *On a class of sublinear quasilinear elliptic problems*. Proc. Amer. Math. Soc. **131** (2003), 2409-2414.

[HS1] Hai D.D., and Shivaji R. *An existence result on positive solutions for a class of p-Laplacian systems*. Nonlin. Anal. **56**(2004), 1007-1010.

[HS2] Hai D.D., and Shivaji R. *An existence result on positive solutions for a class of semilinear elliptic systems*. Proc. of the Roy. Soc., Edin. **134A**(2004), 137-141.

[HS3] Hai D.D., and Shivaji R. *Positive solutions for semipositone systems in the annulus*. Rocky. Mountain J. of Math. 29 (1999), 1285-1300.

[HS4] Hai D.D., and Shivaji R. *A non-existence result for superlinear semipositone systems*, preprint.

[Hr] Harrison L.G. *Kinetic Theory of Living Pattern*, Cambridge University Press (1993).

[JL] Joseph D., and Lundgren. T. *Quasilinear dirichlet problems driven by positive sources*. Arch. Rat. Mech. Anal. **49** (1973), 241-269.

[KJDBT] Kernevez J.P., Joly G., Duban M.C., Bunow B., and Thomad D. *Hysteresis, oscillations and pattern formation in realistic immobilized enzyme systems*. J. math. Biol. **7**(1979), 41-56.

[KW] J.L Kazdan J.L., and Warner F.F. *Remarks on some quasilinear elliptic equations*. Comm. Pure and Appl. Math. Vol XXVIII (1975), 567-597.

[La] Laetsch T. *The number of solutions of a nonlinear two point boundary value problems*. Indiana Univ. Math. Jour. **20(1)** (1970), 1-13.

[Li] Lions P.L. *On the existence of positive solutions of semilinear elliptic equations*. SIAM Rev. **24** (1982), 441-467.

[Ld] Lloyd N. G. *Degree Theory*. Cambridge University Press, (1978).

[MOS] Maya C., Oruganti S., and Shivaji R. *Positive solutions for classes of p-Laplacian equations*. Diff. and Int. Eqns. **16(6)** (2003), 757-768.

[Pk] Parks J. R. *Criticality criteria for various configurations of a self-heating chemical as functions of activation energy and temperature of assembly*. J. chem. phys. **34**(1961), 46-50.

[Pt] Parter S.V. *Solutions of a differential equation arising in chemical reactor processes*. SIAM J. appl. Math. **26**(1974), 687-716.

[PS] Pucci P., and Serrin J. *The strong maximum principle revised*. J. Diff. Eqns. **196(1)** (2004), 1-66.

[PW] Protter M. H., and Weinberger H. F. *Maximum principle in differential equations*. Prentice Hall (1967).

[Ra] Rabinowitz P. *Minimax methods in critical point theory with applications to differential equations*. Regional Conference Series in Mathematics, 65, Providence R.I. AMS (1986).

[RS] Ramaswamy M., and Shivaji R. *Multiple positive solutions for a classes of p-Laplacian equations*. Diff. and Int. Eqns. **17**(2004), 1255-1261.

[Sa] Sattinger D.H. *Monotone methods in nonlinear elliptic and parabolic boundary value problems*. Indiana Univ. Math. J. **21**, (1971/1972), 979-1000. MR **45:**8969

[Sh] Shivaji R. *A remark on the existence of three solutions via sub-super solutions*. Nonlin. Anal. and Appl., Lecture notes in pure and applied mathematics. **Vol. 109**(1987), 561-566.

[Ta] Tam K. K. *Construction of upper and lower solutions for a problem in combustion theory*. J. math. Analysis Applic. **69**(1979), 131-145.

[Te] Terikas A. *Stability and instability of positive solutions fo semipositone problems*. Proc. AMS. **114(4)** (1992), 1035-1040.