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On existence and global attractivity of periodic solutions of higher order nonlinear difference equations

Justin B. Smith

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By

Justin B. Smith

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By

Justin B. Smith

Approved:

Chuanxi Qian
(Major Professor)

Dinh H. Dang
(Committee Member)

Mohsen Razzaghi
(Committee Member)

Shantia Yarahmadian
(Committee Member)

Robert C. Smith
(Committee Member)

Mohammad Sepehrifar
(Graduate Coordinator)

Rick Travis
Dean
College of Arts and Sciences

Name: Justin B. Smith

Date of Degree: May 01, 2020

Institution: Mississippi State University

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Major Professor: Chuanxi Qian

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Candidate for Degree of Doctor of Philosophy

Difference equations arise in many fields of mathematics, both as discrete analogs of continuous behavior (analysis, numerical approximations) and as independent models for discrete behavior (population dynamics, economics, biology, ecology, etc.). In recent years, many models - especially in mathematical biology - are based on higher order nonlinear difference equations. As a result, there has been much focus on the existence of periodic solutions of certain classes of these equations and the asymptotic behavior of these periodic solutions. In this dissertation, we study the existence and global attractivity of both periodic and quasiperiodic solutions of two different higher order nonlinear difference equations. Both equations arise in biological applications.

Key words: Difference Equation, Global Attractivity, Periodic, Quasiperiodic, Forcing Term

DEDICATION

To Jamie, Hayden, Hudson, and Evan. Thank you for all your love and support during this extended adventure.

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> "Blessed is the man who trusts in the LORD, whose trust is the LORD."
> *Jeremiah 17:7*

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CHAPTER 1

INTRODUCTION

1.1 Background and Definitions

Our purpose in this dissertation is to study the existence and global attractivity of periodic and quasiperiodic solutions of two nonlinear difference equations of order $k+1$. In general, a difference equation of order $k+1$ has the form

$$x(n+1) = F(n, x(n), x(n-1), \cdots, x(n-k)),\ n = 0, 1, \cdots, \tag{1.1}$$

where k is a positive integer, F is a continuous function. Equations of the form (1.1) are important in many current applications as the next state of the model, the $(n+1)$ term, depends on the k previous states of the model instead of only its current state. Some higher order difference equations are studied as discrete analogs of delay differential equations, but there are many difference equations that model diverse physiological phenomena in their own right.

The following definitions and theorems from [35,36] will be useful in defining and proving results in Chapters 2 and 3.

Definition 1 (Periodic Solution)

A solution $\{x(n)\}_{n\geq -k}$ of (1.1) is said to be eventually periodic with period $p \in \mathbb{N}$ if there exists some $n_0 \geq -k$ such that

$$x(n+p) = x(n),\ n \geq n_0. \tag{1.2}$$

If $n_0 = -k$, then we simply say the solution is periodic. As convention dictates, we will refer to both solutions as periodic solutions.

Definition 2 (Quasiperiodic Solution)

A solution $\{x(n)\}$ of (1.1) is said to be quasiperiodic if there exist sequences $\{p(n)\}$ and $\{q(n)\}$ such that $\{p(n)\}$ is periodic with period $\omega \in \mathbb{N}$, $\{q(n)\} \to 0$ as $n \to \infty$, and

$$x(n) = p(n) + q(n),\ n = 0, 1, \cdots . \tag{1.3}$$

Definition 3 (Oscillate)

A sequence $\{x(n)\}$ is said to oscillate about zero, or oscillate, if the terms of $x(n)$ are not all eventually positive or eventually negative. Otherwise, the sequence is called nonoscillatory. For sequences $\{x(n)\}$ and $\{y(n)\}$, $\{x(n)\}$ is said to oscillate about $\{y(n)\}$ if the sequence $\{x(n) - y(n)\}$ oscillates about zero.

Definition 4 (Global Attractor)

A sequence $\{\tilde{x}(n)\}$ is said to be the global attractor of solutions of Equation (1.1) if $\{\tilde{x}(n)\}$ is a solution of Equation (1.1), and for every solution $\{x(n)\}$ of Equation (1.1),

$$\lim_{n\to\infty} (x(n) - \tilde{x}(n)) = 0. \tag{1.4}$$

Theorem 1 (Schauder Fixed Point Theorem)

Let X be a normed vector space, and let $\Lambda \subseteq X$ be a non-empty, compact, and convex set. Suppose $T : \Lambda \to \Lambda$ is continuous. Then T has a fixed point in Λ, i.e. there exists $x \in \Lambda$ such that $Tx = x$.

Definition 5 (Convex Subset)

Let X be a normed vector space. A subset $\Lambda \subseteq X$ is convex if for any $x, y \in \Lambda$, $(1 - t)x + ty \in \Lambda$ for $0 \le t \le 1$.

1.2 Plan and Approach

In this dissertation we study the existence of periodic and/or quasiperiodic solutions, as well as the global attractivity of these solutions, in two different classes of higher order difference equations. In Chapter 2 our focus will be on the equation

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)),\ n = 1, 2, \cdots , \tag{1.5}$$

where $f(n, x), g(n, x) : \{0, 1, \cdots\} \times [0, \infty) \to [0, \infty)$ are continuous functions in x and periodic functions in n with period p, and k is a nonnegative integer. We will show the existence of a periodic solution of Equation (1.5) under certain conditions and obtain a sufficient condition such that this periodic solution is unique and the global attractor of all nonnegative solutions of Equation (1.5).

In Chapter 3 our focus will be on the equation

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)) + b(n),\ n = 0, 1, \cdots , \tag{1.6}$$

where where $f(n, x), g(n, x) : \{0, 1, \cdots\} \times [0, \infty) \to [0, \infty)$ are continuous functions in x and periodic functions in n with period ω, $\{b(n)\}$ is a real sequence, and k is a nonnegative integer.

Equation (1.5) can be considered the corresponding homogeneous equation of Equation (1.6). In Chapter 3 we will use results from Chapter 2 and obtain a sufficient condition such that all nonnegative solutions of Equation (1.6) converge to the unique periodic solution of Equation (1.5), i.e. the unique periodic solution of Equation (1.5) is the global attractor of all nonnegative solutions of Equation (1.6). Both Equation (1.5) and (1.6) emerge from results in mathematical biology.

In Chapter 2, we will construct a linear operator that satisfies the hypotheses of the Schauder Fixed Point Theorem. We will then show that this fixed point is a periodic solution of Equation (1.5). We then prove that every nonnegative nonoscillatory solution of (1.5) converges to this periodic solution. Next we show that every nonnegative oscillatory solution of (1.5) also converges to this periodic solution.

In Chapter 3, we will use the results from Chapter 2, as well as nontrivial results necessary to deal with the nonhomogeneous Equation (1.6), in order to follow a similar process as the one described above in proving that every nonnegative solution of (1.6) converges to the periodic solution of Equation (1.5).

Detailed proofs are provided in the following chapters. Applications from mathematical biology, as well as numerical examples to further illustrate these results, will be given in both chapters. The results in Chapter 2 have been published in [35]; the results in Chapter 3 have been published in [36].

CHAPTER 2

EXISTENCE AND GLOBAL ATTRACTIVITY OF PERIODIC SOLUTIONS IN A HIGHER ORDER DIFFERENCE EQUATION

2.1 Introduction

In this chapter, we consider the existence and global attractivity of periodic solutions of the following higher order difference equation

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)),\ n = 0, 1, \cdots, \tag{2.1}$$

where $f(n, x,), g(n, x) : \{0, 1, \cdots\} \times [0, \infty) \to [0, \infty)$ are continuous functions in x and periodic functions with period p in n, and k is a nonnegative integer.

The existence of periodic solutions of difference equations has been studied by numerous authors, and many interesting results have been obtained, see, for example [1, 4, 12, 13, 17–19, 27, 37, 40, 44–47, 53, 55] and the references cited therein. However, to the best of our knowledge, not much work has been done for equations of the form (2.1) on this topic. In addition, the study of global attractivity of periodic solutions of difference equations is also relatively scarce. Recently, the global attractivity of periodic solutions of the following difference equation

$$x(n+1) = a(n)x(n) + g(n, x(n-k)),\ n = 0, 1, \cdots, \tag{2.2}$$

which is a special case of Equation (2.1) with $f(n,x) = a(n)x$, has been studied in [32], along with some applications to difference equations derived from mathematical biology. However, the results previously obtained cannot be applied to cases when $f(n,x)$ is a nonlinear function in x such as the equation

$$x(n+1) = \frac{a(n)x^2(n)}{b(n)+x(n)} + c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}},\ n = 0, 1, \cdots, \tag{2.3}$$

where $\{a(n)\}$, $\{b(n)\}$, $\{c(n)\}$, $\{q(n)\}$, and $\{r(n)\}$ are nonnegative periodic sequences with period p, and k is a nonnegative integer. When $a(n) \equiv a, b(n) \equiv b, c(n) \equiv c, q(n) \equiv q$, and $r(n) \equiv r$ are all positive constants and $k = 0$, Equation (2.3) reduces to

$$x(n+1) = \frac{ax^2(n)}{b+x(n)} + c\frac{e^{r-qx(n)}}{1+e^{r-qx(n)}},\ n = 0, 1, \cdots. \tag{2.4}$$

Equation (2.4) is a biological model derived from the evaluation of a perennial grass [57]. The boundedness and the persistence of positive solutions, the existence, the attractivity, and the global asymptotic stability of the unique positive equilibrium and the existence of periodic solutions of Equation (2.4) have been studied in [27] and [32]. Other recent work on the attractivity of periodic solutions can be seen in [46] and [47] and references cited therein.

In the next section, we first show that under certain conditions, Equation (2.1) has a nonnegative perodic solution $\{\tilde{x}(n)\}$ with period p by employing Schauder's Fixed Point Theorem. We then establish a sufficient condition for $\{\tilde{x}(n)\}$ to be a global attractor of all nonnegative solutions of the equation. For the proof of the global attractivity, we develop the method used in [32]. Related methods have been used in a recent paper [33] for the global attractivity of equilibria of a nonlinear difference equation. In addition, some related equations have been studied by the first author of [32] and collaborators for some time, see [10, 11, 26, 30].

In Section 2.3, we apply the results obtained in Section 2.2 to Equation (2.3) to establish some sufficient conditions for the existence of a periodic solution $\{\tilde{x}(n)\}$and for $\{\tilde{x}(n)\}$to be a global attractor of all nonnegative solutions of the equation. Some interesting special cases of Equation (2.3) are also discussed. In addition, we show that our results can be applied to the following Ricatti difference equation

$$x(n+1) = \frac{\alpha(n)x(n) + \beta(n)}{\gamma(n)x(n) + \delta(n)},\ n = 0, 1, \cdots, \tag{2.5}$$

where $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\gamma(n)\}$, and $\{\delta(n)\}$ are nonnegative periodic sequences with period p. Ricatti difference equations appear in mathematical biology, e.g. the discrete logistic model without delay is a Ricatti difference equation, see [4, 28, 29, 40] . Various properties and applications of Riccati difference equations have been studied by numerous authors, see for example, [17, 18] and the references cited therein. However, it seems that there are not many results on the existence and global attractivity of periodic solutions of these kinds of equations.

2.2 Main Results

The following theorem provides a sufficient condition for the existence of nonnegative periodic solutions of Equation (2.1). For the sake of convenience, we adopt the notation $\sum_{i=m}^{n} s(i) = 1$ and $\sum_{i=m}^{n} s(i) = 0$ whenver $\{s(n)\}$ is a real sequence and $m > n$ in the following discussion.

Theorem 2 (Existence of a Periodic Solution)

Assume that there is a nonnegative periodic sequence $\{a(n)\}$ *with period* p *such that*

$$\hat{a} = \prod_{i=0}^{p-1} a(i) < 1, \ and \ f(n,x) \leq a(n)x \ for \ n = 0, 1, \cdots, p-1 \ and \ x \geq 0 \tag{2.6}$$

and that $f(n,x) - a(n)x$ *is nonincreasing in* x. *Suppose also that* $g(n,x)$ *is nonincreasing in* x *and that there is a positive constant* B *such that*

$$\sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) [f(j,B) - a(j)B + g(j,B)] \geq 0, \ n = 0, 1, \cdots, p-1 \tag{2.7}$$

and

$$\frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) g(j,0) \leq B, \ n = 0, 1, \cdots, p-1. \tag{2.8}$$

Then Equation (2.1) has a nonnegative periodic solution $\{\tilde{x}(n)\}$ *with period* p.

Proof: Let $\mathbf{x} = \{x(n)\}_{n=-k}^{\infty}$ be a real sequence and let

$$X = \{\mathbf{x} : \mathbf{x} \text{ satisfies } x(n+p) = x(n), \ n \geq -k\}. \tag{2.9}$$

Then X is a normed vector space with the usual linear operations and norm

$$||\mathbf{x}|| = \sup_{0 \leq n \leq p-1} |x(n)|. \tag{2.10}$$

Let Λ be a subset of X defined by

$$\Lambda = \{\mathbf{x} : \mathbf{x} \in X \text{ with } 0 \leq x(n) \leq B, \ n \geq -k\}. \tag{2.11}$$

First we show Λ is convex. Let $\{x(n)\}, \{y(n)\} \in \Lambda$, $t \in [0,1]$. Then as

$$(1-t)x(n+p) + ty(n+p) = (1-t)x(n) + ty(n) \text{ and}$$

$$0 \leq (1-t)x(n) + ty(n) \leq (1-t)B + tB = B,$$

$$(1-t)x(n) + y(n) \in \Lambda.$$

Thus, Λ is a convex subset of X. Next we show that Λ is a compact set. We first see that every sequence $\{x(n)\} \in \Lambda$ has a convergent subsequence, namely $\{x_p(n)\} \subset \{x(n)\}$, where $x_p(n) = x(p_n)$ and $p_n = np$ for $n \geq -k$. As every sequence in Λ has a convergent subsequence, Λ is sequentially compact. As sequentially compact implies compact in a metric space, Λ is a compact set.

We now define a mapping T on Λ as follows: for each $\mathbf{x} = \{x(n)\} \in \Lambda$,

$$Tx(n) = \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) [f(j, x(j)) - a(j)x(j) + g(j, x(j-k))]. \tag{2.12}$$

Clearly T is continuous as f and g are periodic in n and continuous in x. We next show that $T : \Lambda \to \Lambda$. In fact, by noting that $f(n, x) - a(n)x$ and $g(n, x)$ are nonincreasing in x, and that (2.7) and (2.8) hold, it is easy to see that

$$Tx(n) \geq \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) [f(j, B) - a(j)B + g(j, B)] \geq 0 \tag{2.13}$$

and

$$Tx(n) \leq \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) g(j, 0) \leq B. \tag{2.14}$$

Next observe that

$$\begin{aligned} Tx(n+p) &= \frac{1}{1-\hat{a}} \sum_{j=n+p}^{n+2p-1} \left(\prod_{i=j+1}^{n+2p-1} a(i) \right) [f(j, x(j)) - a(j)x(j) + g(j, x(j-k))] \\ &= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i+p) \right) [f(j+p, x(j+p)) - a(j+p)x(j+p) \\ &\quad +g(j+p, x(j+p-k))] \\ &= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) [f(j, x(j)) - a(j)x(j) + g(j, x(j-k))] \\ &= Tx(n). \end{aligned} \tag{2.15}$$

Thus by the Schauder Fixed Point Theorem, T has a fixed point $\tilde{\mathbf{x}} = \{x(n)\} \in \Lambda$. We claim that $\tilde{\mathbf{x}}$ is a solution of Equation (2.1). In fact, by noting

$$\prod_{i=n+1}^{n+p} a(i) = \hat{a}, \prod_{n+p+1}^{n+p} a(i) = 1, T\tilde{x}(n) = \tilde{x}(n) \tag{2.16}$$

and that f, g, and $\{\tilde{x}(n)\}$ are periodic in n with period p, we see that

$$\begin{aligned}
T\tilde{x}(n+1) &= \frac{1}{1-\hat{a}} \sum_{j=n+1}^{n+p} \left(\prod_{j+1}^{n+p} a(i) \right) [f(j, \tilde{x}(j)) - a(j)\tilde{x}(j) \\
&\quad + g(j, \tilde{x}(j-k))] \\
&= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{j+1}^{n+p} a(i) \right) [f(j, \tilde{x}(j)) - a(j)\tilde{x}(j) + g(j, \tilde{x}(j-k))] \\
&\quad - \frac{1}{1-\hat{a}} \left(\prod_{n+1}^{n+p} a(i) \right) [f(n, \tilde{x}(n)) - a(n)\tilde{x}(n) + g(n, \tilde{x}(n-k))] \\
&\quad + \frac{1}{1-\hat{a}} \left(\prod_{n+p+1}^{n+p} a(i) \right) [f(n+p, \tilde{x}(n+p)) - a(n+p)\tilde{x}(n+p) \\
&\quad + g(n+p, \tilde{x}(n+p-k))] \\
&= \frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{j+1}^{n+p} a(i) \right) [f(j, \tilde{x}(j)) - a(j)\tilde{x}(j) + g(j, \tilde{x}(j-k))] \\
&\quad - \frac{\hat{a}}{1-\hat{a}} [f(n, \tilde{x}(n)) - a(n)\tilde{x}(n) + g(n, \tilde{x}(n-k))] \\
&\quad + \frac{1}{1-\hat{a}} [f(n, \tilde{x}(n)) - a(n)\tilde{x}(n) + g(n, \tilde{x}(n-k))] \\
&= \frac{a(n+p)}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{j+1}^{n+p} a(i) \right) [f(j, \tilde{x}(j)) - a(j)\tilde{x}(j) + g(j, \tilde{x}(j-k))] \\
&\quad f(n, \tilde{x}(n)) - a(n)\tilde{x}(n) + g(n, \tilde{x}(n-k)) \\
&= a(n)T\tilde{x}(n) + f(n, \tilde{x}(n)) - a(n)\tilde{x}(n) + g(n, \tilde{x}(n-k)) \\
&= f(n, \tilde{x}(n)) + g(n, \tilde{x}(n-k)) \\
&= f(n, T\tilde{x}(n)) + g(n, T\tilde{x}(n-k)).
\end{aligned} \tag{2.17}$$

Hence, $\{T\tilde{x}(n)\}$ satisfies Equation (2.1) and so $\{T\tilde{x}(n)\}$, that is $\{\tilde{x}(n)\}$, is a periodic solution of Equation (2.1) with period p. The proof is complete.

The following theorem provides a sufficient condition for a periodic solution of Equation (2.1) to be a global attractor of all nonnegative solutions of Equation (2.1). This theorem is an extension and improvement of the corresponding result obtained in [32] for Equation (2.2). We relax the condition

$$0 < a(n) \leq 1 \text{ and } a(n) \not\equiv 1,\ n = 0, 1, \cdots, p-1 \tag{2.18}$$

assumed in [32] to the more general condition

$$\prod_{i=0}^{p-1} a(i) < 1 \text{ and } a(n) \geq 0,\ n = 0, 1, \cdots, p-1 \tag{2.19}$$

which has been assumed above in the hypothesis of Theorem 2 for the existence of periodic solutions. This relaxation of the condition improves upon the previous work by making the result more applicable.

Theorem 3 (Uniqueness and Global Attractivity of a Periodic Solution)

Assume that $f(n, x)$ is nondecreasing in x and that there is a nonnegative sequence $\{a(n)\}$ with period p such that (2.6) holds and $f(n, x) - a(n)x$ is nonincreasing in x. Suppose also that $g(n, x)$ is nonincreasing in x and there is a nonnegative periodic sequence $\{L(n)\}$ with period p such that

$$|g(n, x) - g(n, y)| \leq L(n)\,|x - y|\,,\ n = 0, 1, \cdots, p-1 \tag{2.20}$$

and that either

$$a(n) \leq 1 \ \textit{and} \ \sum_{j=n}^{n+k} \left(\prod_{i=j+1}^{n+k+p-1} a(i) \right) L(j) < 1,\ n = 0, 1, \cdots, p-1. \tag{2.21}$$

or

$$\sum_{j=n}^{n+k+p-1}\left(\prod_{i=j+1}^{n+k+p-1} a(i)\right) L(j) < 1,\ n = 0, 1, \cdots, p-1. \tag{2.22}$$

If Equation (2.1) has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p, the $\{\tilde{x}(n)\}$ is the only periodic solution of Equation (2.1) and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.1). That is,

$$\lim_{n\to\infty} x(n) - \tilde{x}(n) = 0. \tag{2.23}$$

Proof: Clearly, if we can show that every nonnegative solution of Equation (2.1) converges to $\{\tilde{x}(n)\}$, then $\{\tilde{x}(n)\}$ is the unique periodic solution. Let $y(n) = x(n) - \tilde{x}(n)$. Then $\{y(n)\}$ satisfies the equation

$$y(n+1) + \tilde{x}(n+1) = f(n, y(n) + \tilde{x}(n)) + g(n, y(n-k) + \tilde{x}(n-k)). \tag{2.24}$$

Since $\{\tilde{x}(n)\}$ is a solution of Equation (2.1),

$$\tilde{x}(n+1) = f(n, \tilde{x}) + g(n, \tilde{x}(n-k)). \tag{2.25}$$

Hence it follows that

$$\begin{aligned} y(n+1) &= f(n, y(n) + \tilde{x}(n)) - f(n, \tilde{x}(n)) \\ &\quad + g(n, y(n-k) + \tilde{x}(n-k)) - g(n, \tilde{x}(n-k)). \end{aligned} \tag{2.26}$$

First assume that $\{x(n)\}$ does not oscillate about $\{\tilde{x}(n)\}$. Then $\{y(n)\}$ is either eventually positive or eventually negative. We assume that $\{y(n)\}$ is eventually positive. The proof for the case that $\{y(n)\}$ is eventually negative is similar and will be omitted. Hence, there exists a positive integer n_0 such that $y(n) > 0,\ n \geq n_0$. Then by noting that g is nonincreasing in x, it follows from (2.26) that

$$y(n+1) \leq f(n, y(n) + \tilde{x}(n)) - f(n, \tilde{x}(n)),\ n > n_0 + k. \tag{2.27}$$

Since $f(n,x) - a(n)x$ is also nonincreasing in x,

$$f(n, y(n) + \tilde{x}(n)) - a(n)(y(n) + \tilde{x}(n)) \leq f(n, \tilde{x}(n)) - a(n)\tilde{x}(n),\ n \geq n_0. \tag{2.28}$$

Hence, (2.27) and (2.28) yield

$$y(n+1) \leq a(n)(y(n) + \tilde{x}(n)) - a(n)\tilde{x}(n) = a(n)y(n),\ n \geq n_0 + k \tag{2.29}$$

and so it follows that

$$y(n) \leq \left(\prod_{i=n_0+k}^{n-1} a(i) \right) y(n_0 + k),\ n \geq n_0 + k. \tag{2.30}$$

By noting $a(n) \geq 0$, $a(n)$ is p-periodic and $\prod_{i=0}^{p-1} a(i) < 0$, we see that

$$\prod_{i=n_0+k}^{n-1} a(i) \to 0 \text{ as } n \to \infty \tag{2.31}$$

Hence $y(n) \to 0$ as $n \to \infty$ and so (2.23) holds.

Next, assume that $\{x(n)\}$ oscillates about $\{\tilde{x}(n)\}$ and so $\{y(n)\}$ oscillates about zero. Then there is an increasing sequence $\{n_t\}$ of positive integers with $n_1 \geq k$ such that $y(n_1) \leq 0$ and for $t = 1, 2, \cdots$,

$$y(n) > 0 \text{ for } n_{2t-1} < n \leq n_{2t} \tag{2.32}$$

and

$$y(n) \leq 0 \text{ for } n_{2t} < n \leq n_{2t+1}. \tag{2.33}$$

We claim that when $n_1 < n \le n_2$,

$$y(n) \le \sum_{j=n_1}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) \left[g(j, y(j-k) + \tilde{x}(j-k)) - g(j, \tilde{x}(j-k)) \right]. \tag{2.34}$$

In fact, from (2.26) we see that

$$\begin{aligned} y(n_1+1) &= f(n_1, y(n_1) + \tilde{x}(n_1)) - f(n_1, \tilde{x}(n_1)) \\ &\quad + g(n_1, y(n_1-k) + \tilde{x}(n_1-k)) - g(n_1, \tilde{x}(n_1-k)). \end{aligned} \tag{2.35}$$

Since $y(n_1) \le 0$ and $f(n,x)$ is nondecreasing in x, $f(n_1, y(n_1) + \tilde{x}(n_1)) \le f(n_1, \tilde{x}(n_1))$. Hence, it follows from (2.35) that

$$y(n_1+1) \le g(n_1, y(n_1-k) + \tilde{x}(n_1-k)) - g(n_1, \tilde{x}(n_1-k)), \tag{2.36}$$

that is, (2.34) holds when $n = n_1 + 1$. Next, assume that (2.34) holds when $n = m$ with $n_1 < m < n_2$,

$$y(m) \le \sum_{j=n_1}^{m-1} \left(\prod_{i=j+1}^{m-1} a(i) \right) \left[g(j, y(j-k) + \tilde{x}(j-k)) - g(j, \tilde{x}(j-k)) \right]. \tag{2.37}$$

From (2.26) we see that

$$\begin{aligned} y(m+1) - a(m)y(m) &= f(m, y(m) + \tilde{x}(m)) - f(m, \tilde{x}(m)) - a(m)y(m) \\ &+ g(m, y(m-k) + \tilde{x}(m-k)) - g(m, \tilde{x}(m-k)). \end{aligned} \tag{2.38}$$

Since $f(n,x) - a(n)x$ is nonincreasing in x and $y(m) > 0$,

$$f(m, y(m) + \tilde{x}(m)) - f(m, \tilde{x}(m)) - a(m)y(m) \le 0. \tag{2.39}$$

Hence it follows from (2.38) that

$$\begin{aligned} y(m+1) &\leq a(m)\sum_{j=n_1}^{m-1}\left(\prod_{i=j+1}^{m-1}a(i)\right)[g(j,y(j-k)+\tilde{x}(j-k))-g(j,\tilde{x}(j-k))] \\ &\quad +g(m,y(m-k)+\tilde{x}(m-k))-g(m,\tilde{x}(m-k)) \\ &= \sum_{j=n_1}^{m}\left(\prod_{i=j+1}^{m}a(i)\right)[g(j,y(j-k)+\tilde{x}(j-k)) \\ &\quad -g(j,\tilde{x}(j-k))], \end{aligned} \tag{2.40}$$

that is, (2.34) holds when $n = m+1$. Therefore, by mathematical induction, (2.34) holds when $n_1 < n \leq n_2$.

Since $g(n,x)$ is periodic in n with period p and (2.20) holds, we see that

$$|g(n,x)-g(n,y)| \leq L(n)|x-y|,\ n=0,1,\cdots. \tag{2.41}$$

Then it follows from (2.34) that when $n_1 < n \leq n_2$

$$\begin{aligned} y(n) &\leq \sum_{j=n_1}^{n-1}\left(\prod_{i=j+1}^{n-1}a(i)\right)|g(j,y(j-k)+\tilde{x}(j-k))-g(j,\tilde{x}(j-k))| \\ &\leq \sum_{j=n_1}^{n-1}\left(\prod_{i=j+1}^{n-1}a(i)\right)L(j)\,|y(j-k)|\,. \end{aligned} \tag{2.42}$$

First, assume that (2.21) holds. By noting the periodic property of $\{a(n)\}$ and $\{L(n)\}$, we see that there is a positive constant $c < 1$ such that

$$\sum_{j=n}^{n+k}\left(\prod_{i=j+1}^{n+k}a(i)\right)L(j) \leq c,\ n=0,1,\cdots. \tag{2.43}$$

We claim that

$$|y(n)| \leq c\max_{n_1-k\leq s\leq n_1}|y(s)| \text{ for } n_1 < n \leq n_2. \tag{2.44}$$

To this end, consider the two cases $n_2 \le n_1 + k + 1$ and $n_2 > n_1 + k + 1$, respectively. When $n_2 \le n_1 + k + 1$, then for any $n_1 < n \le n_2$, $n - k - 1 \le n_1$ and so (2.42) yields

$$\begin{aligned} y(n) &\le \sum_{j=n_1}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) L(j) \max_{n_1 - k \le s \le n_1} |y(s)| \\ &\le \sum_{j=n-k-1}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) L(j) \max_{n_1 - k \le s \le n_1} |y(s)| \end{aligned} \tag{2.45}$$

Then by noting (2.43), we see that (2.44) holds. Next, consider the case $n_2 > n_1 + k + 1$. When $n_1 < n \le n_1 + k + 1$, as we have shown above, (2.44) holds. Hence, we only need to show that (2.44) holds when $n_1 + k + 1 < n \le n_2$. To this end, first we show that

$$y(n) \le a(n-1)y(n-1),\ n_1 + k + 1 < n \le n_2. \tag{2.46}$$

In fact, by noting that when $n_1 + k + 1 < n \le n_2, y(n-1) > 0$ and $f(n,x) - a(n)x$ is nonincreasing in x, we see that

$$\begin{aligned} f(n-1, y(n-1) + \tilde{x}(n-1)) &- a(n-1)(y(n-1) + \tilde{x}(n-1)) \\ &\le f(n-1, \tilde{x}(n-1)) - a(n-1)\tilde{x}(n-1) \end{aligned} \tag{2.47}$$

which yields

$$f(n-1, y(n-1) + \tilde{x}(n-1)) - f(n-1, \tilde{x}(n-1)) \le a(n-1)y(n-1). \tag{2.48}$$

In addition, by noting that when $n_1 + k + 1 < n \le n_2, y(n-1-k) > 0$ and $g(\cdot, x)$ is nonincreasing in x, we see that

$$g(n-1, y(n-1-k) + \tilde{x}(n-1-k)) - g(n-1, \tilde{x}(n-1-k)) \le 0. \tag{2.49}$$

Then it follows from (2.26) that

$$\begin{aligned} y(n) &= f(n-1, y(n-1)+\tilde{x}(n-1)) - f(n-1, \tilde{x}(n-1)) \\ &\quad + g(n-1, y(n-1-k)+\tilde{x}(n-1-k)) - g(n-1, \tilde{x}(n-1-k)) \\ &\leq a(n-1)y(n-1) \end{aligned} \tag{2.50}$$

and so (2.46) holds. Then by noting $a(n) \in [0,1]$, it follows that

$$y(n_2) \leq y(n_2-1) \leq \cdots \leq y(n_1+1+k), \tag{2.51}$$

which implies that (2.44) holds when $n_1+k+1 < n \leq n_2$. Hence for any $n_1 < n < n_2$, (2.44) holds. By a similar argument, we will show that

$$y(n) \geq -c \max_{n_2-k \leq s \leq n_2} |y(s)|\,,\ n_2 < n \leq n_3. \tag{2.52}$$

To this end, we first claim that when $n_2 < n \leq n_3$,

$$y(n) \geq \sum_{j=n_2}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) \left[g(j, y(j-k)+\tilde{x}(j-k)) - g(j, \tilde{x}(j-k)) \right]. \tag{2.53}$$

From (2.26) we see that

$$\begin{aligned} y(n_2+1) &= f(n_2, y(n_2)+\tilde{x}(n_2)) - f(n_2, \tilde{x}(n_2)) \\ &\quad + g(n_2, y(n_2-k)+\tilde{x}(n_2-k)) - g(n_2, \tilde{x}(n_2-k)). \end{aligned} \tag{2.54}$$

Since $y(n_2) > 0$ and $f(n,x)$ is nondecreasing in x, $f(n_2, y(n_2)+\tilde{x}(n_2)$. Hence it follows from (2.54) that

$$y(n_2+1) \geq g(n_2, y(n_2-k)+\tilde{x}(n_2-k)) - g(n_2, \tilde{x}(n_2-k)), \tag{2.55}$$

that is, (2.53) holds when $n = n_2 + 1$. Next, assume that (2.53) holds when $n = m$ with $n_2 < m < n_3$,

$$y(m) \geq \sum_{j=n_2}^{m-1} \left(\prod_{i=j+1}^{m-1} a(i) \right) \left[g(j, y(j-k) + \tilde{x}(j-k)) - g(j, \tilde{x}(j-k)) \right]. \tag{2.56}$$

From (2.26) we see that

$$\begin{aligned} y(m+1) - a(m)y(m) &= f(m, y(m) + \tilde{x}(n)) - f(m, \tilde{x}(m)) - a(m)y(m) \\ &+ g(m, y(m-k) + \tilde{x}(m-k)) - g(m, \tilde{x}(m-k)). \end{aligned} \tag{2.57}$$

Since $f(n, x) - a(n)x$ is nonincreasing in x and $y(m) \leq 0$,

$$f(m, y(m) + \tilde{x}(m)) - a(m)(y(m) + \tilde{x}(m)) \geq f(m, \tilde{x}(m)) - a(m)\tilde{x}(m) \tag{2.58}$$

which yields

$$f(m, y(m) + \tilde{x}(m)) - f(m, \tilde{x}(m)) - a(m)y(m) \geq 0. \tag{2.59}$$

Hence, it follows from (2.57) that

$$y(m+1) \geq a(m)y(m) + g(m, y(m-k) + \tilde{x}(m-k)) - g(m, \tilde{x}(m-k)). \tag{2.60}$$

Then by noting (2.56) we see that

$$\begin{aligned} y(m+1) &\geq a(m) \sum_{j=n_2}^{m-1} \left(\prod_{i=j+1}^{m-1} a(i) \right) \left[g(j, y(j-k) + \tilde{x}(j-k)) - g(j, \tilde{x}(j-k)) \right] \\ &\quad + g(m, y(m-k) + \tilde{x}(m-k)) - g(m, \tilde{x}(m-k)) \\ &= \sum_{j=n_2}^{m} \left(\prod_{i=j+1}^{m} a(i) \right) \left[g(j, y(j-k) + \tilde{x}(j-k)) - g(j, \tilde{x}(j-k)) \right], \end{aligned} \tag{2.61}$$

that is, (2.53) holds when $n = m+1$. Therefore, by mathematical induction, (2.53) holds when $n_2 < n \leq n_3$.

As before, since $g(n,x)$ is periodic in n with period p and (2.20) holds, we see that

$$|g(n,x) - g(n,y)| \leq L(n)\,|x-y|\,,\ n = 0, 1, \cdots . \tag{2.62}$$

It follows from (2.53) that when $n_2 < n \leq n_3$, as $y(n) \leq 0$,

$$\begin{aligned} |y(n)| &= -y(n) \leq \left| \sum_{j=n_2}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) [g(j, y(j-k) + \tilde{x}(j-k)) - g(j, \tilde{x}(j-k))] \right| \\ &\leq \sum_{j=n_2}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) |g(j, y(j-k) + \tilde{x}(j-k)) - g(j, \tilde{x}(j-k))| \\ &\leq \sum_{j=n_2}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) L(j)\,|y(j-k)|\,. \end{aligned} \tag{2.63}$$

To show (2.52) holds, we consider the two cases $n_3 \leq n_2 + k + 1$ and $n_3 > n_2 + k + 1$, respectively. When $n_3 \leq n_2 + k + 1$, then for any $n_2 < n \leq\leq n_3$, $n - k - 1 \leq n_2$ and so (2.63) yields

$$\begin{aligned} -y(n) &\leq \sum_{j=n_2}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) L(j) \max_{n_2-k \leq s \leq n_2} |y(s)| \\ &\leq \sum_{j=n-k-1}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) L(j) \max_{n_2-k \leq s \leq n_2} |y(s)|\,. \end{aligned} \tag{2.64}$$

By noting (2.43)

$$-y(n) \leq c \max_{n_2-k \leq s \leq n_2} |y(s)|\,, \tag{2.65}$$

thus (2.52) holds. Next, consider the case that $n_3 > n_2 + k + 1$. For $n_2 < n \leq n_2 + k + 1$, as we have shown above, (2.52) holds. Hence, we only need to show that (2.52) holds when $n_2 + k + 1 < n \leq n_3$. To this end, we first show that

$$y(n) \geq a(n-1)(y(n-1),\ n_2 + k + 1 < n \leq n_3. \tag{2.66}$$

By noting that when $n_2 + k + 1 < n \leq n_3$, $y(n-1) \leq 0$ and $f(n, x) - a(n)x$ is nonincreasing in x, we see that

$$\begin{aligned} & f(n-1, y(n-1) + \tilde{x}(n-1)) - a(n-1)(y(n-1) + \tilde{x}(n-1)) \\ & \qquad \geq f(n-1, \tilde{x}(n-1)) - a(n-1)\tilde{x}(n-1) \end{aligned} \tag{2.67}$$

which yields

$$f(n-1, y(n-1) + \tilde{x}(n-1)) - f(n-1, \tilde{x}(n-1)) \geq a(n-1)y(n-1). \tag{2.68}$$

In addition, by noting that when $n_2 + k + 1 < n \leq n_3$, $y(n-1-k) \leq 0$ and $g(\cdot, x)$ is nonincreasing in x, we see that

$$g(n-1, y(n-1-k) + \tilde{x}(n-1-k)) - g(n-1, \tilde{x}(n-1-k)) \geq 0. \tag{2.69}$$

Then it follows from (2.26) that

$$\begin{aligned} y(n) &= f(n-1, y(n-1) + \tilde{x}(n-1)) - f(n-1, \tilde{x}(n-1)) \\ &\quad + g(n-1, y(n-1-k) + \tilde{x}(n-1-k)) - g(n-1, \tilde{x}(n-1-k)) \\ &\geq a(n-1)y(n-1) \end{aligned} \tag{2.70}$$

and so (2.66) holds. Then by noting that $a(n) \in [0, 1]$, as $y(n) \leq 0$ it follows that

$$y(n_3) \geq y(n_3 - 1) \geq \cdots \geq y(n_2 + 1 + k), \tag{2.71}$$

which implies that (2.52) holds when $n_2 + k + 1 < n \leq n + 3$. Thus, for any $n_2 < n \leq n_3$, (2.52) holds.

If $n_2 - k > n_1$, we see that when $n_2 - k \leq n \leq n_2$, (2.44) holds and so

$$\max_{n_2 - k \leq s \leq n_2} |y(s)| \leq c \max_{n_1 - k \leq s \leq n_1} |y(s)| \leq \max_{n_1 - k \leq s \leq n_1} |y(s)|. \tag{2.72}$$

If $n_2 - k < n_1$, we see that (2.44) holds when $n_1 < n \le n_2$; while when $n_2 - k \le n \le n_1$, by noting $n_1 - k < n_2 - k$, we see that

$$|y(n)| \le \max_{n_1 - k \le s \le n_1} |y(s)|. \tag{2.73}$$

Hence, from the above discussion, we see that

$$\max_{n_2 - k \le s \le n_2} |y(s)| \le \max_{n_1 - k \le s \le n_1} |y(s)| \tag{2.74}$$

and so it follows from (2.52) that when $n_2 < n \le n_3$,

$$y(n) \ge -c \max_{n_1 - k \le s \le n_1} |y(s)|. \tag{2.75}$$

By combining (2.44) and (2.75), we see that

$$|y(n)| \le c \max_{n_1 - k \le s \le n_1} |y(s)|, \ n_1 < n \le n_3. \tag{2.76}$$

Then by the method of steps, we may show that

$$|y(n)| \le c \max_{n_1 - k \le s \le n_1} |y(s)|, \ n > n_1. \tag{2.77}$$

Now, by choosing an $n_{r_2} \in \{n_r\}$ with $n_{r_2} > n_1 + k$ and then by using a similar argument, we may show that

$$|y(n)| \le c^2 \max_{n_1 - k \le s \le n_1} |y(s)|, \ n > n_{r_2}. \tag{2.78}$$

Finally, by induction, we may show that for any positive integer $m > 1$,

$$|y(n)| \le c^m \max_{n_1 - k \le s \le n_1} |y(s)|, \ n > n_{r_m}. \tag{2.79}$$

Since $0 < c < 1$, we see that $y(n) \to 0$ as $n \to \infty$. It follows that (2.23) holds.

Next, assume that (2.22) holds. By the periodic property of $\{a(n)\}$ and $\{L(n)\}$, there is a positive constant $d < 1$ such that

$$\sum_{j=n}^{n+k+p-1} \left(\prod_{i=j+1}^{n+k+p-1} a(i) \right) L(j) \le d, \; n = 0, 1, \cdots . \tag{2.80}$$

We claim that when $n_1 < n \le n_2$,

$$y(n) \le d \max_{n_1-k \le s \le n_1+p-1} |y(s)| . \tag{2.81}$$

To this end, consider the two cases $n_2 \le n_1 + k + p$ and $n_2 > n_1 + k + p$, respectively. When $n_2 \le n_1 + k + p$, then for any $n_1 < n \le n_2$, $n - k - p \le n_1$, and so (2.42) yields

$$\begin{aligned} y(n) &\le \sum_{j=n_1}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) L(j) \max_{n_1-k \le s \le n_1+p-1} |y(s)| \\ &\le \sum_{j=n-k-p}^{n-1} \left(\prod_{i=j+1}^{n-1} a(i) \right) L(j) \max_{n_1-k \le s \le n_1+p-1} |y(s)| . \end{aligned} \tag{2.82}$$

Then by noting (2.80) we see that (2.81) holds. Next, consider the case $n_2 > n_1 + k + p$. When $n_1 < n \le n_1 + k + p$, as we have shown above, (2.81) holds. Hence, we only need to show that (2.81) holds when $n_1 + k + p < n \le n_2$. First by the same argument used for the case (2.21) above, we have

$$y(n) \le a(n-1)y(n-1), \; n_1 + k + p < n \le n_2. \tag{2.83}$$

Hence,

$$\begin{aligned} y(n_1+k+p) &\le a(n_1+k+p)y(n_1+k+p) \\ &\le a(n_1+k+p)a(n_1+k+p-1)y(n_1+k+p-1) \\ &\vdots \\ &\le \left(\prod_{i=j+1}^{n-1} a(i)\right) y(n_1+k+1) \\ &< y(n_1+k+1), \end{aligned} \tag{2.84}$$

and similarly,

$$y(n_1+k+p+2) < y(n_1+k+2), \cdots, y(n_2) < y(n_2-p). \tag{2.85}$$

Since $y(n)$ satisfies (2.81) when $n_1 < n \le n_1+k+p$, from (2.84) and (2.85) we see that $y(n)$ also satisfies (2.75) when $n_1+k+p < n \le n_2$. Hence, (2.75) holds when $n_1 < n \le n_2$. By another argument similar to that used when (2.21) holds, we may show that there is a subsequence $\{n_{t_l}\}$ of $\{n_t\}$ with $n_{t_{l+1}} \ge n_{t_l} + k,\ l = 1, 2, \cdots$ such that

$$|y(n)| \le d^l \max_{n_1-k \le s \le n_1+p-1} |y(s)|,\ n > n_{t_l}. \tag{2.86}$$

Since $0 < d < 1$, we see that $y(n) \to 0$ as $n \to \infty$, and it follows that (2.23) holds. The proof is complete.

By combining Theorems 2 and 3, we have the following result immediately.

Theorem 4 (Existence, Uniqueness, and Global Attractivity of a Periodic Solution)

Assume that $f(n, x)$ is nondecreasing in x and that there is a nonnegative sequence $\{a(n)\}$ with period p such that (2.6) holds and $f(n, x) - a(n)x$ is nonincreasing in x. Suppose also that $g(n, x)$ is nonincreasing in x and there is a positive constant B and nonnegative *periodic sequence $\{L(n)\}$ with period p such that (2.7), (2.8), (2.20) and either (2.21) or (2.22) hold. Then Equation (2.1) has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.1).*

When $f(n,x) = a(n)x$, where $\{a(n)\}$ is a nonnegative periodic sequence with period p, Equation (2.1) reduces to Equation (2.2). Clearly, (2.7) holds for any positive B, and (2.8) holds when B is large. Hence, the following conclusion is a direct consequence of Theorem 4.

Corollary 1

Assume that $\prod_{i=0}^{p-1} a(i) < 1$, $g(n,x)$ is nonincreasing in x and that there is a nonnegative periodic sequence $\{L(n)\}$ with period p such that (2.5) and either (2.21) or (2.22) hold. Then Equation (2.2) has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.2).

When g is free of x, i.e. $g(n,x) = b(n)$, where $\{b(n)\}$ is a nonnegative periodic sequence with period p, Equation (2.1) reduces to the first order equation

$$x(n+1) = f(n, x(n)) + b(n). \tag{2.87}$$

Clearly, (2.20) and (2.22) hold with $L(n) \equiv 0$. (2.7) and (2.8) become

$$\sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) [f(j,B) - a(j)B + b(j)] \geq 0, \; n = 0, 1, \cdots, p-1 \tag{2.88}$$

and

$$\frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i) \right) b(j) \leq B, \; n = 0, 1, \cdots, p-1 \tag{2.89}$$

respectively. Hence, the following conclusion is a direct consequence of Theorem 4.

Corollary 2

Assume that $f(n,x)$ is nondecreasing in x and there is a nonnegative sequence $\{a(n)\}$ with period p such that (2.6) holds and $f(n,x) - a(n)x$ is nonincreasing in x, and that there is a positive constant B such that (2.88) and (2.89) hold. Then Equation (2.87) has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.87).

In particular, when $f(n,x) = a(n)x$ and $g(n,x) = b(n)$ where $\{a(n)\}$ and $\{b(n)\}$ are nonnegative periodic sequences with period p, Equation (2.1) reduces to the following first order linear equation:

$$x(n+1) = a(n)x(n) + b(n). \tag{2.90}$$

From Corollaries 1 and 2 we have the following conclusion immediately.

Corollary 3

Assume that $\prod_{i=0}^{p-1} a(i) < 1$. Then Equation (2.90) has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.90).

Remark 1 Recently, bounded and periodic solutions of the linear first order difference equation have been studied extensively in [46]. Several interesting results are obtained, one of which is the following: if $\{a(n)\}$ and $\{b(n)\}$ are two periodic sequences with period p and $\prod_{i=0}^{p-1} a(i)$ is different from zero and one, then (2.90) has a unique p-periodic solution $\{\tilde{x}(n)\}$. Furthermore, if $\left|\prod_{i=0}^{p-1} a(i)\right| < 1$, then every solution of (2.90) converges geometrically to $\{\tilde{x}(n)\}$ as $n \to \infty$, and it is getting away geometrically from $\{\tilde{x}(n)\}$ as $n \to -\infty$. Comparing this result with Corollary 3 we see that $\{a(n)\}$ and $\{b(n)\}$ are not required to be nonnegative, making the conclusion stronger in this result. However, $a(n) \neq 0,\ n = 0, 1, \cdots$ is not required in Corollary 3.

2.3 Applications and Numerical Examples

In this section, we apply results found in the previous sections of this chapter to equations derived from mathematical biology. A numerical example is also given to further demonstrate these results. First consider the following equation mentioned in Section 2.1

$$x(n+1) = \frac{a(n)x^2(n)}{b(n)+x(n)} + c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}},\ n = 0, 1, \cdots \tag{2.91}$$

where $\{a(n)\}$, $\{b(n)\}$, $\{c(n)\}$, $\{q(n)\}$, and $\{r(n)\}$ are nonnegative periodic sequences with period p, and k is a nonnegative integer. Equation (2.91) is in the form of Equation (2.1) with

$$f(n,x) = \frac{a(n)x^2}{b(n)+x} \text{ and } g(n,x) = c(n)\frac{e^{r(n)-q(n)x}}{1+e^{r(n)-q(n)x}}. \tag{2.92}$$

Clearly, f is nondecreasing in x and $f(n,x) \leq a(n)x$. Noting

$$\frac{d}{dx}(f(n,x) - a(n)x)' = -\frac{a(n)b^2(n)}{(b(n)+x)^2} \leq 0, \tag{2.93}$$

we see that $f(n,x) - a(n)x$ is nonincreasing in x. Observe that

$$\frac{d}{dx}(g(n,x)) = -c(n)q(n)\frac{e^{r(n)-q(n)x}}{(1+e^{r(n)-q(n)x})^2} \tag{2.94}$$

and

$$\frac{d^2}{dx^2}(g(n,x)) = c(n)q^2(n)\frac{e^{r(n)-q(n)x}(1-e^{r(n)-q(n)x})}{(1+e^{r(n)-q(n)x})^3}. \tag{2.95}$$

Clearly, $g(n,x)$ is nonincreasing in x, and for each n, $\left|\frac{dg(n,x)}{dx}\right|$ takes a maximum when $x = \frac{r(n)}{q(n)}$, and $\left|\frac{dg(n,x)}{dx}\right|_{x=\frac{r(n)}{q(n)}} = \frac{c(n)q(n)}{4}$. Hence, $g(n,x)$ is L-Lipschitz in x with $L(n) = \frac{c(n)q(n)}{4}$ for each n. By Theorem 4, we have the following conclusing immediately.

Theorem 5

Assume that $\prod_{i=0}^{p-1} a(i) < 1$, and that there is a positive B such that

$$\sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i)\right) \left[c(j)\frac{e^{r(j)-q(j)B}}{1+e^{r(j)-q(j)B}} - \frac{a(j)b(j)B}{b(j)+B}\right] \geq 0,\ n = 0, 1, \cdots, p-1 \quad (2.96)$$

and

$$\frac{1}{1-\hat{a}} \sum_{j=n}^{n+p-1} \left(\prod_{i=j+1}^{n+p-1} a(i)\right) c(j)\frac{e^{r(j)}}{1+e^{r(j)}} \leq B,\ n = 0, 1, \cdots, p-1 \quad (2.97)$$

where $\hat{a} = \prod_{i=0}^{p-1} a(i)$. Then Equation (2.91) has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p. Furthermore, if either

$$a(n) \leq 1 \text{ and } \sum_{j=n}^{n+k} \left(\prod_{i=j+1}^{n+k} a(i)\right) \frac{c(j)q(j)}{4} < 1,\ n = 0, 1, \cdots, p-1, \quad (2.98)$$

or

$$\sum_{j=n}^{n+k+p-1} \left(\prod_{i=j+1}^{n+k+p-1} a(i)\right) \frac{c(j)q(j)}{4} < 1,\ n = 0, 1, \cdots, p-1, \quad (2.99)$$

then Equation (2.91) has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.91).

Clearly, if

$$c(n)\frac{e^{r(n)-q(n)}}{1+e^{r(n)-q(n)}} \geq \frac{a(n)b(n)}{b(n)+1},\ n = 0, 1, \cdots, p-1 \quad (2.100)$$

and

$$c(n)\frac{e^{r(n)}}{1+e^{r(n)}} \leq \frac{1-\hat{a}}{\sum_{j=n}^{n+p-1}\left(\prod_{i=j+1}^{n+p-1} a(i)\right)},\ n = 0, 1, \cdots, p-1 \quad (2.101)$$

then (2.96) and (2.97) hold with $B = 1$. Hence, the following corollary is a direct consequence of Theorem 5.

Corollary 4

Assume that $\prod_{i=0}^{p-1} a(i) < 1$, *and that (2.100) and (2.101) hold. Then Equation (2.91) has a nonnegative periodic solution* $\{\tilde{x}(n)\}$. *Furthermore, if either (2.98) or (2.99) holds, then* $\{\tilde{x}(n)\}$ *is the only periodic solution of Equation (2.91) and it is a global attractor of all nonnegative solutions of Equation (2.91).*

Next, we consider some special cases of Equation (2.91). If $a(n)b(n) \equiv 0$, then (2.96) holds for any B and (2.97) holds when B is large. Hence the following corollary is a direct consequence of Theorem 5.

Corollary 5

Assume that $a(n)b(n) \equiv 0$ *and* $\prod_{i=0}^{p-1} a(i) < 1$. *Then Equation (2.91) has a nonnegative periodic solution* $\{\tilde{x}(n)\}$. *Furthermore, if either (2.98) or (2.99) holds, then* $\{\tilde{x}(n)\}$ *is the only periodic solution and* $\{\tilde{x}(n)\}$ *is a global attractor of all nonnegative solutions of Equation (2.91).*

In particular, when $a(n) \equiv 0$, (2.91) reduces to

$$x(n+1) = c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}},\ n = 0, 1, \cdots . \tag{2.102}$$

The following is a direct consequence of Corollary 5.

Corollary 6

Equation (2.102) has a unique nonnegative periodic solution $\{\tilde{x}(n)\}$ *which is also a global attractor of all nonnegative solutions of Equation (2.102).*

When $b(n) \equiv 0$, Equation (2.91) reduces to

$$x(n+1) = a(n)x(n) + c(n)\frac{e^{r(n)-q(n)x(n-k)}}{1+e^{r(n)-q(n)x(n-k)}},\ n = 0, 1, \cdots . \tag{2.103}$$

The following result comes from Corollary 5 immediately.

Corollary 7

Assume $\prod_{i=0}^{p-1} a(i) < 1$. *Then Equation (2.103) has a nonnegative periodic solution* $\{\tilde{x}(n)\}$. *If either (2.98) or (2.99) holds, then* $\{\tilde{x}(n)\}$ *is the only periodic solution, and* $\{\tilde{x}(n)\}$ *is a global attractor of all nonnegative solutions of Equation (2.103).*

In addition, when $q(n) \equiv 0$, Equation (2.91) reduces to

$$x(n+1) = \frac{a(n)x^2(n)}{b(n)+x(n)} + c(n)\frac{e^{r(n)}}{1+e^{r(n)}}, \ n = 0, 1, \cdots . \tag{2.104}$$

In this case, (2.99) holds automatically. Hence, the following conclusion is a direct consequence of Theorem 5 and Corollary 4.

Corollary 8

Assume that $\prod_{i=0}^{p-1} a(i) < 1$. *Suppose also either there is a positive constant* B *such that (2.96) and (2.97) hold with* $q(n) \equiv 0$, *or in particular (2.100) and (2.101) hold with* $q(n) \equiv 0$. *Then Equation (2.104) has a unique nonnegative periodic solutions* $\{\tilde{x}(n)\}$ *with period* p *and* $\{\tilde{x}(n)\}$ *is a global attractor of all nonnegative solutions of Equation (2.104).*

Next, consider the difference equation

$$x(n+1) = \frac{\alpha(n)x(n)}{\gamma(n)x(n)+\delta(n)} + \frac{\beta(n)}{\mu(n)x(n-k)+\eta(n)}, \ n = 0, 1, \cdots \tag{2.105}$$

where $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\gamma(n)\}$, $\{\delta(n)\}$, $\{\mu(n)\}$, and $\{\eta(n)\}$ are nonnegative periodic sequences with period p. Equation (2.105) is in the form of Equation (2.1) with

$$f(n,x) = \frac{\alpha(n)x(n)}{\gamma(n)x(n)+\delta(n)} \text{ and } g(n,x) = \frac{\beta(n)}{\mu(n)x+\eta(n)}. \tag{2.106}$$

Clearly, f is nondecreasing in x. Noting

$$\frac{d}{dx}\left(f(n,x)-\frac{\alpha(n)}{\delta(n)}x\right)=-\frac{\alpha(n)\gamma(n)x(\gamma(n)x+2\delta(n))}{\delta(n)(\gamma(n)x+\delta(n))^2}\leq 0, \tag{2.107}$$

we see that $f(n,x)-\frac{\alpha(n)}{\delta(n)}x$ is nonincreasing in x. Since

$$\frac{d}{dx}(g(n,x))=-\frac{\beta(n)\mu(n)}{(\mu(n)x+\eta(n))^2}, \tag{2.108}$$

we see that for each n,

$$\max_{x\geq 0}\left|\frac{dg(n,x)}{dx}\right|=\frac{\beta(n)\mu(n)}{\eta^2(n)}. \tag{2.109}$$

Hence, by Theorem 4, we have the following result immediately.

Theorem 6

Assume that

$$\delta(n)\neq 0 \text{ and } \prod_{i=0}^{p-1}\frac{\alpha(i)}{\delta(i)}<1, \tag{2.110}$$

and that there is a positve constant B such that

$$\sum_{j=n}^{n+p-1}\left(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\right)\left[\frac{\beta(j)}{\mu(j)B+\eta(j)}-\frac{\alpha(j)\gamma(j)B^2}{\delta(j)(\gamma(j)B+\delta(j))}\right]\geq 0,\ n=0,1,\cdots,p-1 \tag{2.111}$$

and

$$\frac{1}{1-\hat{a}}\sum_{j=n}^{n+p-1}\left(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\right)\frac{\beta(j)}{\eta(j)}\leq B,\ n=0,1,\cdots,p-1 \tag{2.112}$$

where $\hat{a}=\prod_{i=0}^{p-1}\frac{\alpha(i)}{\delta(i)}$. Then Equation (2.105) has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p. Furthermore, if either

$$\frac{\alpha(n)}{\delta(n)}\leq 1 \text{ and} \sum_{j=n}^{n+k}\left(\prod_{i=j+1}^{n+k}\frac{\alpha(i)}{\delta(i)}\right)\frac{\beta(j)\mu(j)}{\eta^2(j)}<1,\ n=0,1,\cdots,p-1, \tag{2.113}$$

or

$$\sum_{j=n}^{n+k+p-1}\left(\prod_{j=i+1}^{n+k+p-1}\frac{\alpha(i)}{\delta(i)}\right)\frac{\beta(j)\mu(j)}{\eta^2(j)}<1,\ n=0,1,\cdots,p-1, \tag{2.114}$$

then Equation (2.105) has a unique periodic solution $\{\tilde{x}(n)\}$ *with period* p *and* $\{\tilde{x}(n)\}$ *is a global attractor of all nonnegative periodic solutions of Equation (2.105).*

When $\mu(n)\equiv\gamma(n),\eta(n)\equiv\delta(n)$ and $k=0$, Equation (2.105) reduces to the following Ricatti equation

$$x(n+1)=\frac{\alpha(n)x(n)+\beta(n)}{\gamma(n)x(n)+\delta(n)},\ n=0,1,\cdots. \tag{2.115}$$

(2.111), (2.112), (2.113), and (2.114) reduce to

$$\sum_{j=n}^{n+p-1}\left(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\right)\left[\frac{\beta(j)\delta(j)-\alpha(j)\gamma(j)B^2}{\delta(j)(\gamma(j)B+\delta(j))}\right]\geq 0,\ n=0,1,\cdots,p-1, \tag{2.116}$$

$$\frac{1}{1-\hat{a}}\sum_{j=n}^{n+p-1}\left(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\right)\frac{\beta(j)}{\delta(j)}\leq B,\ n=0,1,\cdots,p-1, \tag{2.117}$$

$$\frac{\alpha(n)}{\delta(n)}\leq 1 \text{ and } \frac{\beta(n)\gamma(n)}{\delta^2(n)}<1,\ n=0,1,\cdots,p-1, \tag{2.118}$$

and

$$\sum_{j=n}^{n+p-1}\left(\prod_{j=i+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\right)\frac{\beta(j)\gamma(j)}{\delta^2(j)}<1,\ n=0,1,\cdots,p-1, \tag{2.119}$$

respectively. Hence, the following is a direct consequence of Theorem 6.

Corollary 9

Assume that (2.110) holds and that there is a positive constant B *such that (2.116) and (2.117) hold. Then Equation (2.115) has a nonnegative periodic solution* $\{\tilde{x}(n)\}$ *with period* p*. Furthermore, if either (2.118) or (2.119) holds, then Equation (2.115) has a unique nonnegative periodic solution* $\{\tilde{x}(n)\}$ *with period* p *and* $\{\tilde{x}(n)\}$ *is a global attractor of all nonnegative solutions of Equation (2.115).*

Clearly, if

$$\alpha(n)\gamma(n) \leq \beta(n)\delta(n) \leq \frac{1-\hat{a}}{\sum_{j=n}^{n+p-1}\left(\prod_{i=j+1}^{n+p-1}\frac{\alpha(i)}{\delta(i)}\right)},\ n=0,1,\cdots \tag{2.120}$$

then (2.116) and (2.117) hold with $B = 1$. Hence, the following conclusion comes from Corollary 9 directly.

Corollary 10

Assume that (2.110) and (2.120) hold. Then Equation (2.115) has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period p. Furthermore, if either (2.117) or (2.118) holds, then $\{\tilde{x}(n)\}$ is the only periodic solution, and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.115).

If $\alpha(n)\gamma(n) \equiv 0$, then (2.116) holds for any B and 2.117 holds when B is large. Hence, the following corollary is a direct result of Theorem 6.

Corollary 11

Assume that $\alpha(n)\gamma(n) \equiv 0$ and (2.110) holds. Then Equation (2.115) has a nonnegative periodic solution $\{\tilde{x}(n)\}$. Furthermore, if either (2.118) or (2.119) holds, then $\{\tilde{x}(n)\}$ is the only periodic solution of eqnref2.58 and $\{\tilde{x}(n)\}$ is a global attractor of all nonnegative solutions of Equation (2.115).

In particular, when $\alpha(n) \equiv 0$, Equation (2.115) reduces to

$$x(n+1) = \frac{\beta(n)}{\gamma(n)x(n)+\delta(n)},\ n=0,1,\cdots. \tag{2.121}$$

Clearly, (2.110) holds when $\delta(n) \neq 0$. Hence, from Corollary 8 we have the following conclusion immediately.

Corollary 12

Assume that $\delta(n) \neq 0$. *Then Equation (2.121) has a nonnegative periodic solution* $\{\tilde{x}(n)\}$ *with period* p. *Furthermore, if either (2.117) or (2.118) holds, then* $\{\tilde{x}(n)\}$ *is the only periodic solution of Equation (2.121) and* $\{\tilde{x}(n)\}$ *is a global attractor of all nonnegative solutions of Equation (2.121).*

Example 1

Consider the equation

$$\begin{aligned} &x(n+1) = \\ &\frac{(0.1\sin(\frac{\pi n}{4})+0.1)x^2(n)}{(\sin(\frac{\pi n}{4})+1.1)+x(n)} \\ &+\left(0.5\cos\left(\frac{\pi n}{4}\right)+0.6\right)\frac{e^{0.7\sin\left(\frac{\pi n}{4}\right)+0.75-\left(0.2\cos\left(\frac{\pi n}{4}\right)+0.25\right)x(n-3)}}{1+e^{0.7\sin\left(\frac{\pi n}{4}\right)+0.75-\left(0.2\cos\left(\frac{\pi n}{4}\right)+0.25\right)x(n-3)}}. \end{aligned} \tag{2.122}$$

Equation (2.122) takes the form of Equation (2.91) with $a(n) = 0.1\sin(\frac{\pi n}{4}) + 0.1$, $b(n) = \sin(\frac{\pi n}{4}) + 1.1$, $c(n) = 0.5\cos\left(\frac{\pi n}{4}\right) + 0.6$, $r(n) = 0.7\sin\left(\frac{\pi n}{4}\right) + 0.75$, $q(n) = 0.2\cos\left(\frac{\pi n}{4}\right) + 0.25$, and $k = 3$, where $\{a(n)\}$, $\{b(n)\}$, $\{(n)\}$, $\{r(n)\}$, and $\{q(n)\}$ are periodic sequences with period $p = 8$.

As $a(6) = 0$, $\prod_{i=0}^{7} a(i) = 0 < 1$. We verify numerically that (2.96) and (2.97) hold with $B = 1$ and (2.99) holds. By Theorem 5, Equation (2.122) has a nonnegative periodic solution $\{\tilde{x}(n)\}$ with period $p = 8$ that is a global attractor of all nonnegative periodic solutions of Equation (2.122). One such solution with initial data

$$x(-3) = 0.5, x(-2) = 2, x(-1) = 0.75, x(0) = 1.25 \tag{2.123}$$

is shown in Figure 2.1.

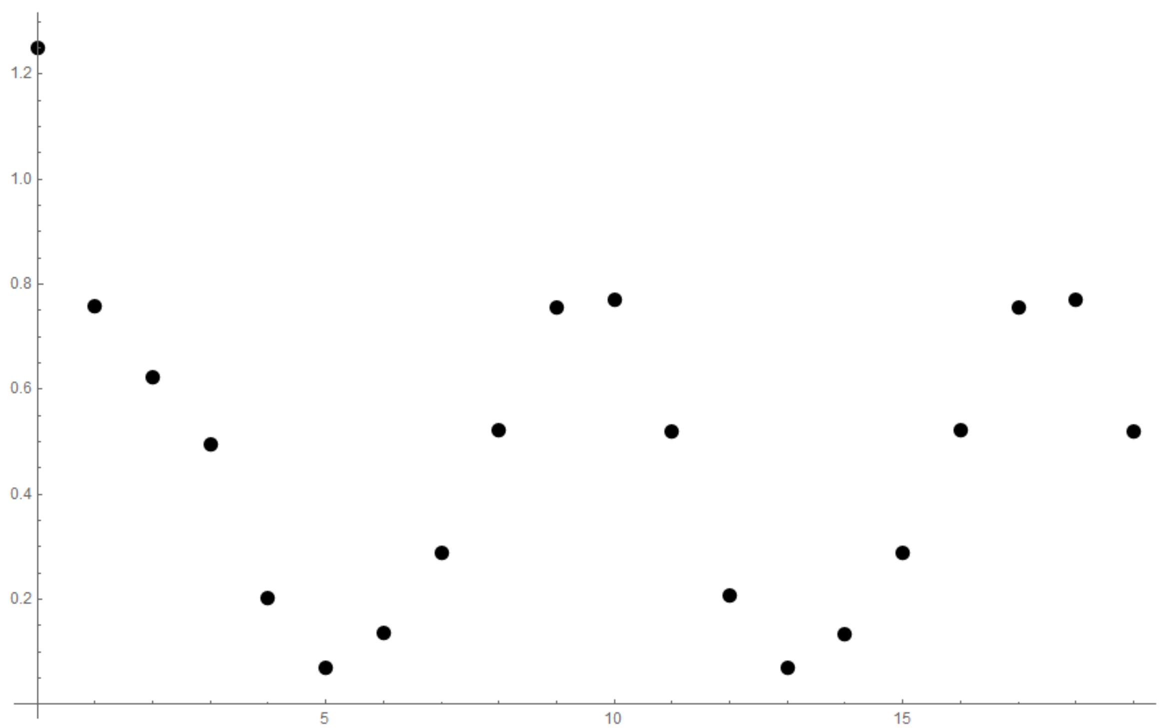

Figure 2.1

Graph of a solution $\{x(n)\},\ n = 0, 1, \cdots, 19$ of Equation (2.122) with initial data (2.123)

CHAPTER 3

QUASI-PERIODIC SOLUTIONS OF FORCED HIGHER ORDER NONLINEAR DIFFERENCE EQUATIONS

3.1 Introduction

Consider the following nonlinear difference equation of order $k+1$ with forcing term $b(n)$

$$x(n+1) = f(n, x(n)) + g(n, x(n-k)) + b(n),\ n = 0, 1, \cdots \tag{3.1}$$

where $f(n,x), g(n,x) : \{0, 1, \cdots\} \times [0, \infty) \to [0, \infty)$ are continuous functions in x and periodic functions with period ω in n, $\{b(n)\}$ is a real sequence, and k is a nonnegative integer.

Our aim in the chapter is to study the quasi-periodicity of solutions of Equation (3.1) in the sense that quasi-periodicity was defined in Definition 2 of Chapter 1. That is, we say that a solution $\{x(n)\}$ of Equation (3.1) is quasi-periodic with period ω if there exist sequences $\{p(n)\}$ and $\{q(n)\}$ such that $\{p(n)\}$ is periodic with period ω and $\{q(n)\}$ converges to zero as $n \to \infty$ and $x(n) = p(n) + q(n), n = 0, 1, \cdots$.

By using, among others, some methods and ideas related to the linear first-order difference equation, in the next section we show that under proper conditions every solution of Equation (3.1) is quasi-periodic with period ω. More specifically, we show that under proper conditions, every solution $\{x(n)\}$ of Equation (3.1) satisfies

$$\lim_{n\to\infty}(x(n)-\tilde{y}(n))=0 \tag{3.2}$$

where $\{\tilde{y}(n)\}$ is a periodic solution with period ω of the associated difference equation Equation (2.1) without forcing term discussed in the previous chapter.

While there has been much progress made in the study of the existence and global attractivity of periodic solutions of Equation (3.1), the quasi-periodicity of solutions of Equation (3.1) is relatively scarce. In order to study this phenomenon, we note the results of Theorem 2 discussed in the previous chapter for the existence of a periodic solution $\tilde{y}(t)$ of Equation (2.1)) (some new results related to those in [35] have been recently presented in [50]). We will make use of this theorem in the next section to guarantee a periodic solution of Equation (2.1), a prerequisite for the existence of quasi-periodic solutions of Equation (3.1). In following sections we show that our main results may be applied to some difference equations derived from applications.

3.2 Main Results

For the sake of convenience, we adopt the notation $\prod_{i=m}^{n}\rho(i)=1$ and $\sum_{i=m}^{n}\rho(i)=0$ whenever $\{\rho(n)\}$ is a real sequence and $m>n$ in the following discussion.

The following lemma - which is needed in the proof of our main result - is found in a related form, though given without proof, in some papers dealing with the linear first-order difference equation (see, for example, [40] and [46] and the related references therein). The proof is given here for the sake of completeness.

Lemma 1

Assume that $\{a(n)\}$ *is a nonnegative periodic sequence with period* ω *and* $\{b(n)\}$ *is a real sequence. If*

$$\prod_{i=0}^{\omega-1} a(i) < 1 \; and \; b(n) \to 0 \; as \; n \to \infty, \tag{3.3}$$

then

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) b(i) \to 0 \text{ as } n \to \infty. \tag{3.4}$$

Proof: First we show that there is a positive constant A such that

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) \leq A, \; n = 0, 1, \cdots . \tag{3.5}$$

Observe that for any $n \geq 0$, there are nonnegative integers m and l such that

$$n = m\omega + l, \; 0 \leq l \leq \omega - 1. \tag{3.6}$$

Then

$$\begin{aligned}
\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(j) \right) &= \sum_{i=0}^{m\omega+l} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\
&= \sum_{i=0}^{\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) + \sum_{i=\omega}^{2\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) \\
&\quad + \cdots + \sum_{i=(m-1)\omega}^{m\omega-1} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right) + \sum_{i=m\omega}^{m\omega+l} \left(\prod_{j=i+1}^{m\omega+l} a(j) \right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}a(j)\right) &= \\
&= \prod_{j=\omega}^{m\omega+l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right)+\prod_{j=2\omega}^{m\omega+l}a(j)\sum_{i=\omega}^{2\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) \\
&\quad +\cdots+\prod_{j=m\omega}^{m\omega+l}a(j)\sum_{i=(m-1)\omega}^{m\omega-1}\left(\prod_{j=i+1}^{m\omega-1}a(j)\right)+\sum_{i=0}^{l}\left(\prod_{j=i+1}^{l}a(j)\right) \\
&= \prod_{j=\omega}^{m\omega-1}a(j)\prod_{j=m\omega}^{m\omega+l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) \\
&\quad +\prod_{j=2\omega}^{m\omega-1}a(j)\prod_{j=m\omega}^{m\omega+l}a(j)\sum_{i=\omega}^{2\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) \\
&\quad +\cdots+\prod_{j=m\omega}^{m\omega-1}a(j)\prod_{j=m\omega}^{m\omega+l}a(j)\sum_{i=(m-1)\omega}^{m\omega-1}\left(\prod_{j=i+1}^{m\omega-1}a(j)\right) \\
&\quad +\sum_{i=0}^{l}\left(\prod_{j=i+1}^{l}a(j)\right) \\
&= \left(\prod_{i=0}^{\omega-1}a(j)\right)^{m-1}\prod_{i=0}^{l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) \\
&\quad +\left(\prod_{i=0}^{\omega-1}a(j)\right)^{m-2}\prod_{i=0}^{l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) \\
&\quad +\cdots+\prod_{i=0}^{l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right)+\sum_{i=0}^{l}\left(\prod_{j=i+1}^{l}a(j)\right) \\
&= \left(\prod_{i=0}^{\omega-1}a(i)\right)^{m-1}\prod_{i=0}^{l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) \\
&\quad +\left(\prod_{i=0}^{\omega-1}a(j)\right)^{m-2}\prod_{i=0}^{l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) \\
&\vdots \\
&\quad +1\cdot\prod_{i=0}^{l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right)+\sum_{i=0}^{l}\left(\prod_{j=i+1}^{l}a(j)\right), \qquad (3.7)
\end{aligned}$$

so

$$\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}a(j)\right) = \frac{1-\left(\prod_{j=0}^{\omega-1}a(j)\right)^{m}}{1-\prod_{j=0}^{\omega-1}a(j)}\prod_{j=0}^{l}a(j)\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) + \sum_{i=0}^{l}\left(\prod_{j=i+1}^{l}a(j)\right). \tag{3.8}$$

Thus for $l = 0, 1, \cdots, \omega - 1$

$$\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}a(j)\right) \le \frac{\prod_{j=0}^{l}a(j)}{1-\prod_{j=0}^{\omega-1}a(j)}\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) + \sum_{i=0}^{l}\left(\prod_{j=i+1}^{l}a(j)\right). \tag{3.9}$$

Let

$$A_1 = \max_{0\le l\le \omega-1}\prod_{j=0}^{l}a(j),\ A_2 = \max_{0\le l\le\omega-1}\sum_{i=0}^{l}\left(\prod_{j=i+1}^{l}a(j)\right) \tag{3.10}$$

and

$$A = \frac{A_1}{1-\prod_{j=0}^{\omega-1}a(j)}\sum_{i=0}^{\omega-1}\left(\prod_{j=i+1}^{\omega-1}a(j)\right) + A_2. \tag{3.11}$$

Then from (3.9) we see that (3.5) holds. Next, we show that (3.4) holds. Since $b(n) \to 0$ as $n \to \infty$, there is a positive constant $C(\ge A)$ such that

$$|b(n)| \le C,\ n \ge 0 \tag{3.12}$$

and for each $\epsilon > 0$, there is a positive integer N_1 such that

$$|b(n)| < \frac{\epsilon}{2C},\ n > N_1. \tag{3.13}$$

Hence, by noting (3.5), we see that

$$\sum_{i=N_1+1}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)|b(i)| \le \sum_{i=N_1+1}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)\frac{\epsilon}{2C} \le A\frac{\epsilon}{2C} \le \epsilon/2,\ n > N_1. \tag{3.14}$$

Since for each $t = 1, 2, \cdots N_1 + 1$, $\prod_{j=t}^{n} a(j) \to 0$ as $n \to \infty$, there is a positive integer $N_2(> N_1)$ such that

$$\prod_{j=t}^{n} a(j) < \frac{\epsilon}{2(N_1+1)C},\ n > N_2,\ t = 1, 2, \cdots N_1 + 1. \tag{3.15}$$

Hence,

$$\sum_{i=0}^{N_1}\left(\prod_{j=i+1}^{n} a(j)\right)|b(i)| \le \sum_{i=0}^{N_1}\left(\prod_{j=i+1}^{n} a(j)\right) C \le (N_1+1)\frac{\epsilon}{2(N_1+1)C}C = \epsilon/2,\ n > N_2. \tag{3.16}$$

Then it follows that

$$\begin{aligned}\left|\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} a(j)\right)b(i)\right| &= \left|\sum_{i=0}^{N_1}\left(\prod_{j=i+1}^{n} a(j)\right)b(i) + \sum_{i=N_1+1}^{n}\left(\prod_{j=i+1}^{n} a(j)\right)b(i)\right| \\ &\le \sum_{i=0}^{N_1}\left(\prod_{j=i+1}^{n} a(j)\right)|b(i)| + \sum_{i=N_1+1}^{n}\left(\prod_{j=i+1}^{n} a(j)\right)|b(i)| \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,\ n > N_2 \end{aligned} \tag{3.17}$$

which yields (3.4). The proof is complete.

Now, consider the linear difference equation

$$u(n+1) = a(n)u(n) + b(n),\ n = 0, 1, \cdots \tag{3.18}$$

where $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 2.1. Assume that $\{u(n)\}$ is a solution of Eq.(2.5). It is known that the general solution to the equation is

$$u(n+1) = \left(\prod_{j=0}^{n} a(j)\right)u(0) + \sum_{i=0}^{n}\left(\prod_{j=i+1}^{n} a(j)\right)b(i),\ n = 0, 1, \cdots, \tag{3.19}$$

which is frequently used in the literature (see, e.g., recent papers [43,46,48,49], as well as many related references therein, where some applications to ordinary and partial difference equations, as well as many historical facts on the equation and related solvable ones can be found). Clearly, by noting the periodicity of $\{a(n)\}$ and (3.3), we see that

$$\left(\prod_{j=0}^{n} a(j)\right) u(0) \to 0 \text{ as } n \to \infty. \tag{3.20}$$

Hence, the following conclusion comes from Lemma 1 immediately.

Corollary 13

Assume that $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 1. Then every solution $\{u(n)\}$ of Equation (3.18) converges to zero as $n \to \infty$.

The following corollary is about the difference inequality

$$v(n+1) \le a(n)v(n) + b(n),\ n = 0, 1, \cdots \tag{3.21}$$

Assume that $\{v(n)\}$ is a nonnegative solution of (3.21). Clearly, $\{v(n)\}$ satisfies

$$0 \le v(n) \le u(n),\ n = 0, 1, \cdots \tag{3.22}$$

where $\{u(n)\}$ is the solution of Equation (3.18) with $u(0) = v(0)$. Hence, the following conclusion is a direct consequence of Corollary 13.

Corollary 14

Assume that $\{a(n)\}$ and $\{b(n)\}$ satisfy the hypotheses in Lemma 1. Then every nonnegative solution $\{v(n)\}$ of (3.21) converges to zero as $n \to \infty$.

The following lemma is straightforward but will be referenced multiple times in the main result.

Lemma 2

Suppose $f(n,x)$, $g(n,x)$ are real functions and that $\{a(n)\}$ is a real sequence, and assume $f(n,x) - a(n)x$ and $g(n,x)$ are nonincreasing. Then for any $y \geq 0$,

$$f(n, x+y) - f(n,x) \leq a(n)y \tag{3.23}$$

and

$$f(n, x+y) - f(n,x) + g(n, x+y) - g(n,x) \leq a(n)y. \tag{3.24}$$

Proof: Let $y \geq 0$. As $f(n,x) - a(n)x$ is nonincreasing we have

$$f(n, x+y) - a(n)(x+y) \leq f(n,x) - a(n)x. \tag{3.25}$$

Thus, $f(n, x+y) - f(n,x) \leq a(n)y$. As $g(n,x)$ is nonincreasing, we see that $g(n, x+y) - g(n,x) \leq 0$. Combining the above inequalities completes the proof.

The following theorem is our main result.

Theorem 7

Consider Equation (3.1) and assume that $f(n,x)$ is nondecreasing in x. Suppose that $\{a(n)\}$ is a nonnegative periodic sequence with period ω, and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (3.3), $f(n,x) \leq a(n)x$ and $f(n,x) - a(n)x$ is nonincreasing in x. Suppose also that $g(n,x)$ is nonincreasing in x and there is a positive constant B such that (2.7) and (2.8) are satisfied. Suppose there is a nonnegative sequence $\{L(n)\}$ with period ω such that

$$|g(n,x) - g(n,y)| \leq L(n)|x-y|, \; n = 0, 1, \cdots, \omega - 1 \tag{3.26}$$

and that either

$$a(n) \leq 1 \; and \; \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) L(i) < 1, \; n = 0, 1, \cdots, \omega - 1 \tag{3.27}$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) L(i) < 1, \; n = 0, 1, \cdots, \omega - 1. \tag{3.28}$$

Then every solution $\{x(n)\}$ *of Equation (3.1) satisfies*

$$\lim_{n \to \infty} (x(n) - \tilde{y}(n)) = 0 \tag{3.29}$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution of Equation (2.1) with period* ω*.*

Proof: In view of Theorem 4, we know that Equation (2.1) has a unique periodic solution $\{\tilde{y}(n)\}$. Let $z(n) = x(n) - \tilde{y}(n)$. Then $\{z(n)\}$ satisfies

$$z(n+1) + \tilde{y}(n+1) = f(n, z(n) + \tilde{y}(n)) + g(n, z(n-k) + \tilde{y}(n-k)) + b(n), \; n = 0, 1, \cdots. \tag{3.30}$$

Since $\{\tilde{y}(n)\}$ is a solution of Equation (2.1), $\tilde{y}(n+1) = f(n, \tilde{y}(n)) + g(n, \tilde{y}(n-k))$. Hence, it follows that

$$\begin{aligned} z(n+1) &= f(n, z(n) + \tilde{y}(n)) - f(n, \tilde{y}(n)) \\ &\quad + g(n, z(n-k) + \tilde{y}(n-k)) - g(n, \tilde{y}(n-k)) + b(n), \; n \geq 0. \end{aligned} \tag{3.31}$$

Clearly, to complete the proof of the theorem and show that (3.29) holds, it suffices to show that every solution $\{z(n)\}$ of Equation (3.31) tends to zero as $n \to \infty$. First assume that $\{z(n)\}$ is a nonoscillatory solution of Equation (3.31). Then $\{z(n)\}$ is either eventually positive or eventually negative. We assume that $\{z(n)\}$ is eventually positive. The proof for the case that $\{z(n)\}$ is eventually negative is similar and will be omitted. Hence, there is a positive integer n_0 such that $z(n) > 0$ for $n \geq n_0$. Then by noting $f(n, x) - a(n)x$ and $g(n, x)$ are nonincreasing in x, it follows from Lemma 2 and (3.31) that

$$z(n+1) \leq a(n)z(n) + b(n),\ n \geq n_0 + k$$

and so by Corollary 14, $z(n) \to 0$ as $n \to \infty$.

Next, assume that $\{z(n)\}$ is an oscillatory solution of Equation (3.31). Then there is an increasing sequence $\{n_t\}$ of positive integers such that $y(n_1) \leq 0$ and for $\tau = 1, 2, \cdots$,

$$\begin{cases} y(n) > 0 \text{ when } \ n_{2\tau-1} < n \leq n_{2\tau} \text{ and} \\ y(n) \leq 0 \text{ when } \quad n_{2\tau} < n \leq n_{2\tau+1}. \end{cases} \tag{3.32}$$

Case 1. Assume that (3.27) holds. Then there is a positive number μ such that

$$\mu < 1 \text{ and } \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) L(i) \leq \mu,\ n = 0, 1, \cdots . \tag{3.33}$$

We show that

$$z(n) \leq \mu \max_{n_1 - k \leq l \leq n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|,\ n_1 < n \leq n_2. \tag{3.34}$$

In fact, from (3.32) we see that $z(n_1) \leq 0$ and $z(n) > 0$, $n_1 < n \leq n_2$. As $f(n, x) - a(n)x$ is nonincreasing in x, from Lemma 2 we see that $f(n, z(n) + \tilde{y}(n)) - f(n, \tilde{y}(n)) \leq a(n)z(n)$ and (3.31) becomes

$$z(n+1) \leq a(n)z(n) + g(n, z(n-k) + \tilde{y}(n-k)) - g(n, \tilde{y}(n-k)) + b(n). \tag{3.35}$$

Then by using (3.26), it follows that when $n_1 < n \le n_2$,

$$\begin{aligned}
z(n) &= \left(\prod_{j=n_1}^{n-1} a(j)\right) z(n_1) \\
&\quad + \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) [g(i, z(i-k) + \tilde{y}(i-k)) - g(i, \tilde{y}(i-k)) + b(i)] \\
&\le \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |g(i, z(i-k) + \tilde{y}(i-k)) \\
&\quad - g(i, \tilde{y}(i-k))| + \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)| \\
&\le \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) L(i)|z(i-k)| + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)|. \qquad (3.36)
\end{aligned}$$

Now, consider two cases $n_2 \le n_1 + k + 1$ and $n_2 > n_1 + k + 1$, respectively. When $n_2 \le n_1 + k + 1$, for any $n_1 < n \le n_2$, $n - k - 1 \le n_1$ and so (3.36) yields

$$\begin{aligned}
z(n) &\le \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) L(i) \max_{n_1-k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)| \\
&\le \sum_{i=n-k-1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) L(i) \max_{n_1-k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)| \\
&\le \mu \max_{n_1-k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)|. \qquad (3.37)
\end{aligned}$$

Hence, (3.34) holds in this case. Next, consider the case that $n_2 > n_1 + k + 1$. When $n_1 < n \le n_1 + k + 1$, as we have shown above, (3.34) holds. In particular,

$$z(n_1 + k + 1) \le \mu \max_{n_1-k \le l \le n_1} \{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j)\right) |b(i)|. \qquad (3.38)$$

When $n_1 + k + 1 < n \leq n_2$, by noting $z(n-k-1) > 0$, (3.38) holds and Lemma 2, (3.31) yields

$$\begin{aligned}
z(n) &\leq a(n-1)z(n-1) + b(n-1) \\
&= \left(\prod_{j=n_1+k+1}^{n-1} a(j)\right) z(n_1+k+1) + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) b(i) \\
&\leq \left(\prod_{j=n_1+k+1}^{n-1} a(j)\right) \left(\mu \max_{n_1-k\leq l\leq n_1}\{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j)\right) |b(i)|\right) \\
&\quad + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) b(i) \\
&\leq \mu \max_{n_1-k\leq l\leq n_1}\{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)| + \sum_{i=n_1+k+1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) b(i) \\
&= \mu \max_{n_1-k\leq l\leq n_1}\{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)| \qquad (3.39)
\end{aligned}$$

and so $z(n)$ satisfies (3.34). Hence for any case, (3.34) holds. Then by a similar argument, we may show that

$$z(n) \geq -\left[\mu \max_{n_2-k\leq l\leq n_2}\{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)|\right],\ n_2 < n \leq n_3, \qquad (3.40)$$

and in general,

$$|z(n)| \leq \mu B(t) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)|,\ n_t < n \leq n_{t+1}. \qquad (3.41)$$

where

$$B(t) = \max_{n_t-k\leq l\leq n_t}\{|z(l)|\},\ t = 1, 2, \cdots. \qquad (3.42)$$

Since $b(n) \to 0$ as $n \to \infty$, $|b(n)| \to 0$ as $n \to \infty$. Then it follows from Lemma 1,

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(i)\right) |b(i)| \to 0 \text{ as } n \to \infty. \qquad (3.43)$$

Hence, from (3.41) we see that if $B(t) \to 0$ as $t \to \infty$, then $z(n) \to 0$ as $n \to \infty$. In the following, we assume that $B(t) \not\to 0$ as $t \to \infty$. Then there is a subsequence $\{B(t_s)\}$ of $\{B(t)\}$ such that

$$B(t_s) \geq \eta, \ s = 1, 2, \cdots \tag{3.44}$$

where η is a positive constant.

By noting (3.43) again, we may choose a positive number δ such that

$$\mu + \delta < 1 \tag{3.45}$$

and a subsequence $\{n_{t_{s_r}}\}$ of $\{n_{t_s}\}$ such that for each $r = 1, 2, \cdots$,

$$n_{t_{s_{r+1}}} - n_{t_{s_r}} \geq 1 + 2k \tag{3.46}$$

and

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(i) \right) |b(i)| < \eta\delta^r, \ n \geq n_{t_{s_r}} - 1. \tag{3.47}$$

We claim that

$$B(t) \leq B(t_{s_r}) \text{ for } t \geq t_{s_r}, \ r = 1, 2, \cdots . \tag{3.48}$$

In fact, if $n_{t_{s_r}+1} - k > n_{t_{s_r}}$, we see that when $n_{t_{s_r}+1} - k \leq n \leq n_{t_{s_r}+1}$, it follows from (3.41) and (3.47) that

$$|z(n)| \leq \mu B(t_{s_r}) + \eta\delta^r \leq (\mu + \delta^r)B(t_{s_r}) \leq B(t_{s_r}). \tag{3.49}$$

If $n_{t_{s_r}+1} - k \leq n_{t_{s_r}}$ we see that (3.49) holds when $n_{t_{s_r}} < n \leq n_{t_{s_r}+1}$; while when $n_{t_{s_r}+1} - k \leq n \leq n_{t_{s_r}}$, by noting $n_{t_{s_r}} - k < n_{t_{s_r}+1} - k$, we see that

$$|z(n)| \leq \max_{n_{t_{s_r}} - k \leq l \leq n_{t_{s_r}}} \{|z(l)|\} = B(t_{s_r}). \tag{3.50}$$

Hence, from the above discussion we see that for any case when $n_{t_{s_r}+1} - k \le n \le n_{t_{s_r}+1}$,

$$|z(n)| \le B(t_{s_r}) \tag{3.51}$$

and so

$$B(t_{s_r}+1) = \max_{n_{t_{s_r}+1}-k \le l \le n_{t_{s_r}+1}} \{|z(l)|\} \le B(t_{s_r}). \tag{3.52}$$

Then by a similar argument and induction, we may show that for any $l \ge 1$,

$$B(t_{s_r}+l) \le B(t_{s_r}) \tag{3.53}$$

that is, (3.48) holds. Then it follows from (3.41) and (3.48) that

$$|z(n)| \le \mu B(t_{s_r}) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|,\ n > n_{s_r}. \tag{3.54}$$

Next, we show that

$$|z(n)| \le (\mu+\delta)^r B(t_{s_1}),\ n > n_{t_{s_1}},\ r = 1, 2, \cdots. \tag{3.55}$$

When $r = 1$, from (3.47) and (3.54) we see that

$$|z(n)| \le \mu B(t_{s_1}) + \eta\delta \le (\mu+\delta) B(t_{s_1}), n > n_{t_{s_1}} \tag{3.56}$$

which satisfies (3.55) with $r = 1$. Assume that when $r = m$, (3.55) holds, that is,

$$|z(n)| \le (\mu+\delta)^m B(t_{s_1}),\ n > n_{t_{s_m}}. \tag{3.57}$$

Then from (3.54) and (3.57) we see that when $n > n_{t_{s_{m+1}}}$,

$$\begin{aligned} |z(n)| &\le \mu B(t_{s_{m+1}}) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\le \mu(\mu+\delta)^m B(t_{s_1}) + \eta\delta^{m+1} \\ &\le (\mu(\mu+\delta)^m + \delta^{m+1}) B(t_{s_1}) \\ &\le (\mu+\delta)^{m+1} B(t_{s_1}) \end{aligned} \tag{3.58}$$

which satisfies (3.55) with $r = m + 1$. Hence, by induction, (3.55) holds. Clearly, (3.55) implies that $z(n) \to 0$ as $n \to \infty$.

Case 2. Assume that (3.28) holds. Then there is a positive number ν such that

$$\nu < 1 \text{ and } \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) L(i) \le \nu, \; n = 0, 1, \cdots . \tag{3.59}$$

We claim that

$$z(n) \le \nu \max_{n_1-k \le l \le n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|, \; n_1 < n \le n_2. \tag{3.60}$$

First, from the proof of Case 1, we see that when $n_1 < n \le n_2$, (3.36) holds. Next, consider two cases $n_2 \le n_1 + k + \omega$ and $n_2 > n_1 + k + \omega$, respectively. When $n_2 \le n_1 + k + \omega$, for any $n_1 < n \le n_2$, $n - k - \omega \le n_1$ and so (3.36) yields

$$\begin{aligned} z(n) &\le \sum_{i=n_1}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k \le l \le n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\le \sum_{i=n-k-\omega}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) L(i) \max_{n_1-k \le l \le n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)| \\ &\le \nu \max_{n_1-k \le l \le n_1+\omega-1} \{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j) \right) |b(i)|. \end{aligned} \tag{3.61}$$

Hence, (3.60) holds in this case. Next, consider the case that $n_2 > n_1 + k + \omega$. When $n_1 < n \le n_1 + k + \omega$, as we have shown above, (3.60) holds. Hence, we only need to show that (3.60) holds also when $n_1 + k + \omega < n \le n_2$. In fact, by noting that when $n_1 + k + 1 < n \le n_2$, $z(n - k - 1) > 0$, and the result of Lemma 2, (3.31) yields

$$z(n) \le a(n-1)z(n-1) + b(n-1), \; n_1 + k + 1 < n \le n_2. \tag{3.62}$$

Hence, it follows from (3.61) and (3.62) that

$$
\begin{aligned}
z(n_1+k+\omega+1) &\le \left(\prod_{j=n_1+k+1}^{n_1+k+\omega} a(j)\right) z(n_1+k+1) + \sum_{i=n_1+k+1}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right) |b(i)| \\
&\le \left(\prod_{j=n_1+k+1}^{n_1+k+\omega} a(j)\right) \cdot \nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} \\
&\quad + \left(\prod_{j=n_1+k+1}^{n_1+k+\omega} a(j)\right) \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k} a(j)\right) |b(i)| \\
&\quad + \sum_{i=n_1+k+1}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right) |b(i)| \\
&\le \nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_1+k} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right) |b(i)| \\
&\quad + \sum_{i=n_1+k+1}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right) |b(i)| \\
&= \nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_1+k+\omega} \left(\prod_{j=i+1}^{n_1+k+\omega} a(j)\right) |b(i)| \qquad (3.63)
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
z(n_1+k+\omega+2) &\le \nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_1+k+\omega+1} \left(\prod_{j=i+1}^{n_1+k+\omega+1} a(j)\right) |b(i)| \\
&\vdots \\
z(n_2) &\le \nu \max_{n_1-k\le l\le n_1+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n_2-1} \left(\prod_{j=i+1}^{n_2-1} a(j)\right) |b(i)|. \qquad (3.64)
\end{aligned}
$$

Hence for any case, (2.25) holds. Then by a similar argument, we may show that

$$
z(n) \ge -\left[\nu \max_{n_2-k\le l\le n_2+\omega-1}\{|z(l)|\} + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)|\right],\ n_2 < n \le n_3, \quad (3.65)
$$

and in general,

$$
|z(n)| \le \mu C(t) + \sum_{i=0}^{n-1} \left(\prod_{j=i+1}^{n-1} a(j)\right) |b(i)|,\ n_t < n \le n_{t+1}. \quad (3.66)
$$

where

$$C(t) = \max_{n_t - k \le l \le n_t + \omega - 1} \{|z(l)|\}, \ t = 1, 2, \cdots . \tag{3.67}$$

Then by an argument similar to that for Case 1, we may show the following:

If $C(t) \to 0$ as $t \to \infty$, then $z(n) \to 0$ as $n \to \infty$; If $C(t) \not\to 0$ as $t \to \infty$, then there is a subsequence $\{C(t_s)\}$ of $\{C(t)\}$ such that

$$C(t_s) \ge \eta, \ s = 1, 2, \cdots \tag{3.68}$$

where η is a positive constant. A positive number δ such that

$$\nu + \delta < 1 \tag{3.69}$$

and a subsequence $\{n_{t_{s_r}}\}$ of $\{n_{t_s}\}$ such that for each $r = 1, 2, \cdots$,

$$n_{t_{s_{r+1}}} - n_{t_{s_r}} \ge 1 + 2k \tag{3.70}$$

could be chosen such that

$$\sum_{i=0}^{n} \left(\prod_{j=i+1}^{n} a(i) \right) |b(i)| < \eta \delta^r, \ n \ge n_{t_{s_r}} - 1 \tag{3.71}$$

and

$$|z(n)| \le (\mu + \delta)^r C(t_{s_1}), \ n > n_{t_{s_1}}, \ r = 1, 2, \cdots . \tag{3.72}$$

Clearly, the above inequalities imply that $z(n) \to 0$ as $n \to \infty$. The proof is complete.

When $g(n, x) = p(n)h(x)$, where $\{p(n)\}$ is a nonnegative periodic sequence with period ω and h is a nonnegative continuous function, Equation (3.1) becomes

$$x(n + 1) = f(n, x(n)) + p(n)h(x(n - k)) + b(n), \ n = 0, 1, \cdots \tag{3.73}$$

and the following result is a direct consequence of Theorem 7.

Corollary 15

Consider Equation (3.73) and assume that $f(n,x)$ is nondecreasing in x. Assume also that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (3.3), $f(n,x) \leq a(n)x$ and that $f(n,x) - a(n)x$ is nonincreasing in x. Suppose that h is nonincreasing and L-Lipschitz and that there is a positive constant B such that

$$\sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) [f(i,B) - a(i)B + p(i)h(B)] \geq 0, \ n = 0, 1, \cdots, \omega - 1 \qquad (3.74)$$

and

$$\frac{1}{1 - \prod_{j=0}^{\omega-1} a(j)} \sum_{i=n}^{n+\omega-1} \left(\prod_{j=i+1}^{n+\omega-1} a(j) \right) p(i)h(0) \leq B, \ n = 0, 1, \cdots, \omega - 1. \qquad (3.75)$$

Suppose also that either

$$a(n) \leq 1 \ and \ L \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) p(i) < 1, \ n = 0, 1, \cdots, \omega - 1 \qquad (3.76)$$

or

$$L \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) p(i) < 1, \ n = 0, 1, \cdots, \omega - 1. \qquad (3.77)$$

Then every solution $\{x(n)\}$ of Equation (3.73) satisfies

$$\lim_{n\to\infty} (x(n) - \tilde{y}(n)) = 0 \qquad (3.78)$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1) = f(n, y(n)) + p(n)h(y(n-k)), \ n = 0, 1, \cdots. \qquad (3.79)$$

When $f(n,x) = a(n)x(n)$, Equation (3.73) becomes

$$x(n+1) = a(n)x(n) + p(n)h(x(n-k)) + b(n), \ n = 0, 1, \cdots. \qquad (3.80)$$

(3.74) is satisfied for any $B > 0$ and (3.75) holds for B large enough. Thus the following result is a direct consequence of Corollary 15.

Corollary 16

Consider Equation (3.80) and assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (3.3). Suppose also that $h(x)$ is nonincreasing and L-Lipschitz, and that either

$$a(n) \leq 1 \text{ and } L \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) p(i) < 1, \; n = 0, 1, \cdots, \omega - 1 \tag{3.81}$$

or

$$L \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) p(i) < 1, \; n = 0, 1, \cdots, \omega - 1. \tag{3.82}$$

Then every solution $\{x(n)\}$ of Equation (3.80) satisfies

$$\lim_{n \to \infty} (x(n) - \tilde{y}(n)) = 0$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1) = a(n)y(n) + p(n)h(y(n-k)), \; n = 0, 1, \cdots . \tag{3.83}$$

In particular, when $h(x) \equiv 1$, Equation (3.73) reduces to the first order linear equation

$$x(n+1) = a(n)x(n) + p(n) + b(n), \; n = 0, 1, \cdots . \tag{3.84}$$

Since we may choose $L = 0$, (3.81) and (3.82) hold. Hence, from Corollary 16, we have the following result immediately.

Corollary 17

Consider Equation (3.84) and assume that $\{a(n)\}$ is a nonnegative periodic sequence with period ω and $\{b(n)\}$ is a real sequence such that $\{a(n)\}$ and $\{b(n)\}$ satisfy (3.3). Then every solution $\{x(n)\}$ of Equation (3.84) satisfies

$$\lim_{n\to\infty}(x(n)-\tilde{y}(n))=0 \tag{3.85}$$

where $\{\tilde{y}(n)\}$ is the unique periodic solution with period ω of the equation

$$y(n+1)=a(n)y(n)+p(n),\ n=0,1,\cdots. \tag{3.86}$$

Remark 2 When $a(n)\equiv a$ and $p(n)\equiv p$ are nonnegative constants, Equations (3.84) and (3.86) become

$$x(n+1)=ax(n)+p+b(n),\ n=0,1,\cdots. \tag{3.87}$$

and

$$y(n+1)=ay(n)+p,\ n=0,1,\cdots. \tag{3.88}$$

respectively. The nonnegative periodic solution $\{\tilde{y}(n)\}$ of Equation (3.88) becomes the nonnegative equilibrium point $\bar{y}=\frac{p}{1-a}$. Then by Corollary 17, when $a<1$, every nonnegative solution $\{x(n)\}$ of Equation (3.87) converges to $\bar{y}$ as $n\to\infty$. In fact, in this case, the solution of Equation (3.87) is

$$x(n)=a^n x(0)+p\frac{1-a^n}{1-a}+\sum_{i=0}^{n-1}\left(\prod_{j=i+1}^{n-1}a(j)\right)b(i),\ n=1,2,\ldots. \tag{3.89}$$

By noting (3.3) and Lemma 1, we know that $\sum_{i=0}^{n}\left(\prod_{j=i+1}^{n}a(j)\right)b(i)\to 0$ as $n\to\infty$ and so

$$x(n)\to\frac{p}{1-a}\text{ as }n\to\infty. \tag{3.90}$$

Remark 3 Clearly, Corollary 17 implies that for the equation

$$x(n+1) = a(n)x(n) + q(n),\ n = 0, 1, \cdots . \tag{3.91}$$

where $\{a(n)\}$ is nonnegative and periodic with period ω, and $\{q(n)\}$ is nonnegative and quasi-periodic with period ω, if $\sum_{i=0}^{\omega-1} a(j) < 1$, then every nonnegative solution of the equation is quasi-periodic with period ω.

3.3 Applications and Numerical Examples

In this section, we apply our results obtained in Section 3.2 to some equations derived from mathematical biology. In applications, there are often external factors - known or unknown - that affect the mathematical model. Two such factors that have been studied in related models are migration and subsets of populations which become isolated and unchanged by density-dependent effects, see [21, 56] and references cited therein.

Consider the difference equations

$$x(n+1) = \frac{a(n)x^2(n)}{x(n)+\delta(n)} + \frac{\nu(n)\rho(n)\sigma(n)}{1+e^{\beta(n)x(n-k)-\alpha(n)}} + b(n),\ n = 0, 1, \cdots , \tag{3.92}$$

$$x(n+1) = a(n)x(n) + \beta(n)e^{-\sigma(n)x(n-k)} + b(n),\ n = 0, 1, \cdots \tag{3.93}$$

and

$$x(n+1) = a(n)x(n) + \frac{\beta(n)}{1+x^{\gamma}(n-k)} + b(n),\ n = 0, 1, \cdots \tag{3.94}$$

where $\{a(n)\}$, $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\nu(n)\}$, $\{\delta(n)\}$, $\{\rho(n)\}$, $\{\sigma(n)\}$ are nonnegative periodic sequences with period ω, $\{b(n)\}$ is a real sequence, γ is a positive constant and k is a nonnegative integer. When $a(n) \equiv a$, $\alpha(n) \equiv \alpha$, $\beta(n) \equiv \beta$, $\nu(n) \equiv \nu$, $\delta(n) \equiv \delta$, $\rho(n) \equiv \rho$

and $\sigma(n) \equiv \sigma$ are nonnegative constants and $b(n) \equiv 0$, Equations (3.92), (3.93), and (3.94) reduce to

$$x(n+1) = \frac{ax^2(n)}{x(n)+\delta} + \frac{\nu\rho\sigma}{1+e^{\beta x(n-k)-\alpha}},\ n = 0, 1, \cdots, \tag{3.95}$$

$$x(n+1) = ax(n) + \beta e^{-\sigma x(n-k)},\ n = 0, 1, \cdots \tag{3.96}$$

and

$$x(n+1) = ax(n) + \frac{\beta}{1+x^{\gamma}(n-k)},\ n = 0, 1, \cdots \tag{3.97}$$

respectively. Equation (3.95) is derived from a model of the energy cost for new leaf growth in citrus crops, see [59]. When $b(n) \not\equiv 0$, $\{b(n)\}$ may represent defoliation that does not occur naturally or is not considered natural defoliation by the model parameters. A similar equation is given for the litter mass in perennial grasses, and the results that follow will apply directly to this model, see [57]. Equation (3.96) is a discrete version of a model of the survival of red blood cells in an animal [58], and Equation (3.97) is a discrete analog of a model that has been used to study blood cell production [20]. The global attractivity of positive solutions of Equations (3.96) and (3.97) and some extensions of them has been studied by numerous authors, see for example [7, 9, 14, 15, 17, 31, 34] and references cited therein. When $b(n) \not\equiv 0$, $\{b(n)\}$ may represent the medical replacement of blood cells or administration of antibodies, see [3, 16] and references cited therein.

Suppose $\{b(n)\}$ is quasi-periodic, that is, there exist real sequences $\{q(n)\}$ and $\{r(n)\}$ such that $\{q(n)\}$ is periodic with period ω, $\{r(n)\}$ is such that $r(n) \to 0$ as $n \to \infty$, and $b(n) = q(n) + r(n)$. Then Equations (3.92), (3.93), and (3.94) become

$$x(n+1) = \frac{a(n)x^2(n)}{x(n)+\delta(n)} + \frac{\gamma(n)\rho(n)\sigma(n)}{1+e^{\beta(n)x(n-k)-\alpha(n)}} + q(n) + r(n),\ n = 0, 1, \cdots, \tag{3.98}$$

$$x(n+1) = a(n)x(n) + \beta(n)e^{-\sigma(n)x(n-k)} + q(n) + r(n),\ n = 0, 1, \cdots \tag{3.99}$$

and

$$x(n+1) = a(n)x(n) + \frac{\beta(n)}{1 + x^{\gamma}(n-k)} + q(n) + r(n),\ n = 0, 1, \cdots \tag{3.100}$$

respectively.

First, consider Equation (3.98). It is of the form of Equation (3.1) with

$$f(n, x) = \frac{a(n)x^2}{x + \delta(n)} \text{ and } g(n, x) = \frac{\nu(n)\rho(n)\sigma(n)}{1 + e^{\beta(n)x - \alpha(n)}} + q(n). \tag{3.101}$$

As

$$\frac{df}{dx} = \frac{a(n)x(x + 2\delta(n))}{(x + \delta(n))^2},\ x \geq 0, \tag{3.102}$$

we see that $f(n, x)$ is nondecreasing in x. We next note that

$$f(n, x) - a(n)x = \frac{-a(n)\delta(n)x}{x + \delta(n)},\ x \geq 0 \tag{3.103}$$

and

$$\frac{d}{dx}(f(n, x) - a(n)x) = \frac{-a(n)\delta^2(n)}{(x + \delta(n))^2},\ x \geq 0, \tag{3.104}$$

thus $f(n, x) \leq a(n)x$ and $f(n, x) - a(n)x$ is nonincreasing in x. As

$$\frac{dg}{dx} = -\beta(n)\nu(n)\rho(n)\sigma(n)\frac{e^{\beta(n)x - \alpha(n)}}{(1 + e^{\beta(n)x - \alpha(n)})^2},\ x \geq 0 \tag{3.105}$$

and

$$\frac{d^2g}{dx^2} = -\beta^2(n)\nu(n)\rho(n)\sigma(n)\frac{e^{\beta(n)x - \alpha(n)}(1 - e^{\beta(n)x - \alpha(n)})}{(1 + e^{\beta(n)x - \alpha(n)})^3},\ x \geq 0, \tag{3.106}$$

we see that $g(n, x)$ is nonincreasing in x, and for each n, $\left|\frac{dg(n,x)}{dx}\right|$ achieves a maximum when $x = \frac{\alpha(n)}{\beta(n)}$, and

$$\left|\frac{dg(n, x)}{dx}\right|_{x = \frac{\alpha(n)}{\beta(n)}} = \frac{\beta(n)\nu(n)\rho(n)\sigma(n)}{4}. \tag{3.107}$$

Thus $g(n,x)$ is L-Lipschitz with $L(n) = \frac{\beta(n)\nu(n)\rho(n)\sigma(n)}{4}$. Hence, we have the following conclusion from Theorem 7.

Corollary 18

Assume that

$$\hat{a} = \prod_{j=0}^{\omega-1} a(j) < 1. \tag{3.108}$$

Suppose there exists a positive constant B *such that*

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \left[q(i) + \frac{\nu(i)\rho(i)\sigma(i)}{1+e^{B\cdot\beta(i)-\alpha(i)}} - \frac{B^2 a(i)\delta(i)}{B+\delta(i)} \right] \geq 0, \; n = 0, 1, \cdots, \omega-1 \tag{3.109}$$

and

$$\frac{1}{1-\hat{a}} \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \left(\frac{\nu(i)\rho(i)\sigma(i)}{1+e^{-\alpha(i)}} + q(i) \right) \leq B, \; n = 0, 1, \cdots, \omega-1. \tag{3.110}$$

Suppose also that either

$$a(n) \leq 1 \text{ and } \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) \beta(i)\nu(i)\rho(i)\sigma(i) < 4, \; n = 0, 1, \cdots, \omega-1 \tag{3.111}$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \beta(i)\nu(i)\rho(i)\sigma(i) < 4, \; n = 0, 1, \cdots, \omega-1. \tag{3.112}$$

Then every solution $\{x(n)\}$ *of Equation (3.98) satisfies*

$$\lim_{n\to\infty} (x(n) - \tilde{y}(n)) = 0 \tag{3.113}$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution with period* ω *of the following equation*

$$y(n+1) = \frac{a(n)y^2(n)}{y(n)+\delta(n)} + \frac{\gamma(n)\rho(n)\sigma(n)}{1+e^{\beta(n)y(n-k)-\alpha(n)}} + q(n), \; n = 0, 1, \cdots. \tag{3.114}$$

Next consider Equation (3.99). It is in the form of Equation (3.1) with

$$f(n,x) = a(n)x \text{ and } g(n,x) = \beta(n)e^{-\sigma(n)x} + q(n). \tag{3.115}$$

(2.7) is satisfied for any $B > 0$ and (2.8) holds for B large enough. Observing

$$\frac{dg}{dx} = -\beta(n)\sigma(n)e^{-\sigma(n)x},\ x \geq 0, \tag{3.116}$$

we see that $g(n,x)$ is nonincreasing in x and

$$\left|\frac{dg}{dx}\right| \leq \beta(n)\sigma(n) \text{ for } x \geq 0, \tag{3.117}$$

which implies that for each n, $g(n,x)$ is L-Lipschitz with $L(n) = \beta(n)\sigma(n)$. Hence, we have the following conclusion from Theorem 7.

Corollary 19

Assume that

$$\prod_{j=0}^{\omega-1} a(j) < 1$$

and that either

$$a(n) \leq 1 \text{ and } \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right)\beta(i)\sigma(i) < 1,\ n = 0,1,\cdots,\omega-1 \tag{3.118}$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\beta(i)\sigma(i) < 1,\ n = 0,1,\cdots,\omega-1. \tag{3.119}$$

Then every solution $\{x(n)\}$ *of Equation (3.99) satisfies*

$$\lim_{n\to\infty}(x(n) - \tilde{y}(n)) = 0 \tag{3.120}$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution with period* ω *of the following equation*

$$y(n+1) = a(n)y(n) + \beta(n)e^{-\sigma(n)y(n-k)},\ n = 0,1,\cdots. \tag{3.121}$$

Next, consider Equation (3.100). It is in the form of Equation (3.1) with

$$f(n,x) = a(n)x \text{ and } g(n,x) = \frac{\beta(n)}{1+x^{\gamma}} + q(n). \tag{3.122}$$

Again, (2.7) is satisfied for any $B > 0$ and (2.8) hold for B large enough. Observing that

$$\frac{dg}{dx} = -\beta(n)\frac{\gamma x^{\gamma-1}}{(1+x^{\gamma})^2} \text{ and } \frac{d^2 g}{dx^2} = \beta(n)\frac{\gamma x^{\gamma-2}((\gamma+1)x^{\gamma} - (\gamma-1))}{(1+\gamma)^3} \tag{3.123}$$

we see that for each n, when $\gamma = 1$,

$$\left|\frac{dg}{dx}\right| \leq \left|\frac{dg}{dx}\right|_{x=0} = \beta(n) \text{ for } x \geq 0 \tag{3.124}$$

and when $\gamma > 1$, $\left|\frac{dg}{dx}\right|$ attains its maximum at $x^* = \left(\frac{\gamma-1}{\gamma+1}\right)^{1/\gamma}$ and

$$\left|\frac{dg}{dx}\right|_{x=x^*} = \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma}\beta(n),\ n = 0, 1, \ldots, \omega-1. \tag{3.125}$$

Hence, $g(n,x)$ is L-Lipschitz with

$$L(n) = \begin{cases} \beta(n), & \gamma = 1, \\ \frac{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}}{4\gamma}\beta(n), & \gamma > 1. \end{cases} \tag{3.126}$$

It follows from Theorem 7 that the following conclusion holds.

Corollary 20

Assume that

$$\prod_{j=0}^{\omega-1} a(j) < 1. \tag{3.127}$$

Suppose also that when $\gamma = 1$, *either*

$$a(n) \leq 1 \textit{ and } \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right)\beta(i) < 1,\ n = 0, 1, \cdots, \omega-1 \tag{3.128}$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\beta(i) < 1,\ n=0,1,\cdots,\omega-1; \tag{3.129}$$

when $\gamma > 1$, *either*

$$a(n) \le 1 \text{ and } \sum_{i=n}^{n+k}\left(\prod_{j=i+1}^{n+k} a(j)\right)\beta(i) < \frac{4\gamma}{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}},\ n=0,1,\cdots,\omega-1 \tag{3.130}$$

or

$$\sum_{i=n}^{n+k+\omega-1}\left(\prod_{j=i+1}^{n+k+\omega-1} a(j)\right)\beta(i) < \frac{4\gamma}{(\gamma-1)^{\frac{\gamma-1}{\gamma}}(\gamma+1)^{\frac{\gamma+1}{\gamma}}},\ n=0,1,\cdots,\omega-1. \tag{3.131}$$

Then every solution $\{x(n)\}$ *of Equation (3.100) satisfies*

$$\lim_{n\to\infty}(x(n)-\tilde{y}(n)) = 0 \tag{3.132}$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution of with period* ω *of the following equation*

$$y(n+1) = a(n)y(n) + \frac{\beta(n)}{1+y^{\gamma}(n-k)},\ n=0,1,\cdots. \tag{3.133}$$

Consider the equation

$$x(n+1) = \frac{\alpha(n)x(n)}{\delta(n)+(\gamma(n)+\beta(n)x(n))^p} + \frac{\lambda(n)}{\delta(n)+(\gamma(n)+\beta(n)x(n-k))^p} + b(n),\ n=0,1,\cdots \tag{3.134}$$

where $\{\alpha(n)\}, \{\beta(n)\}, \{\gamma(n)\}, \{\delta(n)\}, \{\lambda(n)\}$ are nonnegative periodic sequences with period ω, $p > 0$, k is a nonnegative integer and $b(n)$ is a real sequence. When $\alpha(n) \equiv \alpha$, $\beta(n) \equiv \beta$ are positive constants, $\gamma(n) \equiv 1, \delta(n) \equiv 0, \lambda(n) \equiv 0$, and $b(n) \equiv 0$, we see that Equation (3.134) reduces to

$$x(n+1) = \frac{\alpha x(n)}{(1+\beta x(n))^p},\ n=0,1,\cdots \tag{3.135}$$

which models the population for a large number of species, such as the Colorado Beetle [23]. When $p = 1$, Equation (3.135) reduces in form to the classic Beverton/Holt model [5]. When $b(n) \not\equiv 0$, $\{b(n)\}$ may represent migration or a subset of the population that is independent of density-dependent effects.

If $\{b(n)\}$ is quasi-periodic, then we write $b(n) = q(n) + r(n)$, where $q(n)$ is periodic with period ω and $r(n) \to 0$. Equation (3.134) becomes

$$\begin{aligned} x(n+1) &= \frac{\alpha(n)x(n)}{\delta(n) + (\gamma(n) + \beta(n)x(n))^p} + \frac{\lambda(n)}{\delta(n) + (\gamma(n) + \beta(n)x(n-k))^p} \\ &\quad +q(n) + r(n),\ n = 0, 1, \cdots . \end{aligned} \tag{3.136}$$

Equation (3.136) is of the form of Equation (3.1) with

$$f(n, x) = \frac{\alpha(n)x}{\delta(n) + (\gamma(n) + \beta(n)x)^p} \tag{3.137}$$

and

$$g(n, x) = \frac{\lambda(n)}{\delta(n) + (\gamma(n) + \beta(n)x)^p} + q(n). \tag{3.138}$$

As

$$\frac{df}{dx} = \frac{\alpha(n)\delta(n) + \alpha(n)\left(\gamma(n) + \beta(n)x\right)^{p-1}\left[\gamma(n) + \beta(n)x(1-p)\right]}{\left(\delta(j) + (\gamma(n) + \beta(n)x)^p\right)^2}, \tag{3.139}$$

if $p = 1$ then we see that

$$\frac{df}{dx} = \frac{\alpha(n)(\delta(n) + \gamma(n))}{(\delta(n) + \gamma(n) + \beta(n)x)^2} \geq 0 \text{ for } x \geq 0. \tag{3.140}$$

If $p \neq 1$, $f(n, x)$ is nonincreasing when $\gamma(n) + \beta(n)x(1-p) \geq 0$. For $p < 1$, $f(n, x)$ is nondecreasing provided x satisfies the inequality $x \geq \frac{-\gamma(n)}{\beta(n)(1-p)}$. This is true for all $x \geq 0$, $\beta(n) \neq 0$. For $p > 1$, $f(n, x)$ is nondecreasing provided x satisfies the inequality

$x \leq \frac{\gamma(n)}{\beta(n)(p-1)}$. As we do not desire to restrict the upper bound of solutions x, we will only consider $p \leq 1$ for Equation (3.136).

Next, assume that $\delta(n) \geq 1$. Then for $x \geq 0$,

$$f(n,x) - \alpha(n)x = \frac{-\alpha(n)x\left[(\gamma(n)+\beta(n)x)^p + \delta(n) - 1\right]}{(\gamma(n)+\beta(n)x)^p + \delta(n)}, \tag{3.141}$$

and

$$(f(n,x) - a(n)x)' =$$

$$\frac{-\alpha(n)\left[((\gamma(n)+\beta(n)x)^p+\delta(n)-1)((\gamma(n)+\beta(n)x)^p+\delta(n))+p\beta(n)x(\gamma(n)+\beta(n)x)^{p-1}\right]}{((\gamma(n)+\beta(n)x)^p+\delta(n))^2}. \tag{3.142}$$

Thus, $f(n,x) \leq \alpha(n)x$ and $f(n,x) - \alpha(n)x$ is nonincreasing in x. As

$$\frac{dg}{dx} = -\frac{p\beta(n)\lambda(n)(\gamma(n)+\beta(n)x)^{p-1}}{(\delta(n) + (\gamma(n)+\beta(n)x)^p)^2}, \tag{3.143}$$

$$\frac{d^2g}{dx^2} = \frac{p(1-p)\beta^2(n)\lambda(n)(\gamma(n)+\beta(n)x)^{p-2}}{(\delta(n) + (\gamma(n)+\beta(n)x)^p)^2} \text{ for } p \neq 1, \tag{3.144}$$

and

$$\frac{d^2g}{dx^2} = \frac{2\beta^2(n)\lambda(n)}{(\delta(n)+\gamma(n)+\beta(n)x)^2} \text{ for } p = 1, \tag{3.145}$$

$g(n,x)$ is clearly nonincreasing in x and $|\frac{dg}{dx}|$ achieves a maximum when $x = 0$. Thus we see that $|\frac{dg}{dx}|_{x=0} = \frac{p\beta(n)\lambda(n)\gamma^{p-1}(n)}{(\delta(n)+\gamma^p(n))}$. Hence, $g(n,x)$ is L-Lipschitz with $L(n) = \frac{p\beta(n)\lambda(n)\gamma^{p-1}(n)}{(\delta(n)+\gamma^p(n))}$.

We then have the following conclusion from Theorem 7.

Corollary 21

Assume that

$$\hat{\alpha} = \prod_{j=0}^{\omega-1} \alpha(j) < 1, \{q(n)\} \textit{ is periodic with period } \omega \textit{ and } \lim_{n\to\infty} r(n) = 0, \tag{3.146}$$

and suppose that $g(n,x) = \frac{\lambda(n)}{\delta(n)+(\gamma(n)+\beta(n)x)^p} + q(n) \geq 0$ *for* $x \geq 0$. *Suppose that* $p \leq 1$, $\beta(n) \not\equiv 0$, *and* $\delta(n) \geq 1$, *and suppose also that there exists a positive constant* B *such that for* $n = 0, 1, \cdots, \omega - 1$,

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} \alpha(j) \right) \left[\frac{\lambda(i) - \alpha(i)B[(\gamma(i)+\beta(i)B)^p + \delta(i) - 1]}{\delta(i) + (\gamma(i)+\beta(i)B)^p} + q(i) \right] \geq 0 \quad (3.147)$$

and

$$\frac{1}{1-\hat{\alpha}} \sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} \alpha(j) \right) \left[\frac{\lambda(i)}{\delta(i) + \gamma^p(i)} + q(i) \right] \leq B. \quad (3.148)$$

Suppose also that either

$$\alpha(n) \leq 1 \text{ and } \sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} \alpha(j) \right) \frac{p\beta(i)\lambda(i)\gamma^{p-1}(i)}{(\delta(i)+\gamma^p(i))^2} < 1, \ n = 0, 1, \cdots, \omega - 1 \quad (3.149)$$

or

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} \alpha(j) \right) \frac{p\beta(i)\lambda(i)\gamma^{p-1}(i)}{(\delta(i)+\gamma^p(i))^2} < 1, \ n = 0, 1, \cdots, \omega - 1. \quad (3.150)$$

Then every solution $\{x(n)\}$ *of Equation (3.136) satisfies*

$$\lim_{n\to\infty} (x(n) - \tilde{y}(n)) = 0 \quad (3.151)$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution with period* ω *of the following equation*

$$\begin{aligned} y(n+1) &= \frac{\alpha(n)y(n)}{\delta(n) + (\gamma(n)+\beta(n)y(n))^p} \\ &\quad + \frac{\lambda(n)}{\delta(n) + (\gamma(n)+\beta(n)y(n-k))^p} + q(n), \ n = 0, 1, \cdots. \end{aligned} \quad (3.152)$$

If $\lambda(n) \equiv 0$, Equation (3.136) reduces to

$$x(n+1) = \frac{\alpha(n)x(n)}{\delta(n) + (\gamma(n)+\beta(n)x(n))^p} + q(n) + r(n), n = 0, 1, \cdots. \quad (3.153)$$

Clearly, if there exists positive B such that

$$q(n) - \frac{\alpha(n)B\left[(\gamma(n)+\beta(n)B)^p + \delta(n) - 1\right]}{\delta(n) + (\gamma(n)+\beta(n)B)^p} \geq 0,\ n = 0, 1, \ldots, \omega - 1 \tag{3.154}$$

and

$$q(n) \leq \frac{B(1-\hat{\alpha})}{\sum\limits_{i=n}^{n+\omega-1}\left(\prod\limits_{j=i+1}^{n+\omega-1}\alpha(j)\right)},\ n = 0, 1, \ldots, \omega - 1 \tag{3.155}$$

then (3.147) and (3.148) hold. We see from above that

$$q(n) - \frac{\alpha(n)B\left[(\gamma(n)+\beta(n)B)^p + \delta(n) - 1\right]}{\delta(n) + (\gamma(n)+\beta(n)B)^p} \geq q(n) - \alpha(n)B \tag{3.156}$$

Hence, the following is a direct consequence of Corollary 21.

Corollary 22

Assume that

$$\hat{\alpha} = \prod_{j=0}^{\omega-1}\alpha(j) < 1, \{q(n)\} \text{ is periodic with period } \omega \text{ and } \lim_{n\to\infty} r(n) = 0, \tag{3.157}$$

and suppose that $p \leq 1$, $\beta(n) \not\equiv 0$, *and* $\delta(n) \geq 1$. *Suppose also that there exists a positive constant* B *such that*

$$\alpha(n)B \leq q(n) \leq \frac{B(1-\hat{\alpha})}{\sum\limits_{i=n}^{n+\omega-1}\left(\prod\limits_{j=i+1}^{n+\omega-1}\alpha(j)\right)},\ n = 0, 1, \ldots, \omega - 1. \tag{3.158}$$

Then every solution $\{x(n)\}$ *of Equation (3.153) satisfies*

$$\lim_{n\to\infty}(x(n) - \tilde{y}(n)) = 0 \tag{3.159}$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution with period* ω *of the following equation*

$$y(n+1) = \frac{\alpha(n)y(n)}{\delta(n) + (\gamma(n)+\beta(n)y(n))^p} + q(n),\ n = 0, 1, \cdots. \tag{3.160}$$

Next, consider the equation

$$x(n+1) = \frac{\alpha(n)e^{-\beta(n)x(n-k)}}{\delta(n)+\gamma(n)x(n-k)} + b(n),\ n = 0, 1, \cdots, \tag{3.161}$$

where $\{\alpha(n)\}, \{\beta(n)\}, \{\gamma(n)\}, \{\delta(n)\}$ are nonnegative periodic sequences with period ω and $b(n)$ is a real sequence. In [22], the global stability of the equation

$$x(n+1) = \alpha + \beta x(n-1)e^{-x(n)} \tag{3.162}$$

was studied, where the equation could be viewed as a population model. The global stability of

$$x(n+1) = \frac{\alpha + \beta e^{-x(n)}}{\gamma + x(n-1)},\ n = 0, 1, \cdots \tag{3.163}$$

and

$$x(n+1) = \frac{\alpha e^{-(nx(n)+(n-k)x(n-k))}}{\beta + nx(n) + (n-k)x(n-k)},\ n = 0, 1, \cdots \tag{3.164}$$

were studied in [25] and [24], respectively.

If $\{b(n)\}$ is quasi-periodic, then we write $b(n) = q(n) + r(n)$, where $q(n)$ is periodic with period ω and $r(n) \to 0$. Thus Equation (3.161) becomes

$$x(n+1) = \frac{\alpha(n)e^{-\beta(n)x(n-k)}}{\delta(n)+\gamma(n)x(n-k)} + q(n) + r(n),\ n = 0, 1, \cdots. \tag{3.165}$$

Equation (3.165) is of the form of Equation (3.1) with $f(n,x) \equiv 0$ and $g(n,x) = \frac{\alpha(n)e^{-\beta(n)x}}{\delta(n)+\gamma(n)x}$ $+q(n)$. As

$$\frac{dg}{dx} = \frac{-\alpha(n)e^{-\beta(n)x}\left(\beta(n)\delta(n) + \beta(n)\gamma(n)x + \gamma(n)\right)}{(\delta(n)+\gamma(n)x)^2} \tag{3.166}$$

$|\frac{dg}{dx}|$ achieves a maximum at $x = 0$ for $x \geq 0$ and

$$\left|\frac{dg}{dx}\right|_{x=0} = \frac{\alpha(n)\left(\beta(n)\delta(n) + \gamma(n)\right)}{\delta^2(n)}. \tag{3.167}$$

Thus, $g(n,x)$ is L-Lipschitz with $L(n) = \frac{\alpha(n)(\beta(n)\delta(n)+\gamma(n))}{\delta^2(n)}$. We see that (2.7) holds for $g(n,x) \geq 0$ and (2.8) holds for B large enough. The following is then a direct consequence of Theorem 7.

Corollary 23

Assume that

$$\hat{\alpha} = \prod_{j=0}^{\omega-1} \alpha(j) < 1, \{q(n)\} \text{ is periodic with period } \omega \text{ and } \lim_{n\to\infty} r(n) = 0, \tag{3.168}$$

and suppose that $g(n,x) = \frac{\alpha(n)e^{-\beta(n)x(n)}}{\delta(n)+\gamma(n)x} + q(n) \geq 0$ *for* $x \geq 0$*. Suppose also that either* $a(n) \leq 1$ *and*

$$\sum_{i=n}^{n+k} \left(\prod_{j=i+1}^{n+k} a(j) \right) \frac{\alpha(i)(\beta(i)\delta(i)+\gamma(i))}{\delta^2(i)} < 1, \; n = 0, 1, \cdots, \omega - 1 \tag{3.169}$$

or that $a(n) \not\leq 1$ *and*

$$\sum_{i=n}^{n+k+\omega-1} \left(\prod_{j=i+1}^{n+k+\omega-1} a(j) \right) \frac{\alpha(i)(\beta(i)\delta(i)+\gamma(i))}{\delta^2(i)} < 1, \; n = 0, 1, \cdots, \omega - 1. \tag{3.170}$$

Then every solution $\{x(n)\}$ *of Equation (3.165) satisfies*

$$\lim_{n\to\infty} (x(n) - \tilde{y}(n)) = 0 \tag{3.171}$$

where $\{\tilde{y}(n)\}$ *is the unique periodic solution with period* ω *of the following equation*

$$y(n+1) = \frac{\alpha(n)e^{-\beta(n)y(n-k)}}{\delta(n)+\gamma(n)y(n-k)} + q(n), \; n = 0, 1, \cdots. \tag{3.172}$$

Example 2

Consider the equation

$$\begin{aligned} x(n+1) &= \frac{\left(0.15\sin\left(\frac{\pi n}{3}\right) + 0.86\right)x(n)}{\left(\cos\left(\frac{\pi n}{3}\right)+2\right) + \left[\left(0.2\cos\left(\frac{\pi n}{3}\right)+0.4\right) + (0.25\sin\left(\frac{\pi n}{3}\right)+0.25)x(n)\right]^p} \\ &+ \frac{0.3\sin\left(\frac{\pi n}{3}\right)+0.4}{\left(\cos\left(\frac{\pi n}{3}\right)+2\right) + \left[\left(0.2\cos\left(\frac{\pi n}{3}\right)+0.4\right) + (0.25\sin\left(\frac{\pi n}{3}\right)+0.25)x(n-4)\right]^p} \\ &+ \left(0.2\cos\left(\frac{\pi n}{3}\right)+0.3\right) + n^2e^{-n}, \; n = 0, 1, \cdots. \end{aligned} \tag{3.173}$$

Equation (3.173) takes the form of Equation (3.136) with $\alpha(n) = 0.15\sin\left(\frac{\pi n}{3}\right) + 0.86$, $\beta(n) = 0.25\sin\left(\frac{\pi n}{3}\right) + 0.25$, $\gamma(n) = 0.2\cos\left(\frac{\pi n}{3}\right) + 0.4$, $\delta(n) = \cos\left(\frac{\pi n}{3}\right) + 2$, $\lambda(n) = 0.3\sin\left(\frac{\pi n}{3}\right) + 0.4$, $q(n) = 0.2\cos\left(\frac{\pi n}{3}\right) + 0.3$, and $r(n) = n^2 e^{-\frac{1}{n}}$, with $p = 0.75$ and $k = 4$, where $\{\alpha(n)\}$, $\{\beta(n)\}$, $\{\gamma(n)\}$, $\{\delta(n)\}$, and $\{\lambda(n)\}$ are periodic sequences with period $\omega = 6$.

Clearly, $p \leq 1, \beta(n) \not\equiv 0, \delta(n) \geq 1$. It is also clear that $q(n)$ is periodic with period $\omega = 6$ and that $r(n) \to 0$ as $n \to \infty$. It can be numerically verified that $\prod_{j=0}^{\omega-1} \alpha(j) < 1$, that (3.147) and (3.148) hold for $B = 2.7$, and that (3.149) holds. Then by Corollary 21, every solution $\{x(n)\}$ of Equation (3.173) satisfies

$$\lim_{n\to\infty} \left(x(n) - \tilde{y}(n)\right), \tag{3.174}$$

where $\{\tilde{y}(n)\}$ is the unique nonnegative periodic solution with period $\omega = 6$ of the equation

$$\begin{aligned} x(n+1) &= \frac{\left(0.15\sin\left(\frac{\pi n}{3}\right) + 0.86\right) x(n)}{\left(\cos\left(\frac{\pi n}{3}\right) + 2\right) + \left[\left(0.2\cos\left(\frac{\pi n}{3}\right) + 0.4\right) + \left(0.25\sin\left(\frac{\pi n}{3}\right) + 0.25\right)x(n)\right]^p} \\ &+ \frac{0.3\sin\left(\frac{\pi n}{3}\right) + 0.4}{\left(\cos\left(\frac{\pi n}{3}\right) + 2\right) + \left[\left(0.2\cos\left(\frac{\pi n}{3}\right) + 0.4\right) + \left(0.25\sin\left(\frac{\pi n}{3}\right) + 0.25\right)x(n-4)\right]^p} \\ &+ \left(0.2\cos\left(\frac{\pi n}{3}\right) + 0.3\right),\ n = 0, 1, \cdots. \end{aligned} \tag{3.175}$$

One such solution with initial function

$$x(n) = e^{-\frac{1}{n}}, -k \leq n \leq 0 \tag{3.176}$$

is shown in Figure 3.1.

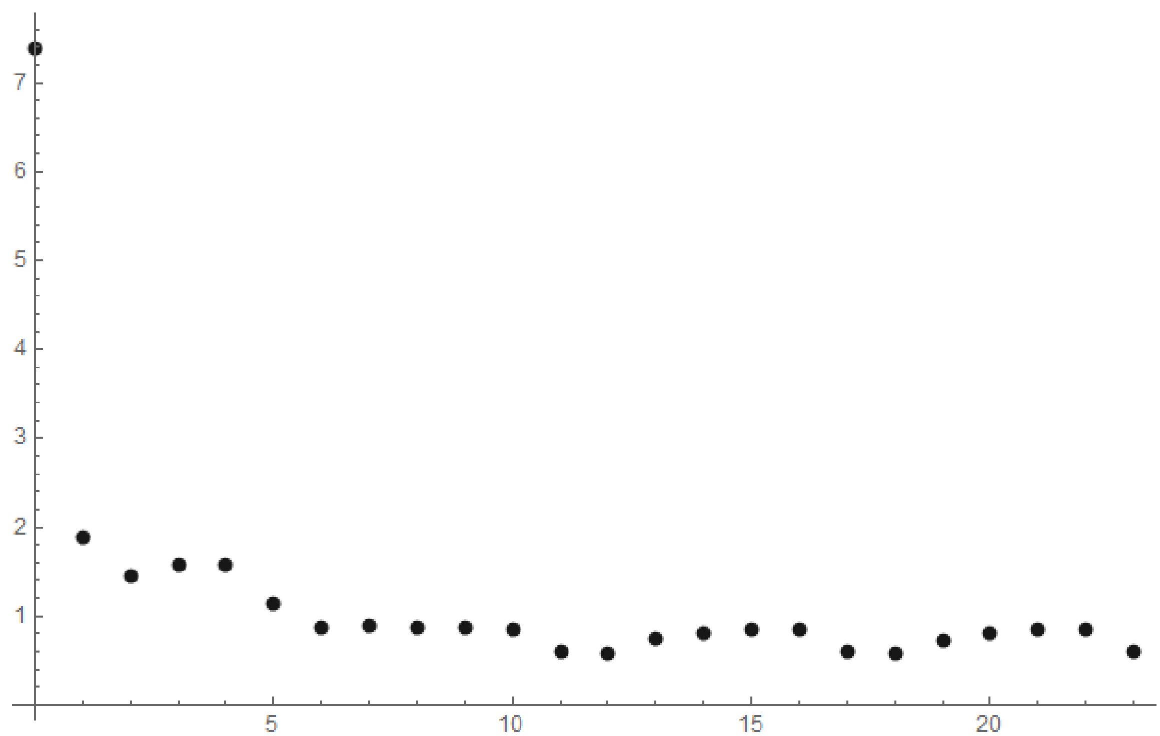

Figure 3.1

Graph of a solution $\{x(n)\},\ n = 0, 1, \cdots, 23$ of Equation (2.122) with initial function (3.176).

CHAPTER 4

FUTURE WORK

We have obtained some sufficient conditions for the existence of a nonnegative periodic solution, and the global attractivity of this periodic solution, of the equation mentioned in Chapter 2. We have also obtained some sufficient conditions for the existence of quasiperiodic solutions of the forced difference equation mentioned in Chapter 3. While both of these equations are derived from biological models, there are many related models where the conditions discussed in the previous chapters fail to apply in their current state. In particular, equations derived from Nicholson's blowfly model and some forms of Mackey-Glass equations do not satisfy the hypotheses of Equation (2.1). For example, consider the equation

$$x(n+1) = \frac{\alpha x(n)}{x(n)+b} + \frac{px(n)}{q+x^m(n)},\ n = 0, 1, \cdots, \tag{4.1}$$

which may be considered a type of discrete analog of an equation studied in [52]. This equation may represent a new fishery model that generalizes Nicholson's blowfly model. Equation (4.1) takes the form of Equation (2.1) with

$$f(n,x) = f(x) = \frac{\alpha x}{x+b} \text{ and } g(n,x) = g(x) = \frac{px}{q+x^m}. \tag{4.2}$$

However, as,

$$g'(x) = \frac{pq + px^m - pmx^m}{(q+x^m)^2}, \tag{4.3}$$

$g(n, x)$ nonincreasing in x for $x \geq 0$ does not hold, and the conditions presented in Chapter 2 for the existence of a periodic solution do not apply. In [52], however, a sufficient condition for the existence of a periodic solution of Equation (4.1) is developed using coincidence degree theory. Is there a way to show the existence of a periodic solution of this equation using Schauder's Fixed Point Theorem? If so, can we then extend these results to a generalized equation with periodic coefficients in place of the fixed parameters in Equation (4.1)? Conversely, is it possible to extend the results from [52] from the existence of a positive periodic solution to the existence of a nonnegative periodic solution using coincidence degree theory? Many authors in recent years have used the Coincidence Degree Theorem to show the existence of positive periodic and asymptotically periodic solutions to delay differential equations and higher order difference equations. However, results extending this idea to nonnegative periodic solutions seem quite rare, see [54].

To the best of our knowledge, there are few results dealing with the existence and global attractivity of periodic solutions using Schauder's Theorem and Lipschitz conditions when g is not restricted to be a nonincreasing function in x (a restriction applied to g in Equations (2.1) and (3.1), but not applicable in Equation (4.1) and similar equations). In this case, g is sometimes referred to as a a "one-hump" function, and there exists some positive constant K such that g is nonincreasing in x for all $x \geq K$. As g is bounded above and eventually decreasing, e.g. in Equation (4.1) $g(n, x)$ has a maximum at $x^* = \left(\frac{q}{m-1}\right)^{\frac{1}{m}}$ for $m > 1$ and is nonincreasing for $x \geq x^*$, it may be possible to show that g can be bounded above by another nonincreasing function in x in order to apply the results from Chapters 2 and 3 in a similar way.

There are also results such as those in [8] dealing with other classes equations related to Equation (2.1). Consider

$$x(n+1) = f(n, x(n)) + \sum_{i=1}^{m} g_i(n, x(n - \tau_i(n))),\ n = 0, 1, \cdots. \tag{4.4}$$

When $f(n, x(n)) = a(n)x(n)$ and $g_i(n, x(n - \tau_i(n))) = \frac{p_i(n)}{1+x^\gamma(t-\tau_i(n))},\ i = 1, 2, \cdots, m$ Equation (4.4) reduces to the equation

$$x(n+1) = a(n)x(n) + \sum_{i=1}^{m} \frac{p_i(n)}{1 + x^\gamma(t - \tau_i(n))},\ n = 0, 1, \cdots, \tag{4.5}$$

which is a type of discretization of the equation studied in [8] and a generalization of Equation (3.94) with forcing term $b(n) \equiv 0$. Clearly, Equation (4.4) is a more general result than Equation (2.1) studied in Chapter 2 as there are multiple delays present and each delay is a sequence instead of a fixed integer. Can the results from Chapter 2 be extended to apply to a model with multiple delays such as Equation (4.4)? If the same assumptions and restrictions placed on $g(n, x)$ in Chapter 2 are applied to each $g_i(n, x)$, it is reasonable that similar results can be obtained through an inductive process.

Along with these questions, it is natural to study the effect of adding a forcing term that does not necessarily exhibit periodic behavior to Equations (4.1) and (4.4) to determine if sufficient conditions can be found to produce quasiperiodic solutions of the corresponding nonhomogeneous equations.

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