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By

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We study the Erdős- Sós conjecture that states that every graph of average degree greater than $k-1$ contains every tree of order $k+1$. While the conjecture was studied for some graphs, it still remains open and of interest after more than 40 years. We study the conjecture for graphs with no $K_{2,s}$ where, $s \geq 2$ and $k > 12(s-1)$. We use the fact that as G contains no $K_{2,s}$, any two distinct vertices in G have at most $s-1$ neighbors in common in proving the results. We have answered in the affirmative that the Erdős- Sós conjecture is true for graphs defined above, thus adding to the list of graphs for which the conjecture is true.

We also study the Cayley Isomorphism Problem that states that for which finite groups H is it true that any two Cayley graphs of H are isomorphic if and only if they are isomorphic by a group automorphism of H? (H is a CI-group with respect to graphs.) Determining whether or not a group is a CI-group with respect to graphs has received considerable attention over the last 40 or so years. In particular, we study the problem for (pq, r)-metacirculant color digraphs where $p < q < r$ and $pq|\ |\alpha|$. We use the fact that Γ is a CI-color digraph of H if and only if given a permutation $\gamma \in S_H$ such that $\gamma^{-1}H_L\gamma \leq \mathrm{Aut}(\Gamma)$, H_L and $\gamma^{-1}H_L\gamma$ are conjugate in $\mathrm{Aut}(\Gamma)$. We consider the Cayley isomorphism problem for a nonabelian group of order pqr, where p, q, r are distinct primes such that $pq/(r-1)$. We show that the results are true, not only for Cayley graphs but for some related classes of non Cayley vertex transitive graphs, thus solving the problem for that case.

Key words: $K_{2,s}$, Erdős-Sós conjecture, Cayley graph, (pq, r)- metacirculant, Cayley Isomorphism problem.

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LIST OF SYMBOLS, ABBREVIATIONS, AND NOMENCLATURE

$G = (V, E)$ A graph with vertex set V and edge set E

T A Tree

$\delta(G)$ Minimum degree of a graph G

$\Delta(G)$ Maximum degree of a graph G

$\Delta_2(G)$ Second highest degree of a graph G

$\mu(G)$ Median degree of a graph G

$t(G)$ Average degree of a graph G

$C_{r,s,t}$ A caterpillar

$K_{2,s}$ A complete bipartite graph with partite sets of size 2 and s respectively

$N_G(v)$ The set of all neighbors in G of v

$\deg_G(v)$ The degree in G of v

$\mathrm{Aut}(\Gamma)$ The Automorphism group of a graph Γ

$\mathbb{S}ym(n)$ The symmetric group on n elements

$\Gamma(m, n)$ (m, n) Metacirculant

$C(S)$ Centralizer of a subset S of a group G

$N(G)$ Normalizer of a subset S of a group G

$\Gamma(G, \Omega)$ Cayley Graph

$\Gamma_1 \wr \Gamma_2$ Wreath product of two graphs Γ_1 and Γ_2

CHAPTER 1

INTRODUCTION

In the beginning stages of my Ph.D., the primary focus of my research was toward extremal graph theory. A typical question in this area is *For a given number of vertices, what is the minimum number of edges that ensures that a graph shall have a subgraph of a specified kind* ? *[21]*

One then determines whether the graph G has the property in question or not. Examples of which include

1. When does a given graph G have a property S ?
2. When does a graph G contain a Hamilton cycle or a Hamilton path ?

The question was originally studied by Erdős and Gallai who looked at the cases where the specific subgraphs were an arc, a polygon (circuit), a path, or a regular graph of degree 1. As a consequence, Erdős and Sós conjectured [20] in 1963 that

every graph of average degree greater than $k-1$ *contains every tree of order* $k+1$.

While the problem in itself is very difficult to prove, (it being considered as an open problem in the first and more recent second editions of Bondy and Murty's *Graph Theory With Applications* [9], [10]), various partial results have been shown over the years and we show in the following chapter that the Erdős - Sós Conjecture is true for graphs containing no $K_{2,s}$ where $s \geq 2$ and $k > 12(s-1)$ that satisfy the Erdős - Sós hypothesis.

As my research progressed, I started looking at problems in algebraic graph theory and specifically, the Cayley Isomorphism problem.

The Cayley Isomorphism problem was first raised by Ádám [1] who conjectured that *for a given* n*, any two circulant graphs of order* n *are isomorphic if and only if they are isomorphic by a group automorphism of* $\mathbb{Z}_n$.

Ádám was actually conjecturing that the list of permutations that need to be checked to test for isomorphism for circulant graphs is as small as possible.

This conjecture was quickly proved to be false by Elspas and Turner [19] in 1970 who gave a counterexample of order 8.

Following the counter example Ádám's conjecture was reformulated in the following way : *For which finite groups* G *is it true that any two Cayley graphs of* G *are isomorphic if and only if they are isomorphic by a group automorphism of* G?

Any group G with the above property is called a *CI-group* with respect to graphs and a group G is a CI-group with respect to graphs only if the list of permutations which must be checked to determine Cayley graphs of the groups that are isomorphic is as small as possible. In recent years there has been considerable interest in which groups are CI-groups with respect to graphs. Also, as a natural analogue, there has been interest in obtaining constructions that show that certain groups are not CI-groups with respect to graphs.

We consider the Cayley Isomorphism Problem for a nonabelian group of order pqr, where p, q, r are distinct primes such that $pq|(r-1)$, not only for Cayley graphs but for some related classes of non Cayley vertex transitive graphs. We show that such Cayley graphs have the CI- property (or an analogous property for non-Cayley graphs) or are Cayley graphs of groups that are 'almost' cyclic groups (or an analogous property for non-Cayley graphs).

In the next chapter, we consider the Erdős-Sós conjecture, while the Isomorphism problem is considered in Chapter 3.

CHAPTER 2

THE ERDŐS- SÓS CONJECTURE

2.1 Introduction

In 1959 Erdős and Gallai [21] looked at a problem posed by P. Turan [21] who had asked the following :

For a given number of vertices, what is the minimum number of edges that ensures that a graph shall have a subgraph of a specified kind ?

Erdős and Gallai studied the cases in which the specified subgraph is an arc, a polygon (circuit), a path, or a regular graph of degree 1. There is another kind of problem, studied by Zarankiewicz and Dirac, in which the existence of specified subgraphs is asserted if the degrees are sufficiently high, (See [21] for further details.). If the degrees of a certain number of the vertices are sufficiently high, then the average degree of the graph tends to go up which then allows for assertion of the existence of specified subgraphs of the required order.

As a consequence, Erdős and Sós conjectured [20] in 1963 that

every graph of average degree greater than $k-1$ *contains every tree of order* $k+1$.

The difficulty of the problem as stated is well justified by its being listed as one of 30 open problems found in Bondy and Murty's original *Graph Theory With Applications* [9] and has been repeated in the open problems list of their latest edition [10] of *Graph Theory With Applications*.

However, approximate versions of the conjecture were proved by Ajtai, Komlós and Szemerédi [2] using the Regularity Lemma. They showed that for sufficiently large k and sufficiently large n, the Erdős - Sós Conjecture is true for all graphs of order n that satisfy the hypothesis.

Partial solutions to the conjecture have been proved by showing either certain classes of graphs that satisfy the hypothesis contain all trees of order $k+1$ or all graphs that satisfy the hypothesis contain special cases of trees. In the former approach, Brandt and Dobson, [11] verified the conjecture for all graphs of girth at least 5. They proved:

Theorem 1

Let G be a graph with girth at least 5 and T be a tree with k edges. If $\delta(G) \geq \dfrac{k}{2}$ and $\Delta(G) > \Delta(T)$, then G contains T as a subgraph.

The Theorem is proved by inductively showing that a resolution (see Definition 13) $T_i \subset T$ (and hence T) is contained in G, that is, if G contains the resolutions $T_1, T_2, ..., T_{i-1}$ (where any star T_1 can be embedded in G by mapping its center to a vertex of maximum degree in G), then G contains T_i.

As it is not difficult to show that every graph G that satisfies the Erdős-Sós hypothesis contains a subgraph of minimum degree at least $\dfrac{k}{2}$ and maximum degree at least k, (the proof for which is shown in Lemma 4), they then showed:

Theorem 2

Every graph of order n and girth at least 5 with more than $\frac{n(k-1)}{2}$ edges contains every tree with k edges as a subgraph.

As a consequence of Theorem 1, Theorem 2 implies that the Erdős-Sós conjecture is true for graphs with girth at least 5. Li and Yin [44] showed that the Erdős-Sós conjecture is true for graphs whose complement contain no C_4. Li, Liu, and Wang [27] showed that the Erdős-Sós conjecture is true for graphs whose complement has girth at least 5. Dobson [17] had extended this result and had showed that the Erdős-Sós conjecture is true for graphs whose complement contains no $K_{2,4}$ which is a stronger result when compared to [44].

Sacle and Woźniak [37] then improved Theorem 2 by proving that the conjecture is true for triangle free graphs of order n satisfying the hypothesis of the conjecture.

Haxell [23] then went on to show the following, thus improving the result of Sacle and Woźniak :

Theorem 3 Let $t > 1$ be an integer and let T be a tree with t edges. Suppose G is a graph with average degree greater than $t-1$ that does not contain a copy of $K_{2,r}$, where $r = \lfloor \frac{k}{18} \rfloor$. Then G contains T as a subgraph.

Subsequently, the author and Dobson [8] have relaxed the bounds in Haxell's paper and have shown, as illustrated in the subsequent pages, the following :

Theorem 4

Let $s > 2$, $k > 12(s-1)$. Every graph with average degree $> k-1$ that contains no $K_{2,s}$ contains every tree of order $k+1$.

In the latter approach, all graphs satisfying the hypothesis of the Erdős-Sós Conjecture were shown to contain the following families of trees :

1. Paths [21],
2. Stars,
3. Caterpillars [31],
4. Spiders of diameter ≤ 4 [42],
5. Trees (of size k) with a vertex adjacent to t leaves, where $t \geq \frac{1}{2}(k-1)$ [38] and
6. Trees of diameter 4 [28].

2.2 Preliminaries

We now define terms that will be used throughout the chapter. For terms not defined here see [6].

Definition 1 A *graph* $G = (V, E)$ consists of a vertex set $V = V(G)$ and an edge set $E = E(G) \subseteq V \times V$ with $V \cap E = \phi$.

An edge $e \in E$ usually denotes $e = (u, v)$ (equivalently $e = uv$), where u, $v \in V$. The number of vertices of a graph G is the order of G, denoted by $|V|$. All graphs considered here will be of finite order.

Two vertices u, $v \in V$ are *adjacent* if $e = uv \in E$. If v is adjacent to u then, v is a *neighbor* of u.

We say that $e = uu \in E$ is a *loop*. A graph G is *simple* if it has no loops or multiple edges. For all our purposes here, G is always a simple graph.

Definition 2 Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We say that G and G' are *isomorphic* if there exists a bijection $f\colon V \to V'$ such that $xy \in E$ if and only if $f(x)f(y) \in E'$ for all $x, y \in V$.

Definition 3 For two graphs $G = (V, E)$ and $G' = (V', E')$, we say that G contains G' ($G' \subseteq G$) if $V' \subseteq V$ and $E' \subseteq E$.

Let $G = (V, E)$ be a graph. The set of all neighbors of a vertex $v \in V$ is denoted by $N_G(v)$.

Definition 4 The *degree* of a vertex $v \in V$, denoted by $\deg_G(v)$ is the order of $N_G(v)$.

The *minimum degree* of a graph G is the number $\delta(G) = \min\{\deg_G(u)\colon u \in V\}$ and the *maximum degree* of a graph G is the number $\Delta(G) = \max\{\deg_G(u)\colon u \in V\}$. The number $d(G)\colon = \frac{1}{|V|}\Sigma_{v\in V} \deg_G(v)$ is the *average degree* of G.

Definition 5 A *path* $P = (V, E)$ is the graph of the form $V\colon = \{x_0, x_1, ..., x_n\}$, $E = \{x_0x_1, x_1x_2, ..., x_{n-1}x_n\}$, and we say that P is an $x_0 - x_n$ path of length n.

Definition 6 The *distance* $\mathrm{dist}_G(u, v)$ in G of two vertices u, v is the length of the shortest $x - y$ path in G.

Definition 7 A graph G is *connected* if there exists a path joining x and y for any pair $x, y \in V$.

Definition 8 For a path $P = (V, E)$ as defined above, a *cycle* C is the graph $P + x_n x_0$, the length of which is again the number of edges in C. The minimum length of a cycle contained in G is called the *girth* of the graph G. An *acyclic* graph is a graph without any cycles.

Definition 9 A graph G is $r-$ *partite* if V admits an r class partition such that each edge in E has its ends in different classes.

If $r = 2$, then G is bipartite and is denoted by $K_{m,n}$, where $m + n$ is the order of G.

Definition 10 A *star* on n vertices is $K_{1,n-1}$.

Definition 11 Let $C_{r,s,t}$ denote the graph that consists a star of size r and a star of size s whose vertices of maximal degree are joined by a path of length t. Such a graph is a an example of a *caterpillar*.

Definition 12 A *tree* T is a connected acyclic graph.

A *leaf* of a tree T is a vertex v such that $\deg_T(v) = 1$, and a *penultimate* vertex of a tree T is a vertex v such that every neighbor of v in T, with at most one exception, is a leaf of T.

Definition 13 A sequence (T_i), $1 \leq i \leq p$, of subtrees is a *resolution* of a tree T if T_1 is a star such that $\Delta(T_1) = \Delta(T)$, $T_p = T$ and T_{i-1} is obtained from T_i by deleting all leaf neighbors of a penultimate vertex of T_i of minimal degree, $2 \leq i \leq p$.

Definition 14 An *inclusion* $f : H \to G$ is an injective function from $V(H)$ to $V(G)$ such that if $e \in E(H)$, then $f(e) \in E(G)$.

Hence if there is an inclusion from H to G, then G contains a subgraph isomorphic to H, namely, $f(H)$.

A fact that will be used implicitly throughout this chapter is that if G is a graph containing no $K_{2,s}$ then for every $u, v \in V(G)$, $u \neq v$, $|N_G(u) \cap N_G(v)| \leq s-1$.

2.3 The Erdős- Sós Conjecture

For the rest of this chapter, we begin by first providing some tools that will be used throughout this chapter and the next. We then proceed to show that the Erdő-Sós conjecture is true for graphs having no $K_{2,s}$ that satisfy the Erdős-Sós hypothesis by considering various cases involving bounds on the maximum degree of T. That is when $\Delta(T) \leq \frac{k}{4}$ and $\Delta(T) > \frac{k}{4}$. For the case where $\Delta(T) > \frac{k}{4}$, bounds on $\Delta_2(T)$ are used.

The number of leaves in a tree of maximum degree k is characterized by the following result [12, Theorem 3.7].

Lemma 1

Let T be a tree with maximum degree $\Delta(T) = k$. If $n_i, i = 1, \ldots, k$ denote the number of vertices of degree i, then $n_1 = n_3 + 2n_4 + 3n_5 + \cdots + (k-2)n_k + 2$.

We shall have need of the following consequence of this result.

Lemma 2

Let T be a tree of order at most k. Then the number of vertices in a path joining any two vertices $u, v \in V(T)$ that are not leaves of T is at most $k - \Delta(T) - \Delta_2(T) + 2$.

PROOF. Let T be a tree that satisfies the hypothesis. Let $u, v \in V(T)$ be two distinct vertices that are not leaves. Then the uv-path P in T is contained in the tree T' obtained from T by deleting every leaf of T. By Lemma 1, T' contains at most $k-(\Delta-2+\Delta_2(T)-2+2) = k-\Delta(T)-\Delta_2(T)+2$ vertices. Thus P contains at most $k-\Delta(T)-\Delta_2(T)+2$ vertices. $\square$

A form of the following result is well-known and the result is straightforward to prove using the greedy algorithm.

Lemma 3

Let G be a graph with $\delta(G) \geq k$ and $u \in V(G)$. Let T be a tree of order $k+1$ with $v \in V(T)$. Then there exists an inclusion $f : T \to G$ such that $f(v) = u$.

PROOF. Let $T' \subset T$ be a maximal subtree such that there exists an inclusion $g \colon T' \to G$ with $g(x) \in V(G)$ for every $x \in T_1$. As $\delta(G) \geq k$ and $\Delta(T_1) \leq \Delta(T)$, $\deg_{T'}(x) \leq \Delta(T') \leq k$, and hence for every $x_i \in V(T) - V(T_1)$, there exists an inclusion $f_i \colon T_i \to G$ where $T_i = T \cup \{x_i\}$ such that $f_i(x_i) \in G$ and $f_i(T_{i-1}) = f_{i-1}(T_{i-1})$ for $i = 2, 3, \ldots, k+1$ thus giving us the required inclusion. $\square$

We will also need the following easy (and fairly well-known) result.

Lemma 4

Let k be a positive integer and G be a graph of average degree greater than $k-1$. Then G contains a subgraph H such that $\delta(H) \geq \dfrac{k}{2}$ and $\Delta(H) \geq k$.

PROOF. Let $H \subseteq G$ be a subgraph such that $\Delta(H) \geq k$ and is minimal with respect to $d(H) \geq k$. Note that as G is of average degree at least $k-!$, such a subgraph exists. If $\delta(H) \geq \frac{k}{2}$, then we are done. Otherwise, let $x \in V(H)$ such that $\deg_H(x) < \frac{k}{2}$. Then

$$
\begin{aligned}
d(H-x) &= \frac{|V(H)|d(H) - 2\deg_H(x)}{|V(H)|-1} &(2.3.1)\\
&= \frac{|V(H)|d(H) - k}{|V(H)|-1} &(2.3.2)\\
&= \frac{|V(H)|d(H) - d(H)}{|V(H)|-1} &(2.3.3)\\
&= \frac{|V(H)|-1}{|V(H)|-1}d(H) &(2.3.4)\\
&= d(H), &(2.3.5)
\end{aligned}
$$

a contradiction. Hence $\delta(H) \geq \frac{k}{2}$. □

Lemma 5

Let $s \geq 2$, $k > 12(s-1)$ and T be a tree of order at most $k+1$. Let G be a graph such that $\delta(G) \geq \dfrac{k}{2}$ and G contains no $K_{2,s}$. If there exists an inclusion $f\colon T \to G$, then at most two vertices in $V(f(T))$ can each have at least $k/2$ neighbors in $f(T)$.

PROOF. Let G and T be defined as above. If there are at least three distinct vertices in $V(T)$, say x, y, z such that $f(x)$, $f(y)$, $f(z)$ each have all of at least $k/2$ neighbors in G, then as G contains on $K_{2,s}$, the number of vertices in $V(f(T))$ is at least $\dfrac{k}{2}+\dfrac{k}{2}+\dfrac{k}{2}-3(s-1) > k+1$, a contradiction to $|V(f(T))| \leq k+1$. Hence at most two vertices in $V(f(T))$ can have at least $k/2$ neighbors in $f(T)$. □ For the rest of this chapter, we consider the following cases based on the bounds of $\Delta(T)$ and $\Delta_2(T)$.

- Case $I : \Delta(T) \leq \dfrac{k}{4}$,
- Case $II : \Delta(T) > \dfrac{k}{4}$ and $\Delta_2(T) \leq \dfrac{k}{4}+3$,
- Case $III : \Delta(T) > \dfrac{k}{4}$ and $\Delta_2(T) > \dfrac{k}{4}+2$

2.3.1 Case I

Lemma 6

Let $s \geq 2$ and $k > 12(s-1)$. Let T be a tree of order $k+1$ such that $\Delta(T) \leq \frac{k}{4}$. Let G be a graph such that $|V(G)| \geq k+1$, $\delta(G) \geq \frac{k}{2}$ and G contains no $K_{2,s}$. Then g contains a subgraph isomorphic to T.

PROOF. Let G and T be defined as above. As $\delta(G) \geq \frac{k}{2}$, by Lemma 3, G contains every tree of order $\frac{k}{2}+1$. Let $T' \subseteq T$ be a maximal subtree such that there exists an inclusion $f\colon T' \to G$. Assume without loss of generality that $T' \neq T$. Let $w \in V(T')$ such that $\deg_{T'}(w) < \deg_T(w)$. As T' is maximal, all of the $\deg_G(f(w))$ neighbors of $f(w)$ in G must be contained in $V(f(T'))$. Choose $y \in V(f(T'))$ such that $yf(w) \in E(G)$ and $\mathrm{dist}_{f(T')}(f(w), y)$ is maximal. Let C be the component of $f(T') - y$ that contains $f(w)$. By choice of y, all of the $\deg_G(f(w)) - 1$ neighbors of $f(w)$ in G except y are contained in C. Let $f(w_1) \in N_{f(T')}(y)$ such that $f(w_1)$ lies on the unique $yf(w)$ path in $f(T')$ (note that $f(w_1) \in V(C)$).

Let $C_1 = f^{-1}(C) + wf^{-1}(y)$. Then there exists an inclusion $g\colon C_1 \to G$ such that $g(u) = f(u)$ for every $u \in C_1$, $\deg_{C_1}(w_1) = \deg_C(w_1) - 1$ and $g(C_1)$ contains all of the $\deg_G(g(w)) = \deg_G(f(w))$ neighbors in G. Furthermore, $\deg_{C_1}(w) = \deg_{T'}(w) + 1$. Let $T_1 \subset T$ be a maximal subtree such that $C_1 \subset T_1$ and there exists an inclusion $g_1\colon T_1 \to G$ such that $g_1(u) = g(u)$ for every $u \in C_1$.

Suppose there exists $z_1 \in \{V(T_1) - V(C_1)\} \cup \{w_1\}$ such that $\deg_{T_1}(z_1) < \deg_T(z_1)$. If there exists a $y_1 \in N_G(g_1(z_1)) - N_{g_1(T_1)}(g_1(z_1))$, then the tree $g_1(T_1) + g_1(z_1)y_1$ is a subgraph of G containing $g_1(T_1)$ and is isomorphic to a subtree of T containing C_1

larger than T_1, contradicting the maximality of T_1. Hence every neighbor in G of $g_1(z_1)$ is contained in $V(g_1(T_1))$. Let C' be the component of $g_1(T_1 - z_1)$ that contains $g_1(w)$.

Suppose that $g_1(z_1)$ is adjacent to some vertex $a \in V(g_1(T_1)) - V(C') \subseteq V(G)$ that is not a neighbor of $g_1(z_1)$ in $g_1(T_1)$. Choose such an a to be of maximal distance in $g_1(T_1)$ from $g_1(z_1)$. Let T_2 be the subtree of $T_1 - g_1^{-1}(a)$ that contains z_1 (and hence w as well) and $g_2 : T_2 \to G$ be the inclusion induced by g_1. Note that $\deg_{T_2}(w) = \deg_{T_1}(w) = \deg_{T'}(w) + 1$ and $\deg_{T_2}(z_1) = \deg_{T_1}(z_1)$. By choice of a, every neighbor in G of $g_2(z_1) = g_1(z_1)$ is contained in $g_2(T_2)$ with the exception of a. Also, as a is not a neighbor of $g_1(w_1) = f(w_1)$ in $g_1(T_1)$, we have that every neighbor of $g_1(w_1)$ is also contained in T_2. Then the tree $g_2(T_2) + g_2(z_1)a$ is isomorphic to a subtree of T' contained in G. Let $T_3 \subseteq T$ be a maximal subtree of T such that there exists an inclusion $g_3 : T_3 \to G$ and $g_2(T_2)+g_2(z_1)a \subseteq g_3(T_3)$. Hence $\deg_{T_3}(w) = \deg_{T_1}(w) = \deg_{T'}(w)+1$ and $\deg_{T_3}(z_1) > \deg_{T_1}(z_1)$. Then every neighbor of $g_3(z_1) = g_2(z_1)$ is contained in $g_3(T_3)$ and every neighbor of $g_3(w) = g_2(w)$ is also contained in $g_3(T_3)$. By Lemma 5, we may conclude that if $x \in V(T_3)$, $w \neq x \neq z_1$, then $\deg_{T_3}(x) = \deg_T(x)$. As $\deg_{T_3}(w) > \deg_{T_1}(w)$ and $\deg_{T_3}(z_1) = \deg_{T_1}(z_1) + 1$, $T_1 \subset T_3$, contradicting our choice of T_1. Hence $g_1(z_1)$ is not adjacent to any vertex in $V(g_1(T_1)) - V(C')$ that is not a neighbor of $g_1(z_1)$ in $g_1(T_1)$. As $\deg_{T_1}(z_1) \leq \frac{k}{4} - 1$ and $g_1(z_1)$ is adjacent in $g_1(T_1)$ to exactly one vertex of C', $g_1(z_1)$ is adjacent in G to at most $\dfrac{k}{4} - 2$ vertices that are not in C'. Thus $g_1(z_1)$ is adjacent in G to at least $\dfrac{k}{4} + 2$ vertices contained in C'. Of course, we still have that every neighbor of $g_1(z_1)$ is a vertex of $g_1(T_1)$.

Suppose that $z_1 = w_1$. As $|N_G(g_1(w)) \cap N_G(g_1(w_1)| \leq s - 1$, there are at most $s - 1$ common neighbors of $g_1(w)$ and $g_1(w_1)$. Also, by choice of y, any neighbor of $g_1(w) = f(w)$ in $g_1(T_1 - z_1)$ not in C' must be a neighbor of $g_1(w_1)$ in G. As $g_1(z_1) = g_1(w_1)$ may be a neighbor of $f(w)$ not in $g_1(T_1 - z_1)$ and y is a neighbor of $g_1(z_1) = g_1(w_1)$ in G contained in C' there are at least $\frac{k}{2} - (s-1) + 1 - 1 = \frac{k}{2} - s + 1$ distinct neighbors of $g_1(w)$ in C' that are distinct from the neighbors of $g_1(z_1) = g_1(w_1)$.

Suppose now that $z_1 \neq w_1$. If z_1 is a neighbor of w_1 in T_1, then every neighbor of $g_1(w)$ is contained in C' with the possible exception of $g_1(z_1)$. Thus there are at least $\frac{k}{2} - (s-1) - 1 = \frac{k}{2} - s$ distinct neighbors of $g_1(w)$ in C' that are distinct from the neighbors of $g_1(z_1)$. If z_1 is not a neighbor of w_1 in T_1, then every neighbor of $g_1(w)$ in $g_1(T_1)$ is contained in C'. Hence there are at least $k/2 - (s-1)$ distinct neighbors of $g_1(w)$ in C' that are distinct from the neighbors of $g_1(z_1)$. In any of the these cases, we have that there are at least $\frac{k}{2} - s$ distinct neighbors of $g_1(w)$ in C' that are distinct from the neighbors of $g_1(z_1)$.

Let $z_2 \in N_{T_1}(z_1)$ such that z_2 is on the unique wz_1 path in T_1. As every neighbor of both $g_1(w)$ and $g_1(w_1)$ are vertices of $g_1(T_1)$, we have by Lemma 5 that there exists a $y_1 \in N_G(g_1(z_2)) - V(g_1(T_1))$. Note that the component C_2 of $g_1(T_1) - g_1(z_1)g_1(z_2) + g_1(z_2)y_1$ that contains $g_1(w)$ contains C' and so contains all of the at least $k/4 + 2$ neighbors of $g_1(z_1)$ that are contained in C' and all of the at least $k/2 - s$ neighbors of $g_1(w)$ in C' that are distinct from neighbors of $g_1(z_1)$.

Let $T_2 \subset T$ be a maximal subtree such that there exists an inclusion $g_2 \colon T_2 \to G$ with $C_2 \subseteq g_2(T_2)$ and $g_1(z_1) \notin V(g_2(T_2))$. Note that such T_2 and g_2 exist as C_2 is isomorphic to a subtree of T with such an inclusion. Suppose that $w_2 \in V(T_2)$ such that $\deg_{T_2}(w_2) < \deg_T(w_2)$ and $w_2 \neq w$. Then $|V(T_2)| \leq k$. If there is no $y_2 \in N_G(g_2(w_2)) - (N_{g_2(T_2)}(g_2(w_2)) \cup \{g_1(z_1)\})$ then all of the $\deg_G(g_2(w_2))$ neighbors of $g_2(w_2)$, except perhaps for one (namely $g_1(z_1)$), are in $g_2(T_2)$. As G contains no $K_{2,s}$, $|N_G(g_2(w_2)) \cap N_G(g_1(z_1))| \leq s-1$ and $|N_G(g_2(w_2)) \cap N_G(g_2(w))| \leq s-1$. The number of vertices in $g_2(T_2)$ is then at least $\frac{k}{2} - 1 + \frac{k}{4} + 2 + \frac{k}{2} - s - 2(s-1) \geq k$. We conclude that such a y_2 exists, or that $|V(T_2)| = k$ and $g_2(T_2)$ is adjacent in G to $g_1(z_1) \notin V(g_2(T_2))$. If y_2 exists, then let $T_3 = g_2(T_2) + g_2(w_2)y_2$. Then there exists an inclusion $g_3 : T_3 \to G$ such that $C_2 \subset g_3(T_3)$ but T_2 is a proper subtree of T_3, a contradiction. Thus in this case $T_2 = T$ and so $T' = T$. If y_2 does not exist, then $g_2(T_2) + g_2(w_2)g_1(z_1)$ is a subgraph of G isomorphic to T, so that $T_1 = T$, a contradiction. □

Lemma 7

Every tree T has a resolution.

PROOF. Clearly T_p exists. Assume that T_i exists. If T_i is a star, then (T_j), $i \leq j \leq p$ is a resolution of T. Otherwise, T_i is not a star and so a path P of maximal length in T_i has length at least 3. Let u and v be penultimate vertices of P, so that $u \neq v$. By choice of P, u and v are adjacent in T_i to exactly one vertex whose degree in T_i is not 1, and so are penultimate vertices of T_i. Removing the leaf neighbors of a penultimate vertex of minimal degree and denoting the resulting tree by T_{i-1}, we have that $\Delta(T_{i-1}) = \Delta(T_i) = \Delta(T)$. The result then follows by induction. □

Observe that a given tree T need not have a unique resolution. Furthermore, if $T_i \subseteq T' \subset T_{i+1}$, then there exists a unique vertex v in T' such that $\deg_{T'}(v) < \deg_{T_{i+1}}(v)$.

Lemma 8

Let $s \geq 2$, $k > 12(s-1)$ and G a graph with no $K_{2,s}$ such that $\delta(G) \geq \frac{k}{2}$. Let T be a tree of order $k+1$ such that $\Delta(T) > \frac{k}{4}$, (T_i) a resolution of T and $T_i \subseteq T' \subset T_{i+1}$ with $\deg_{T'}(u) = \Delta(T)$. Suppose there exists an inclusion $f : T' \to G$ where $\deg_G(f(u)) = k$. Let $w \in V(T')$ be such that $\deg_{T'}(w) < \deg_{T_{i+1}}(w)$. If $f(w)$ is adjacent to a vertex y in $V(f(T'))$, $y \notin N_{f(T')}(f(w))$ such that y is not on the $f(u)f(w)$ path in $f(T')$, then there exists a tree T'', $T' \subset T'' \subseteq T_{i+1}$ and an inclusion $g\colon T'' \to G$ such that $g(v) = f(v)$ for every vertex v in the unique uw path in T_i.

PROOF.

Let T be a tree that satisfies the hypothesis, with (T_i) a resolution. Let $T_i \subseteq T' \subset T_{i+1}$ be such that there is the required inclusion $f : T' \to G$. Clearly if $f(w)$ is adjacent to some vertex not in $V(f(T'))$, then a $T' \subset T'' \subseteq T_{i+1}$ exists with an appropriate embedding g. We thus assume that every neighbor of $f(w)$ in G is in $V(f(T'))$. Choose $y \in V(f(T'))$ such that $f(w)y \in E(G)$, $y \notin N_{f(T')}(f(w))$, y is not on the $f(u)f(w)$ path in $f(T')$ and y is of maximal distance from $f(w)$ in $f(T')$. Let C be the component of $f(T') - y$ that contains $f(w)$. Note that by the choice of C, C contains the $f(u)f(w)$ path in $f(T')$ and all of the $\deg_G(f(w)) - 1$ neighbors of $f(w)$ except y. Let $T_1' = f^{-1}(C) + xw$, where $x \in V(T_{i+1}) - V(T')$. Then there exists an inclusion $g_1 : T_1' \to G$ given by $g_1(v) = f(v)$ for all $v \in V(f^{-1}(C))$ and $g_1(x) = y$. Clearly then $g_1(v) = f(v)$ for every vertex v in the unique uw path in T_i as this path is contained in $f^{-1}(C)$, $g_1(T_1')$ contains every neighbor of $g_1(w) = f(w)$ in G, and $\deg_{T_1'}(w) = \deg_{T'}(w) + 1$. Let T'' be a maximal subtree of T_{i+1} such that $T_1' \subseteq T''$ and there exists an inclusion $g : T'' \to G$ such that $g(v) = g_1(v)$ for all $v \in V(T_1')$.

Suppose that $w_1 \in V(T'')$ such that $w_1 \neq w$ but $\deg_{T''}(w_1) < \deg_{T_{i+1}}(w_1)$. If $w_1 = u$ and there exists no $z \in N_G(g(u)) - N_{g(T'')}(g(u))$, then $g(u)$ and all of it's at least k neighbors in G are in $g(T'')$, so $|V(T'')| \geq k + 1$. Then $T'' = T$ so that w_1 cannot exist. Thus there exists $z \in N_G(g(u)) - N_{g(T'')}(g(u))$. Then $g(T'') + g(u)z$ certainly contains $g(T'')$ (and so $g(T_1')$) and is isomorphic to a larger subtree of T_{i+1} than T'', a contradiction. We will thus assume that $w_1 \neq u$. This then implies that $\deg_{g(T'')}(g(u)) = \deg_T(u)$.

By the maximality of T'', all the neighbors of $g(w_1)$ in G must be contained in $g(T'')$. Furthermore, as G contains no $K_{2,s}$, no vertex of G (other than $f(u) = g(u)$) can be adjacent to more than $s - 1$ neighbors of $g(u) = f(u)$ in G. Thus $g(w_1)$ and $g(w)$ are together adjacent to at most $2(s - 1)$ vertices that are neighbors of $g(u)$ in G. As $\deg_T(u) = \deg_{g(T'')}(g(u)) > \dfrac{k}{4} \geq 3(s - 1)$, there are at least s distinct neighbors of $g(u)$ in $g(T'')$ that are not neighbors of either $g(w_1)$ or $g(w)$. Now, as G contains no $K_{2,s}$, $|N_G(g(w)) \cap N_G(g(w_1))| \leq s-1$. As $\delta(G) \geq \dfrac{k}{2}$, there are at least $\frac{k}{2} + \frac{k}{2} - (s-1) + s = k+1$ vertices in $g(T'')$. However, as $\deg_{T''}(w_1) < \deg_{T_i}(w_1)$, T'' has at most k vertices, a contradiction. Hence if $w_1 \in V(T'')$ such that $\deg_{T''}(w_1) < \deg_{T_i}(w_1)$, then $w_1 = w$. As if $T_i \subseteq T'' \subset T_{i+1}$, then w is the only vertex w in T' for which it is possible that $\deg_{T''}(w) < \deg_{T_{i+1}}(w)$ and $\deg_{T''}(w) > \deg_{T'}(w)$, we have that T' is a proper subgraph of T'' and the result follows. $\square$

2.3.2 Case II

Lemma 9

Let $s \geq 2$, $k > 12(s-1)$ and T be a tree of order $k+1$ such that $\frac{k}{4}+1 \leq \Delta(T)$ and G a graph containing no $K_{2,s}$ such that $\delta(G) \geq \frac{k}{2}$ and $\Delta(G) \geq k$. Let (T_i), $1 \leq i \leq p$ be a resolution of T with $u \in V(T_1)$ such that $\deg_{T_1}(u) = \Delta(T)$. Suppose that $T_i \subseteq T' \subset T_{i+1}$ is maximal such that there exists an inclusion $f : T' \to G$. Let $w \in V(T')$ such that $\deg_{T'}(w) < \deg_T(w)$, and $w_1 \in N_{T'}(w)$ such that $f(w_1)$ is on the $f(u)f(w)$ path in $f(T')$. If $w_1 \neq u$, then there are at least two neighbors of $f(w_1)$ that are not vertices of $f(T')$.

PROOF. If $f(w)$ is adjacent in G to a vertex $y \notin V(f(T'))$, then clearly $f(T') + f(w)y$ is a subgraph of G isomorphic to a subgraph of T that properly contains T'. Hence we may assume without loss of generality that every neighbor of $f(w)$ in G is in $V(f(T'))$. Suppose at most one neighbor of $f(w_1)$ in G is not a vertex of $f(T')$. Hence at least $k/2-1$ neighbors of $f(w_1)$ are in $f(T')$. As G contains no $K_{2,s}$, $|N_G(f(w)) \cap N_G(f(u))| \leq s-1$, $|N_G(f(u)) \cap N_G(f(w_1))| \leq s-1$, and $|N_G(f(w)) \cap N_G(f(w_1))| \leq s-1$. As all of the at least $\frac{k}{2}$ neighbors of w in G are in $V(f(T'))$, the number of vertices in $f(T')$ is at least

$$\frac{k}{2}-1+\frac{k}{2}+\Delta(T)-3(s-1) \geq \frac{k}{2}-1+\frac{k}{2}+k/4+1-3(s-1) > k,$$

contradicting $|V(f(T'))| \leq k$. This establishes the result. □

Lemma 10

Let $s \geq 2$, $k > 12(s-1)$ and T be a tree of order $k+1$ such that $\Delta(T) > k/4$ and $\Delta_2(T) \leq k/4+3$. Let G be a graph such that $\delta(G) \geq \frac{k}{2}$, $\Delta(G) \geq k$ and G contains no $K_{2,s}$. Then G contains a subgraph isomorphic to T.

PROOF. Let (T_i), $1 \leq i \leq p$, be a resolution of T, with $u \in V(T_1)$ such that $\deg_{T_1}(u) = \Delta(T)$. Note that there is an inclusion $g : T_1 \to G$ such that $\deg_G(g(u)) \geq k$ as G contains a vertex of degree at least k. Let $1 \leq i \leq p$ be the largest value of i such that there exists an inclusion $g : T_i \to G$ such that $\deg_G(g(u)) \geq k$. We assume that $i < p$. Let $T_i \subseteq T' \subset T_{i+1}$ be the largest subgraph such that there exists an inclusion $f : T' \to G$ with $\deg_G(f(u)) \geq k$. Let $w \in V(T')$ such that $\deg_{T'}(w) < \deg_{T_{i+1}}(w)$. Then $w \neq u$. Furthermore, by the maximality of T', every neighbor of $f(w)$ in G must be a vertex of $f(T')$, and by maximality of T' and Lemma 8, we have that at least $k/2 - \deg_{T_{i+1}}(w) + 1$ neighbors of $f(w)$ must be on the $f(u)f(w)$-path in $f(T')$. Let $w_1 \in V(T')$ such that $f(w_1)f(w)$ is an edge on the $f(u)f(w)$-path in $f(T')$. As $\Delta_2(T) \leq k/4 + 3$, there are at least $\dfrac{k}{2} - (\dfrac{k}{4} + 3) + 1 = \dfrac{k}{4} - 2 \geq 2$ vertices on the $f(u)f(w)$-path in $f(T')$, so that $w_1 \neq u$. Hence by Lemma 9 there exists $w_1, w_2 \in V(G) - V(f(T'))$ such that $w_j f(w) \in E(G)$, $j = 1, 2$.

Note that $g_j : T_i \to G$ given by $g_j(v) = f(v)$ if $v \neq w$ and $g_j(w) = w_j$ is an inclusion with $\deg_G(g_j(u)) \geq k$, $j = 1, 2$. As T' is maximum, arguing as above we have that each w_j is adjacent in G to at least $\dfrac{k}{2} - \deg_{T_{i+1}}(w) + 1$ vertices of the $g_j(u)g_j(w)$-path in $g_j(T_j)$. As G contains no $K_{2,s}$, we have that $|N(f(w)) \cap N(g_j(w))| \leq s - 1$ for $j = 1, 2$, and $|N(g_1(w) \cap N(g_2(w))| \leq s - 1$. Furthermore, as $f(w_1) = g_j(w_1)$, $j = 1, 2$, is on the $f(w)f(u)$-path in $f(T')$ and also on the $g_j(w)f(u)$-paths in $g_j(T_i)$, $j = 1, 2$, we have that $f(w_1) \in N(f(w)) \cap N(g_1(w)) \cap N(g_2(w))$. We conclude that there are at most $3(s-2)+1$ vertices in the union of the pair-wise intersections of the neighbors of $f(w)$, $g_1(w)$, and $g_2(w)$. Hence there are at least$3(\dfrac{k}{2} - \deg_{T_{i+1}}(w) + 1) - 3(s - 2) - 1$ distinct vertices on the $f(u)f(w_1)$-path in $f(T')$. By Lemma 2, there are at most $k - \Delta(T) - \Delta_2(T) + 2$ vertices on this path. Hence

$3(\frac{k}{2}) - 3\Delta_2(T) + 3 - 3(s-2) - 1 \leq k - 2\Delta_2(T) + 2$, or, equivalently, $\frac{k}{2} - 3(s-1) + 3 \leq \Delta_2(T)$. As $k > 12(s-1)$, $\frac{k}{4} > 3(s-1)$ so that $\frac{k}{4} + 3 < \Delta_2(T)$, a contradiction which establishes the result. □

2.3.3 Case III

Lemma 11

Let $s \geq 2$, $k > 12(s-1)$ and T be a tree of order $k+1$ such that $\frac{k}{4} + 2 < \Delta_2(T)$. Let $u, v \in V(T)$ such that $\deg_T(u) = \Delta(T)$ and $\deg_T(v) = \Delta_2(T)$. Let C be the subtree of T containing the vertices u, v, their respective neighbors in T and the path joining u and v in T. Let G be a graph containing no $K_{2,s}$ such that $\Delta(G) \geq \frac{k}{2}$. Then G contains a subgraph isomorphic to T if and only if G contains a subgraph isomorphic to C.

PROOF. Clearly if G contains a subgraph isomorphic to T then G contains a subgraph isomorphic to C as $C \subseteq T$.

Conversely let T be a tree that satisfies the hypothesis. Let T' be the largest subtree of T that contains C such that there exists an inclusion $f : T' \to G$. We will show that $T' = T$.

Suppose $T' \neq T$. Let $w \in V(T')$ such that $\deg_{T'}(w) < \deg_T(w)$. Note that $u \neq w \neq v$ as $C \subseteq T'$. By the maximality of T', all of the at least $k/2$ neighbors of $f(w)$ in G are in $V(f(T'))$. We now show that $f(w)$ is adjacent in G to a leaf y in $V(f(T'))$ that is not adjacent to $f(w)$ in $f(T')$. By Lemma 1 there are at least

$\Delta(T) - 2 + \Delta_2(T) - 2 + \deg_{f(T')}(f(w)) - 2 + 2 \geq$
$\frac{k}{4} + 1 + \frac{k}{4} + 1 + \deg_{f(T')}(f(w)) = \frac{k}{4} + 2 + \deg_{f(T')}(f(w))$
leaves in $f(T')$. If none of the $(\deg_G(f(w)) - \deg_{f(T')}(f(w)))$ neighbors of $f(w)$ in G that are not neighbors of $f(w)$ in $f(T')$ are leaves of $f(T')$, then the number of vertices in $f(T')$ is at least $(\deg_G(f(w)) - \deg_{f(T')}(f(w))) + \frac{k}{2} + 2 + \deg_{f(T')}(f(w))) > k+1$, contradicting the order of $f(T')$. Hence there exists a $y \in V(f(T'))$ such that y is a leaf in $f(T')$ but $y \notin N_{f(T')}(f(w))$.

Let $z \in f(T')$ such that $zy \in E(f(T'))$. Note that as $y \notin N_{f(T')}(f(w))$, we have that $z \neq f(w)$. If z is adjacent in G to some vertex $x \notin V(f(T'))$, then $T'' = f(T') - zy + zx + f(w)y$ is isomorphic to a subtree of T larger than T' that contains C, which is not possible. Hence every neighbor of z in G is a vertex of $f(T')$. Observe that there exists $v' \in \{u, v\}$ such that $z \neq f(v')$. As $u \neq w \neq v$, $f(v') \neq f(w)$. As G contains no $K_{2,s}$, at most $s-1$ neighbors of z can be neighbors of $f(v')$ and at most $s-1$ neighbors of $f(w)$ can be neighbors of $f(v')$. As $\deg_{T'}(v') \geq \Delta_2(T) \geq \frac{k}{4} + 3$, there are at least $\frac{k}{4} + 3 - 2(s-1)$ neighbors of $f(v')$ in $f(T')$ that are not adjacent to z or $f(w)$. As G contains no $K_{2,s}$, we have that z and $f(w)$ are adjacent to at least $\frac{k}{2} + \frac{k}{2} - (s-1) = k - (s-1)$ vertices of $f(T')$. Hence T' contains at least
$k - (s-1) + \frac{k}{4} + 3 - 2(s-1) = k + 3 + \frac{k}{4} - 3(s-1) \geq k+3$ vertices, a contradiction to the order of T. Thus $T = T'$ and the result follows. □

2.3.4 Main Results

Combining Lemmas 6, 10 and 11 we have the following result.

Theorem 5

Let $s \geq 2$, $k > 12(s-1)$ and G a graph such that $\delta(G) \geq k/2$, $\Delta(G) \geq k$, and G contains no $K_{2,s}$. Then G contains every tree T of order $k+1$ if and only if G contains every $C_{r,s,t}$, where $\frac{k}{4} + 4 \leq r \leq s$ and $r + s + t \leq k$.

The following result proved by Perles [31] shows that every caterpillar $C_{r,s,t}$ is contained in every graph G satisfying the hypothesis of the Erdős-Sós conjecture (Note that every $C_{r,s,t}$ is a subgraph of some caterpillar of order $k+1$.).

Lemma 12

Let G be a graph of average degree at least $k-1$ and C a caterpillar of order $k+1$. Then G contains a subgraph isomorphic to C.

Combining the preceding two results with Lemma 4, we obtain the following partial solution to the Erdős-Sós conjecture.

Corollary 1

Let $s \geq 2$ and $k > 12(s-1)$. Every graph G with average degree larger than $k-1$ that contains no $K_{2,s}$ contains every tree of order $k+1$.

CHAPTER 3

THE CAYLEY ISOMORPHISM PROBLEM

3.1 Introduction

In 1967, Ádám [1] conjectured that :

for a given n, any two circulant graphs of order n are isomorphic if and only if they are isomorphic by a group automorphism of $\mathbb{Z}_n$.

See Definition 30 for the definition of a circulant graph. It is not difficult to show (see the argument after Definition 29) that the image of a circulant graph under a group automorphism of $\mathbb{Z}_n$ is also a circulant graph, so to check whether or not two circulant graphs are isomorphic, one must at the very least check whether a group automorphism of $\mathbb{Z}_n$ is an isomorphism. Thus Ádám was conjecturing that the list of permutations that need to be checked to test for isomorphism for circulant graphs is as small as possible. This was quickly proved to be false by Elspas and Turner [19] in 1970 who gave a counterexample of order 8. Previously, however, Turner [40] had shown that Ádám's Conjecture is true if n is prime, and also showed that a vertex-transitive graph of prime order is isomorphic to a circulant graph. We remark that Turner was not motivated by Ádám's conjecture (it is not mentioned in his paper), but was motivated by an enumeration problem posed by F. Harary [3]. Muzychuck [32] and [33] showed that if n is square free, then $\mathbb{Z}_n$ and $\mathbb{Z}_{2n}$ have the property that *any two Cayley graphs of $\mathbb{Z}_n$ are isomorphic if and only if they are* *isomorphic by a group automorphism of $\mathbb{Z}_n$.* The only other values of n for which the above property is true are 8 and 9 [4]. Ádám's conjecture was quickly generalized to the following problem:

For which finite groups G is it true that any two Cayley graphs of G are isomorphic if and only if they are isomorphic by a group automorphism of G?

A group G with the above property is called a *CI-group* with respect to graphs. As with circulant graphs, the image of a Cayley graph of G under an automorphism of G is also a Cayley graph of G (also, see the argument after Definition 29), so a group G is a CI-group with respect to graphs only if the list of permutations which must be checked to determine isomorphism is as small as possible. In recent years there has been considerable interest in which groups are CI-groups with respect to graphs. The following groups are known to be CI-groups with respect to graphs:

- $\mathbb{Z}_p$ (Turner [40]), $\mathbb{Z}_p^2$ (Godsil [22]), $\mathbb{Z}_p^3$ (Dobson [14] and independently Xu [43]), $\mathbb{Z}_p^4$ (Hirasaka and Muzychuk [24] and independently Morris [30]), p a prime.
- $\mathbb{Z}_{pq}$ (Alspach and Parsons [4]), $\mathbb{Z}_n$, $\mathbb{Z}_{2n}$, and $\mathbb{Z}_{4n}$, n odd and square-free (Muzychuk [32, 33])
- $\mathbb{Z}_p \times \mathbb{Z}_q^2$, q and p distinct primes (Dobson and independently by Kovács and Muzychuk - both proofs are unpublished at this time)
- the dihedral group of order $2p$, p a prime (Babai [7])
- $\langle a, b : a^4 = b^p = 1, a^{-1}ba = b^{-1} \rangle \cong \mathbb{Z}_4 \ltimes \mathbb{Z}_p$ (Li, Lu, and Pálfy [26]).
- $\mathbb{Z}_{p^2}^2 \times \mathbb{Z}_n$ is a CI-group with respect to graphs if $\gcd(np, \phi(np)) = 1$ where ϕ is the Euler phi function $q_1, \ldots, q_s$ are distinct primes and p a prime such that $p^2 < q_1$, $p^2 q_1 q_2 \ldots q_i < q_{i+1}$ for $1 \leq i \leq s-1$ (Dobson [18])
- $\mathbb{Z}_2^3 \times \mathbb{Z}_p$, p a prime, (Dobson, unpublished)

Additionally, many constructions have been obtained showing that certain groups are not CI-groups with respect to graphs. These results are summarized in the following result (see

[25]). Note that this result is stated for digraphs. This is usual, as although Ádám posed his original conjecture for graphs, the problem is usually now considered for digraphs, or more generally still, for color digraphs (digraphs in which the edge set has been partitioned into different colors).

Theorem 6

If G is a CI-group with respect to digraphs, then all Sylow p-subgroups of G are elementary abelian of order at most p^{4p-3} or isomorphic to $\mathbb{Z}_4$ or Q_8. Moreover, $G = U \times V$, where $\gcd(|U|, |V|) = 1$, U is abelian of odd order, and V is one of the following:

1. $\mathbb{Z}_2^d$, $D(M, 2)$, $D(M, 4)$ where M is abelian of odd order;
2. $\mathbb{Z}_4$, Q_8, A_4, $\mathbb{Z}_3 \ltimes Q_8$, $\mathbb{Z}_3^2 \ltimes Q_8$.

At the present time, it seems unrealistic to expect to determine which groups on the above list of possible CI-groups are indeed CI-groups. However, Ádám's original conjecture was concerned with determining necessary and sufficient conditions for two circulant digraphs to be isomorphic. One could then ask for a shortest possible list of isomorphisms to check to determine whether or not two Cayley digraphs of a non-CI-group are isomorphic (and as the group is not a CI-group with respect to digraphs, some element of every shortest possible list must be a non-isomorphism of the group). This problem has been considered for some groups, and in fact Muzychuk [34] has determined necessary and sufficient conditions for two circulant digraphs of order n to be isomorphic. The only other groups which have been considered in this context are the nonabelian groups of order pq, p and q distinct primes, and $\mathbb{Z}_p \times \mathbb{Z}_{p^2}$. The results on the former groups hold for graphs which are not Cayley digraphs, but hold for metacirculant digraphs (see Definition 31) of order

pq. As Cayley digraphs of G are characterized by their automorphism group containing a transitive group isomorphic to G (see the comment after Definition 29), metacirculant graphs are characterized by their automorphism group containing an appropriate transitive subgroup, as shown by Alspach and Parsons [5].

Theorem 7

Let m and n be positive integers with $\alpha \in \mathbb{Z}_n^*$. Define $\rho, \tau : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_n$ by $\rho(i,j) = (i, j+1)$ and $\tau(i,j) = (i+1, \alpha j)$. Then a digraph Γ is an (m,n)-metacirculant digraph if and only if $\langle \rho, \tau \rangle \leq \mathrm{Aut}(\Gamma)$.

We remark that $\langle \rho, \tau \rangle \cong \mathbb{Z}_m \ltimes \mathbb{Z}_n$. Henceforth, we will refer to $\langle \rho, \tau \rangle$ as an (m,n)-*metcyclic group*. Note that by raising τ to an appropriate power relatively prime to m, we may assume without loss of generality that if r divides $|\alpha|$ is a prime, then $r|m$.

For p and q distinct primes, every Cayley digraph of order pq is a metacirculant digraph - this follows as every group of order pq is either cyclic or $\mathbb{Z}_q \ltimes \mathbb{Z}_p$.

Alspach and Parsons [5] also determined sufficient conditions to ensure that an (m,n)-metacirculant digraph is a Cayley digraph.

Theorem 8

Let Γ be a (m,n)-metacirculant digraph with $a = |\alpha|$, and let $c = a/\gcd(a,m)$. If $\gcd(c,m) = 1$, then Γ is a Cayley digraph for the group $\langle \rho, \tau^c \rangle$. Furthermore this group is abelian if $\gcd(a,m) = 1$ and is cyclic if $\gcd(a,m) = 1 = \gcd(m,n)$.

In order to discuss the generalization of the Cayley isomorphism problem to non-Cayley digraphs, we need to discuss transitive groups. So, let G be transitive on a set

X, and let $H = \mathrm{Stab}_G(x) = \{g \in G : g(x) = x\}$. Any transitive group can be though of as the action of G on the left cosets of H (see for example the comments following [13, Lemma 1.6B]). We use the pair (G, H) where $H \leq G$ to denote the permutation group obtained by letting G act by multiplication on the set G/H of left cosets of H. If G acts faithfully by multiplication on G/H, and $\lambda \in \mathrm{Aut}(G)$ such that $\lambda(H) = H$, then $\lambda(gH) \to \lambda(g)H$ is a well-defined permutation of G/H (by Lemma 15), which we denote by $\bar{\lambda}$. We say that Γ is a ***CI-color digraph of*** (G, H) if whenever Γ' is a color digraph with $(G, H) \leq \mathrm{Aut}(X)$, then $\Gamma \cong \Gamma'$ if and only if they are isomorphic by $\bar{\alpha}$ with $\alpha \in \mathrm{Aut}(G)$. Note that if $H = 1$, then this definition agrees with our previous definition of a CI-group with respect to color digraphs.

In [16] (see also [15]) Dobson proved the following result.

Theorem 9

Let Γ be a (q, p)-metacirculant color digraph. Let $|\alpha| = q^a$. Then either Γ is isomorphic to a circulant color digraph of order qp or Γ is a $(\langle \rho, \tau \rangle, \langle \tau^q \rangle)$-CI-color digraph.

In the remainder of this dissertation, we will prove a result analogous to the previous result for (pq, r)-metacirculant color digraphs where $p < q < r$ and $pq|\ |\alpha|$, namely Theorem 11.

3.2 Preliminaries

We begin the chapter with some basic definitions. For terminology not defined here, see [13]

Definition 15 A *group* $G = (G, *)$ is a nonempty set G together with a binary operation $*$ such that

1. (Closure) for $a, b \in G$, $a * b \in G$,
2. (Associativity) for $a, b, c \in G$, $a * (b * c) = (a * b) * c$,
3. (Identity) there is an $e \in G$ such that for every $a \in G$, $a * e = e * a = a$,
4. (Inverse) there is an $a^{-1} \in G$ such that for every $a \in G$, $a * a^{-1} = a^{-1} * a = e$.

For the rest of this chapter, the group $(G, *)$ will be denoted by G, not to be confused with a graph G as defined in the earlier chapters. For all our purposes here, Γ will denote a graph. For all a, $b \in G$, we will denote $a * b$ by ab.

Definition 16 A group G is *abelian* if $ab = ba$ for all $a, b \in G$.

Definition 17 Let $G = (G, *)$ be a group. A non empty subset $H \subseteq G$ is a *subgroup* of G if H is a group under the same binary binary operation $*$.

Definition 18 Let G be a group. The *center* of G (denoted by $Z(G)$) is defined as $Z(G) = \{g \in G\colon gh = hg$ for all $h \in G\}$.

Definition 19 Let G be a group. Let $S \subseteq G$ be a nonempty set. The *centralizer* $C(S)$ of S is defined as $C(S) = \{g \in G\colon gs = sg$ for all $s \in S\}$.

A subgroup $H \leq G$ is *self centralizing* if $C(H) = H$. Note that any abelian group G is self centralizing.

Definition 20 Let G be a group. Let $S \subseteq G$ be a nonempty set. The *normalizer* $N(S)$ of S is defined as $N(S) = \{g \in G\colon gS = Sg\}$.

Definition 21 A *permutation* of a set M is a one-to-one function from M onto M. The set of permutations of M together with the operation composition is a group denoted by $\mathbb{S}ym(M)$.

Definition 22 Let G, H be groups. A map $\phi\colon G \to H$ is a *homomorphism* if for all $g, h \in G$, $\phi(gh) = \phi(g)\phi(h)$. A homomorphism is *injective* if $\phi(g) = \phi(h)$ implies $g = h$, for all g, $h \in G$.

Definition 23 An *automorphism* of a color digraph Γ is a permutation σ of $V(\Gamma)$ which has the property that $uv \in E(\Gamma)$ if and only if $\sigma(u)\sigma(v) \in E(\Gamma)$, and if $C_1, \ldots, C_r$ are the color classes of Γ, then σ permutes the set $\{C_1, \ldots, C_r\}$.

Definition 24 Let G be a permutation group on Ω. If for every $\omega_1, \omega_2 \in \Omega$ there exists $g \in G$ such that $g(\omega_1) = \omega_2$, then G is *transitive* on Ω.

Definition 25 Let G be a transitive permutation group on Ω (that is $G \leq \mathbb{S}ym(\Omega)$). We say that $\Delta \subseteq \Omega$ is a *fixed block* of G if $g(\Delta) = \Delta$ or $g(\Delta) \cap \Delta = \phi$ for all $g \in G$. If Δ is a block of G, then $g(\Delta)$ is also a block of G, and $\{g(\Delta) : g \in G\}$ is a partition of Ω, called a *complete block system of* G. Note that if $\mathcal{B}$ is a complete block system of G, then there is an induced action of G on $\mathcal{B}$. That is, if $g \in G$, we define $g/\mathcal{B} : \mathcal{B} \to \mathcal{B}$ by $g/\mathcal{B}(B_1) = B_2$ if and only if $g(B_1) = B_2$. We set $G/\mathcal{B} = \{g/\mathcal{B} : g \in G\}$.

Definition 26 A permutation group G acting on Ω is *primitive* if the only blocks of G are ϕ and Ω (called the trivial blocks of Ω). If G has non-trivial blocks then G is *imprimitive*

Definition 27 If $\mathcal{B}$ and $\mathcal{C}$ are complete block systems of G with $B \subseteq C$ for every $B \in \mathcal{B}$ and some $C \in \mathcal{C}$, then we write $\mathcal{B} \preceq \mathcal{C}$. If $\mathcal{B} \preceq \mathcal{C}$, then $\mathcal{C}$ induces a complete block system of $G/\mathcal{B}$, denoted $\mathcal{C}/\mathcal{B}$, where a block of $\mathcal{C}/\mathcal{B}$ consists of those blocks of $\mathcal{B}$ whose union is a block of $\mathcal{C}$.

Definition 28 A graph Γ is *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. That is, given any two vertices $u, v \in V(\Gamma)$, there exists a $\sigma \in \mathrm{Aut}(\Gamma)$ such that $\sigma(u) = v$.

Definition 29 Let G be a group with identity 1 and $S \subset G$ with the properties that $x \in S$ implies that $x^{-1} \in S$ and $1 \notin S$. Then the *Cayley graph* $\Gamma = \Gamma(G, S)$ is a simple graph with $V(\Gamma) = G$ and $E(\Gamma) = \{(g, h) \colon g^{-1}h \in S\}$. We say that S is the *connection set* of Γ. Cayley digraphs are defined similarly, except we do not require that $x^{-1} \in S$ if $x \in S$.

Let Γ be a Cayley digraph of G, and $\alpha \in \mathrm{Aut}(G)$. Then $V(\alpha(\Gamma)) = G$ and $\alpha(\Gamma)$ is a Cayley digraph of G with connection set $\alpha(G)$ as $(g', h') \in E(\alpha(\Gamma))$ if and only if $(g', h') = (\alpha(g), \alpha(h))$ for some $g, h \in G$ if and only if $g^{-1}h \in S$ if and only if $\alpha(g^{-1}h) \in \alpha(S)$ if and only if $\alpha(g)^{-1}\alpha(h) \in \alpha(S)$.

Also observe that the function $g_L : G \to G$ given by $g_L(x) = gx$ is an automorphism of a Cayley digraph Γ of G with connection set S as $(a, b) \in E(\Gamma)$ if and only if $a^{-1}b \in S$ if and only if $a^{-1}g^{-1}gb \in S$ if and only if $(ga, gb) \in E(\Gamma)$ if and only if $g_L \in \mathrm{Aut}(\Gamma)$. We set $G_L = \{g_L : g \in G\}$ so that $G_L \leq \mathrm{Aut}(\Gamma)$, and remark that G_L is transitive on G and is the *left-regular representation of* G. Finally, we remark that Sabidussi [36] has shown that the converse of this observation holds. That is, a vertex-transitive digraph Γ is isomorphic to a Cayley digraph of G if and only if G contains a transitive subgroup isomorphic to G.

Definition 30 A *circulant* digraph of order n is a Cayley graph of $\mathbb{Z}_n$, the cyclic group of order n.

Definition 31 Let m, n be positive integers and set $\mu = \lfloor m/2 \rfloor$. Let $\mathbb{Z}_n^*$ be the units of $\mathbb{Z}_n$. Let $V = V(\Gamma) = \{v^i{}_j : i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$ and set $\alpha \in \mathbb{Z}_n^*$. Let $S_0, \ldots, S_\mu$ be subsets of $\mathbb{Z}_n$ satisfying :

1. $0 \notin S_0 = -S_0$,
2. $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$,
3. if m is even then $\alpha^\mu S_\mu = -S_\mu$.

The *metacirculant* graph $\Gamma = \Gamma(m, n, \alpha, S_0, \ldots, S_\mu)$ is the graph with vertex set V and edge set E. We will also refer to Γ as an (m, n)-metacirculant.

Definition 32 Let Ω be a set and $G \leq S_\Omega$ be transitive. The 2-closure of G, denoted by $G^{(2)}$, is defined to be the largest subgroup of S_Ω that has the same orbits as G, acting on $\Omega \times \Omega$. A group G such that $G^{(2)} = G$ is said to be 2-*closed.*

There are also two equivalent definitions of the 2-closure of G, which we now discuss. Let $\mathcal{O}_0, \ldots, \mathcal{O}_t$ the be orbits of G acting on $\Omega \times \Omega$, and assume that $\mathcal{O}_0$ is the trivial orbit $\{(\omega, \omega) : \omega \in \Omega\}$. Define digraphs $\Gamma_1, \ldots, \Gamma_t$ by $V(\Gamma_i) = \Omega$ and $E(\Gamma_i) = \mathcal{O}_i$. The Γ_i are the *orbital digraphs of* G, and $G^{(2)} = \cap_{i=1}^t \mathrm{Aut}(\Gamma_i)$. One may also color in the complete (di)graph the edges of the orbital digraph Γ_i with color i, and adapt the convention that an automorphism must not only map edges to edges but edges of the same color to other edges of the same color. Call the resulting color digraph Γ. Then $\mathrm{Aut}(\Gamma) = \cap_{i=1}^t \mathrm{Aut}(\Gamma_i)$, and we see that a 2-closed group is simply the automorphism group of a color digraph.

Definition 33 Let G be a transitive permutation group that admits a complete block system $\mathcal{B}$ of m blocks of size p, p a prime, where $\mathcal{B}$ is formed by the orbits of some normal subgroup $N \triangleleft G$. Then for each $B \in \mathcal{B}$ there exists $\alpha_B \in N$ such that $\alpha_B|_B$ is a p-cycle. Define an equivalence relation $\equiv$ on the blocks of $\mathcal{B}$ by $B \equiv B'$ if and only if whenever $\alpha \in N$ then $\alpha|_B$ is a p-cycle if and only if $\alpha|_{B'}$ is also a p-cycle. Denote the equivalence classes of $\equiv$ by $C_0, \ldots, C_a$ and let $E_i = \cup_{B \in C_i} B$.

Definition 34 Let G be a group and $H \subseteq G$, a nonempty subset of G. Let $\approx$ be an equivalence relation on H. If $\alpha \approx \beta$ if and only if $g(\alpha) \approx g(\beta)$ for all $\alpha, \beta \in H$ and for all $g \in G$, then $\approx$ is a *G-congruence* on H.

Definition 35 For p a prime, we define the group $\mathrm{AGL}(1, p)$ to be the group of all functions $f : \mathbb{Z}_p \to \mathbb{Z}_p$ such that $f(x) = ax + b$, $a \in \mathbb{Z}_p^*$, $b \in \mathbb{Z}_p$.

Definition 36 Let $G \leq S_X$ and $H \leq S_Y$. We define the *wreath product of G and H*, denoted $G \wr H$ to be the group of all permutations on $X \times Y$ of the form $(x, y) \to (g(x), h_x(y))$, where $g \in G$, and $h_x \in H$. Note that $|G \wr H| = |G| \cdot |H|^{|X|}$.

Definition 37 Let Γ_1 and Γ_2 be digraphs. Define the *wreath product of Γ_1 and Γ_2*, denoted $\Gamma_1 \wr \Gamma_2$, to be the digraph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and

$$E(\Gamma_1 \wr \Gamma_2) = \{\overrightarrow{((x_1, y_1), (x_2, y_2))} : x_1 = x_2 \text{ and } \overrightarrow{y_1 y_2} \in E(\Gamma_2) \text{ or } \overrightarrow{x_1 x_2} \in E(\Gamma_1)\}.$$

We note that $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \leq \mathrm{Aut}(\Gamma_1 \wr \Gamma_2)$, although equality need not hold. We also observe that $\mathrm{Aut}(\Gamma_1) \times \mathrm{Aut}(\Gamma_2)$ with the canonical action is contained in $\mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2) \leq \mathrm{Aut}(\Gamma_1 \wr \Gamma_2)$.

3.3 The Cayley Isomorphism Problem

To begin with, we will first establish all results that will be needed for Theorem 11 that do not depend upon the number of points being permuted being pqr. Additionally, most of the result in this section are already known.

Lemma 13

Let $\rho \in S_{\mathbb{Z}_m \times \mathbb{Z}_n}$ be defined as $\rho(i,j) = (i, j+1)$. Then the centralizer of $\langle\rho\rangle$ in $S_{\mathbb{Z}_m \times \mathbb{Z}_n}$ is $S_m \wr \mathbb{Z}_n$.

PROOF. Let $Z = C_{\mathbb{Z}_m \times \mathbb{Z}_n}(\langle\rho\rangle)$. Then $\langle\rho\rangle \triangleleft Z$ and if $\delta(i,j) = (\sigma(i), j + b_i)$, $\sigma \in S_m$, $b_i \in \mathbb{Z}_n$, then $\delta\rho(i,j) = \delta(i, j+1) = (\sigma(i), j+1+b_1) = \rho(\sigma(i), j+b_1) = \rho\delta(i,j)$, we see that $S_m \wr \mathbb{Z}_n \leq Z$. Thus Z is transitive and so the orbits of $\langle\rho\rangle$ form a complete block system $\mathcal{B}$ of Z. Then $\mathrm{Stab}_Z(B)|_B$ centralizes $\langle\rho\rangle|_B$, and as $\langle\rho\rangle|_B$ is a transitive abelian group and transitive abelian groups are self centralizing[], $\mathrm{Stab}_Z(B)|_B \leq \mathbb{Z}_n$. We then conclude by the Embedding Theorem [29, Theorem 2.6] that $Z \leq S_m \wr \mathbb{Z}_n$, and the result follows. □

The main tool generally used to show that a group G is a CI-group with respect to digraphs is the following result of Babai [7]. A weaker version of this result (which only holds for circulant graphs) can be found in [4].

Lemma 14

For a Cayley color digraph Γ of G, the following are equivalent,

1. Γ is a CI-color digraph of G,
2. Given a permutation $\gamma \in S_G$ such that $\gamma^{-1}G_L\gamma \leq \mathrm{Aut}(\Gamma)$, G_L and $\gamma^{-1}G_L\gamma$ are conjugate in $\mathrm{Aut}(\Gamma)$.

We will be considering the isomorphism problem for some color digraphs which are not Cayley color digraphs, so we first need to generalize Babai's result to this situation. We begin with some group theoretic lemmas that are probably known to many readers. The proofs are given for completeness.

Lemma 15

If G acts faithfully by multiplication on G/H, and $\lambda \in \mathrm{Aut}(G)$ such that $\lambda(H) = H$, then $\lambda(gH) \to \lambda(g)H$ is a well-defined permutation of G/H.

PROOF. To show that $\lambda(gH) \to \lambda(g)H$ is well-defined, suppose $g_1, g_2 \in G$ and that $g_1H = g_2H$. Then $\lambda(g_1H) = \lambda(g_2H)$, and, as $\lambda(H) = H$, $\lambda(g_iH) = \lambda(g_i)H$ for $i = 1, 2$. Hence $\lambda(g_1)H = \lambda(g_1H) = \lambda(g_2H) = \lambda(g_2)H$, and the map $\lambda(gH) \to \lambda(g)H$ is well-defined. That $\lambda(gH) \to \lambda(g)H$ is onto follows from the fact that λ is an automorphism of G and so is onto. Finally, if $\lambda(g_1)H = \lambda(g_2)H$, then $\lambda(g_2^{-1}g_1)H = H$ for all $g \in G$. As G acts faithfully by multiplication on G/H, we must then have that $\lambda(g_2^{-1}g_1) \in H$, so that $g_1 = g_2h$ for some $h \in H$. Thus $g_1H = g_2H$ and $\lambda(gH) \to \lambda(g)H$ is one-to-one, and so a permutation of G/H. □

Notation 1 For G, G/H, and λ as in Lemma 15, we denote the permutation of G/H given by $\lambda(gH) \to \lambda(g)H$ by $\bar{\lambda}$.

Lemma 16

Let G be a group acting by multiplication on the set G/H of left cosets of $H \leq G$, and $\bar{A} = \{\bar{a} : a \in \mathrm{Aut}(G) \text{ and } a(H) = H\}$. Then $N_{S_{G/H}}((G, H)) = \bar{A} \cdot (G, H)$.

PROOF. Let $\delta \in N_{S_{G/H}}((G,H))$. As (G,H) is transitive, there exists $\hat{g} \in (G,H)$ such that $\delta\hat{g}(H) = H$. Let $\beta = \delta\hat{g}$. Define $\lambda : G \to G$ by $\lambda(g) = k$ if and only if $\beta^{-1}g\beta = k$. As $\beta^{-1}H\beta = H$, $\lambda(H) = H$. As $\beta^{-1}gh\beta = \beta^{-1}g\beta \cdot \beta^{-1}h\beta$, we have that $\lambda(gh) = \lambda(g)\lambda(h)$, so that λ is a homomorphism. Clearly λ is one-to-one and onto, so that λ is an automorphism of G. Note that $\beta^{-1}g\beta = \lambda(g)$ for all $g \in G$ and that for $x, g \in G$, $\bar{\lambda}^{-1}g\bar{\lambda}(xH) = \bar{\lambda}^{-1}(g\lambda(x)H) = \lambda^{-1}(g)(xH)$, so that $\bar{\lambda}^{-1}g\bar{\lambda} = \lambda^{-1}(g)$. Thus $\beta^{-1}\bar{\lambda}^{-1}g\bar{\lambda}\beta = \beta^{-1}\lambda^{-1}(g)\beta = g$ so that $\bar{\lambda}\beta$ commutes with g for every $g \in G$. Thus $\bar{\lambda}\beta$ centralizes G so by [13, Lemma 4.2A], we have that $\bar{\lambda}\beta$ is contained in the permutation group K obtained by letting $N_G(H)$ act by left multiplication on H, and is semiregular. Finally, as $\beta(H) = \delta\hat{g}(H) = H$ and $\lambda(H) = H$, we have that $\bar{\lambda}\beta(H) = H$. As $\bar{\lambda}\beta \in K$ and K is semiregular, we must then have that $\bar{\lambda}\beta = 1$ so that $\beta = \bar{\lambda}^{-1}$. Whence $\delta = \beta\hat{g}^{-1} = \bar{\lambda}^{-1}\hat{g}^{-1}$ and the result follows. □

Tan and Tyskevitch [39] generalized Babai's Lemma 14 in the following fashion (a version of the following result for hypergraphs can be found in [16]):

Lemma 17

Let Γ be a vertex-transitive color digraph in $\mathcal{K}$ with $(G,H) \leq \mathrm{Aut}(\Gamma)$. Then the following are equivalent:

1. if Γ_1 is any color digraph in $\mathcal{K}$ with $(G,H) \leq \mathrm{Aut}(\Gamma)$, then $\Gamma \cong \Gamma_1$ if and only if they are isomorphic by a group automorphism $\alpha \in \mathrm{Aut}(G)$ that fixes H and acts on the left cosets of H,
2. whenever $\gamma \in S_{G/H}$ such that $\gamma^{-1}(G,H)\gamma \leq \mathrm{Aut}(\Gamma)$, then (G,H) and $\gamma^{-1}(G,H)\gamma$ are conjugate in $\mathrm{Aut}(\Gamma)$.

PROOF. Let Γ_1 be a color digraph in $\mathcal{K}$ with $(G,H) \leq \mathrm{Aut}(\Gamma_1)$ isomorphic to Γ with $\gamma : \Gamma \to \Gamma_1$ an isomorphism - so $\gamma(\Gamma) = \Gamma_1$. As $(G,H) \leq \mathrm{Aut}(\Gamma_1)$, we have that

$\gamma^{-1}(G,H)\gamma \leq \mathrm{Aut}(\Gamma)$ (note that $\mathrm{Aut}(\Gamma) = \gamma^{-1}\mathrm{Aut}(\Gamma_1)\gamma$). By hypothesis, there exists $\delta \in \mathrm{Aut}(\Gamma)$ such that $\delta^{-1}\gamma^{-1}(G,H)\gamma\delta = (G,H)$. By Lemma 16, we then have that $\gamma\delta \in \bar{A}\cdot(G,H)$, so that $\gamma\delta = \bar{\alpha}g$, $\alpha \in \mathrm{Aut}(G)$, $g \in (G,H)$. Thus $\bar{\alpha} = \gamma\delta g^{-1}$, and $\gamma\delta g^{-1}(\Gamma) = \gamma(\Gamma)$ (as both δ and $g^{-1} \in \mathrm{Aut}(\Gamma)$). Thus (2) implies (1).

Conversely, suppose that if Γ_1 is any color digraph in $\mathcal{K}$ with $(G,H) \leq \mathrm{Aut}(\Gamma)$, then $\Gamma \cong \Gamma_1$ if and only if they are isomorphic by $\bar{\alpha}$, $\alpha \in \mathrm{Aut}(G)$. Let $\gamma \in S_{(G,H)}$ such that $\gamma^{-1}(G,H)\gamma \leq \mathrm{Aut}(X)$. Note then that $\mathrm{Aut}(\gamma(X)) = \gamma\mathrm{Aut}(X)\gamma^{-1}$ as $\gamma^{-1}:\gamma(\Gamma)\to\Gamma$ is an isomorphism. Hence $(G,H) \leq \mathrm{Aut}(\gamma(\Gamma))$. Then there exists $\alpha \in \mathrm{Aut}(G)$ such that $\bar{\alpha}(\Gamma) = \gamma(\Gamma)$. Then $\bar{\alpha}^{-1}\gamma(\Gamma) = \Gamma$ so that $\bar{\alpha}^{-1}\gamma \in \mathrm{Aut}(\Gamma)$. Then

$$(\bar{\alpha}^{-1}\gamma)\gamma^{-1}(G,H)\gamma(\bar{\alpha}^{-1}\gamma)^{-1} = \alpha^{-1}(G,H)\alpha = (G,H)$$

as $\bar{\alpha}$ normalizes (G,H) by Lemma 16. Thus (1) implies (2). □

Definition 38 Let Γ be a color digraph with $(G,H) \leq \mathrm{Aut}(X)$ a transitive group. If Γ' is a color digraph with $(G,H) \leq \mathrm{Aut}(X)$, then $\Gamma \cong \Gamma'$ if and only if they are isomorphic by a group automorphism $\alpha \in \mathrm{Aut}(G)$ that fixes H and acts on the left cosets of H, then we say that X is a *CI-color digraph* of (G,H).

The following result is [14, Lemma 3].

Lemma 18

Let G be as in Definition 33, and $\alpha \in N$ be such that $|\alpha| = p$. Then for each $0 \leq i \leq a$ there exists $\alpha_i \in G^{(2)}$ such that $\alpha_i|_{E_i} = \alpha|_{E_i}$ and $\alpha_i|_{E_j} = 1$ for every $i \neq j$. Furthermore, each E_i is a block of G.

PROOF. In order to show that each E_i is a block of G, it suffices to show that $\{C_i : 0 \leq i \leq a\}$ is a complete block system of $G/\mathcal{B}$, so by [13, Exercise 1.5.4] we need only show that $\equiv$ is a $G/\mathcal{B}$-congruence. Let $B, B' \in \mathcal{B}$ such that $B \equiv B'$. Let $g \in G$. If $g(B) \not\equiv g(B')$, then there exists $\alpha \in \text{fix}_G(\mathcal{B})$ such that, say, $\alpha|_{g(B)}$ is a p-cycle while $\alpha|_{g(B')}$ is not. Then $g^{-1}\alpha g|_B$ is a p-cycle while $g^{-1}\alpha g|_{B'}$ is not, a contradiction.

Let $e = \vec{xy} \in E(\Gamma)$. If $x, y \in E_j$ for some $0 \leq j \leq a$, then clearly $\alpha_i(e) \in E(\Gamma)$ as then either $\alpha_i(e) = e$ (if $i \neq j$) or $\alpha_i(e) = \alpha(e)$ (if $i = j$). Suppose now that $x \in E$ and $y \in E'$, $E, E' \in \mathcal{E}$ with $E \neq E'$. Let $B, B' \in \mathcal{B}$ such that $x \in B \subseteq E$ and $y \in B' \subseteq E'$. As $E \neq E'$, $B \not\equiv B'$, so there exists $\beta \in \text{fix}_{\text{Aut}(\Gamma)}(\mathcal{B})$ such that, say, $\beta|_B$ is a p-cycle while $\beta|_{B'}$ is not. By raising β to an appropriate power relatively prime to p, we may assume without loss of generality that $|\beta| = p$. Applying β to e $p-1$ times, we have that $\vec{x'y} \in E(\Gamma)$ for every $x' \in B$. As $\text{fix}_{\text{Aut}(\Gamma)}(\mathcal{B})|_{B'}$ is transitive, we have that $\vec{x'y'} \in E(\Gamma)$ for every $x' \in B$ and $y' \in B'$. As $\alpha_i(x) \in B$ and $\alpha_i(y) \in B'$, we have that $\alpha_i(e) \in E(\Gamma)$, and the result follows. □

The following result is implicit in [4].

Lemma 19

Let $\tau \in S_{pq}$ be a pq-cycle, and $\gamma \in S_{pq}$ such that $\langle \tau, \gamma^{-1}\tau\gamma \rangle$ admits a complete block system $\mathcal{B}$ of p blocks of size q. If the Sylow q-subgroups of $\text{fix}_{\langle \tau, \gamma^{-1}\tau\gamma \rangle}(\mathcal{B})$ have order q, then $\gamma^{-1}\langle \tau \rangle \gamma$ is conjugate to $\langle \tau \rangle$ in $\langle \tau, \gamma^{-1}\tau\gamma \rangle$.

PROOF. As a transitive abelian group is regular we have that $\text{fix}_{\langle \tau \rangle}(\mathcal{B})$ and $\text{fix}_{\gamma^{-1}\langle \tau \rangle \gamma}(\mathcal{B})$ have order q as both $\langle \tau \rangle/\mathcal{B}$ and $\gamma^{-1}\langle \tau \rangle \gamma/\mathcal{B}$ are transitive abelian groups of degree p. Hence $\langle \tau^p \rangle$ and $\gamma^{-1}\langle \tau^p \rangle \gamma$ are Sylow q-subgroups of $\text{fix}_{\langle \tau, \gamma^{-1}\tau\gamma \rangle}(\mathcal{B})$ and are thus conjugate.

We may thus assume that $\langle \tau^p \rangle = \gamma^{-1}\langle \tau^p \rangle\gamma$. Furthermore, $\langle \tau \rangle/\mathcal{B}$ and $\gamma^{-1}\langle \tau \rangle\gamma/\mathcal{B}$ are Sylow p-subgroups of $\langle \tau, \gamma^{-1}\tau\gamma \rangle/\mathcal{B}$ and are also conjugate. We may thus also assume that $\langle \tau \rangle/\mathcal{B} = \gamma^{-1}\langle \tau \rangle\gamma/\mathcal{B}$. As $\langle \tau^p, \gamma^{-1}\tau^q\gamma \rangle = \gamma^{-1}\langle \tau \rangle\gamma$, we have that $\gamma^{-1}\tau^q\gamma$ centralizes $\langle \tau^p \rangle$. As the centralizer in S_{pq} of $\langle \tau^p \rangle$ is $S_p \wr \mathbb{Z}_q$ by Lemma 13, we have that $\langle \tau, \gamma^{-1}\tau\gamma \rangle \le S_p \wr \mathbb{Z}_q$. As $|S_p \wr \mathbb{Z}_q| = p! \cdot q^p$, we have that a Sylow p-subgroup of $\langle \tau, \gamma^{-1}\tau\gamma \rangle$ has order p. As $\langle \tau^q \rangle$ and $\gamma^{-1}\langle \tau^q \rangle\gamma$ have order p, they are Sylow p-subgroups of $\langle \tau, \gamma^{-1}\tau\gamma \rangle$ and thus conjugate. We thus assume that $\langle \tau^q \rangle = \gamma^{-1}\langle \tau^q \rangle\gamma$. Hence $\gamma^{-1}\langle \tau \rangle\gamma = \langle \tau \rangle$. $\square$

Lemma 20

Let $G \le S_n$ where n is square-free and $(\mathbb{Z}_n)_L \le G$. Then G admits complete block systems $\mathcal{B}$ and $\mathcal{C}$ of m blocks of size k and k blocks of size m, respectively, if and only if G is permutation isomorphic to a subgroup of $S_k \times S_m$.

Proof. Suppose that $G \le S_m \times S_k$. Define $\rho, \tau : \mathbb{Z}_m \times \mathbb{Z}_k \to \mathbb{Z}_m \times \mathbb{Z}_k$ by $\rho(i,j) = (i, j+1)$ and $\tau(i,j) = (i+1, j)$. It is then easy to see that $\rho \in 1_{S_m} \times S_k$, $\tau \in S_m \times 1_{S_k}$, and $\langle \rho, \tau \rangle = (\mathbb{Z}_n)_L \le G$. Then $\langle \rho \rangle^G$, the normal closure of $\langle \rho \rangle$ in G, is a subgroup of $1_{S_m} \times S_k$ and $\langle \tau \rangle^G \le S_k \times 1_{S_m}$, and of course $\langle \rho \rangle^G \triangleleft G$ and $\langle \tau \rangle^G \triangleleft G$. Then the orbits of $\langle \rho \rangle^G$ have order m, and so form $\mathcal{B}$, and the orbits of $\langle \tau \rangle^G$ have order k, and so form $\mathcal{C}$.

Conversely, assume that G admits complete block systems $\mathcal{B}$ and $\mathcal{C}$ of m blocks of size k and k blocks of size m, respectively. For convenience, enumerate the blocks of $\mathcal{B}$ as $\{B_i : i \in \mathbb{Z}_m\}$ and $\mathcal{C}$ as $\{C_j : j \in \mathbb{Z}_k\}$. If $B_i \cap C_j \ne \emptyset$, $i \in \mathbb{Z}_m$, $j \in \mathbb{Z}_k$, then $B_i \cap C_j$ is a block of G. As n is square-free, $\gcd(m,k) = 1$ so that $B_i \cap C_j$ is a singleton. The result then follows by [15, Lemma 2.2] $\square$

Theorem 10

Let $n = p_1 \cdots p_r$ be square-free with each p_i a prime. Suppose that $\gamma \in S_n$ such that $\langle (\mathbb{Z}_n)_L, \gamma^{-1}(\mathbb{Z}_n)_L\gamma\rangle \leq S_{p_1} \times S_{p_2} \times \cdots \times S_{p_r}$. Then there exists $\omega \in \langle (\mathbb{Z}_n)_L, \gamma^{-1}(\mathbb{Z}_n)_L\gamma\rangle$ such that $\omega^{-1}\gamma^{-1}(\mathbb{Z}_n)_L\gamma\omega = (\mathbb{Z}_n)_L$.

PROOF. Let $(\mathbb{Z}_n)_L = \langle x\rangle$ and $y = \gamma^{-1}x\gamma$, so that $\langle (\mathbb{Z}_n)_L, \gamma^{-1}(\mathbb{Z}_n)_L\gamma\rangle = \langle x, y\rangle$. It is then easy to see that $\langle x, y\rangle$ admits a complete block system $\mathcal{B}_i$ consisting of n/p_i blocks of size p_i, $1 \leq i \leq r$. As a transitive abelian group is regular, we must have that both $\mathrm{fix}_{\langle x\rangle}(\mathcal{B}_i)$ and $\mathrm{fix}_{\langle y\rangle}(\mathcal{B}_i)$ are nontrivial p_i-groups of order p_i. As $\langle x, y\rangle \leq S_{p_1} \times \cdots \times S_{p_r}$, we have that $\mathrm{fix}_{\langle x,y\rangle}(\mathcal{B}_i) \leq 1_{S_{p_1}} \times \cdots \times 1_{S_{p_{i-1}}} \times S_{p_i} \times 1_{S_{p_{i+1}}} \times \cdots \times 1_{S_{p_r}}$. We conclude that a Sylow p_i-subgroup of $\mathrm{fix}_{\langle x,y\rangle}(\mathcal{B}_i)$ has order p_i. Hence $\langle x^{n/p_i}\rangle$ and $\langle y^{n/p_i}\rangle$ are Sylow p_i-subgroups of $\mathrm{fix}_{\langle x,y\rangle}(\mathcal{B}_i)$ and are thus conjugate. Let $\omega_1 \in \langle x, y\rangle$ such that $\omega_1^{-1}\langle y^{n/p_1}\rangle\omega_1 = \langle x^{n/p_1}\rangle$ and $y_1 = \omega_1^{-1}y\omega_1$. Note that every element of $\langle x\rangle$ commutes with every element of $\langle x^{n/p_1}\rangle = \langle y_1^{n/p_1}\rangle$ and every element of $\langle y_1\rangle$ commutes with every element of $\langle y_1^{n/p_1}\rangle = \langle x^{n/p_1}\rangle$. Hence $\langle x^{n/p_1}\rangle = \langle y_1^{n/p_1}\rangle$ is contained in the center of $\langle x, y_1\rangle$.

As above, $\langle x^{n/p_2}\rangle$ and $\langle y_1^{n/p_2}\rangle$ are Sylow p_2-subgroups of $\mathrm{fix}_{\langle x,y_1\rangle}(\mathcal{B}_2)$ and are thus conjugate. Let $\omega_2 \in \mathrm{fix}_{\langle x,y_1\rangle}(\mathcal{B}_2)$ such that $\omega_2^{-1}\langle y_1^{n/p_2}\rangle\omega_2 = \langle x^{n/p_2}\rangle$. Let $y_2 = \omega_2^{-1}y_1\omega_2$. As above, we also have that $\langle x^{n/p_2}, y_2^{n/p_2}\rangle$ are contained in the center of $\langle x, y_2\rangle$. Also, as ω_2 commutes with y_1^{n/p_1}, we have that $y_2^{n/p_1} = y_1^{n/p_1}$. Hence $\langle x^{n/p_1}, y^{n/p_1}\rangle$ is still contained in the center of $\langle x, y_2\rangle$. Continuing inductively, we find $\omega_i \in \mathrm{fix}_{\langle x,y_{i-1}\rangle}(\mathcal{B}_i)$ such that $\omega_i^{-1}\langle y_{i-1}^{n/p_i}\rangle\omega_i = \langle x^{n/p_i}\rangle$ and $\langle x^{n/p_j}, y^{n/p_j}\rangle$ is contained in the center of $\langle x, y_i\rangle$ for every $1 \leq j \leq i$, where $y_0 = y$ and for $1 \leq i \leq r-1$, $y_i = \omega_i^{-1}y_{i-1}\omega_i$. Let $\omega = \omega_r\omega_{r-1}\cdots\omega_1$. Then $\omega^{-1}\langle y\rangle\omega = \langle x\rangle$ and the result follows. □

3.4 The Isomorphism Problem for (pq, r)-metacirculant digraphs

In this section we consider the isomorphism problem for (pq, r)-metacirculant digraphs where p, q, r are distinct primes such that $pq|(r-1)$. Let $V = \mathbb{Z}_{pq} \times \mathbb{Z}_r$ and $\alpha \in \mathbb{Z}_r^*$ such that $pq \mid |\alpha|$. Define $\rho, \tau : V \to V$ by $\tau(i,j) = (i+1, \alpha j)$ and $\rho(i,j) = (i, j+1)$. By raising τ to an appropriate power, if need be, we assume without loss of generality that $|\alpha| = p^x q^y$ for some $x, y \geq 1$ so that r does not divide $|\alpha|$. Our main result is the following, and a great deal of the proof consists of a sequence of lemmas which are listed after the proof of the main theorem.

Theorem 11

Let Γ be a (pq, r)-metacirculant color digraph with $|\alpha| = p^x q^y$, $x, y \geq 1$. Then one of the following is true:

1. Γ is isomorphic to a $(pr/\ell, \ell r)$-metacirculant color digraph for $\ell = p$ or q, or
2. Γ is a CI-color digraph of $(\langle \rho, \tau \rangle, \langle \tau^p \rangle)$.

PROOF. We wish to show that either Γ is isomorphic to a $(pq/\ell, \ell r)$-metacirculant color digraph, or to apply Lemma 17. To that end, let $\gamma \in S_V$ such that $\gamma^{-1}\langle \rho, \tau \rangle \gamma \leq \mathrm{Aut}(\Gamma)$, where Γ is a (pq, r)-metacirculant color digraph with $|\alpha| = p^x q^y$, $x, y \geq 1$. Let $p < q$ so that $p < q < r$. By Lemma 21, we may assume without loss of generality that $H = \langle \langle \rho, \tau \rangle, \gamma^{-1}\langle \rho, \tau \rangle \gamma \rangle \leq \mathbb{Z}_p \wr (\mathrm{AGL}(1, q) \wr \mathrm{AGL}(1, r))$, and that H admits complete block systems $\mathcal{B} \prec \mathcal{C}$ such that $\mathcal{B}$ consists of pq blocks of size r formed by the orbits of $\langle \rho \rangle$ and $\mathcal{C}$ consists of p blocks of size qr formed by the orbits of $\langle \tau^p, \rho \rangle$. Now, by Lemma 18 (applied to $N = \mathrm{fix}_H(\mathcal{B})$), the equivalence classes of $\equiv$ form a complete block system $\mathcal{D}$ (and of course $\mathcal{B} \preceq \mathcal{D}$) and $\rho|_D \in \mathrm{Aut}(\Gamma)$ for every $D \in \mathcal{D}$. As $\mathcal{B} \prec \mathcal{D}$ and $H/\mathcal{B}$ contains a regular cyclic subgroup (and so complete block systems with blocks of a given size are unique - see [**?**, Exercise 6.5]), namely $\langle\tau\rangle/\mathcal{B}$, $\mathcal{D}$ consists of pq blocks of size r (in which case $\mathcal{D} = \mathcal{B}$), p blocks of size qr (in which case $\mathcal{D} = \mathcal{C}$), q blocks of size pq (in which case $\mathcal{D}$ is formed by the orbits of $\langle\tau^q, \rho\rangle$), or 1 block of size pqr. We will consider the various possibilities for $\mathcal{D}$ separately.

If $\mathcal{D}$ consists of pq blocks of size r, then Γ is the wreath product of a circulant color digraph of order pq and a circulant color digraph of order r, in which case Γ is a circulant color digraph of order pqr as $\mathbb{Z}_p \ltimes \mathbb{Z}_q \times \mathbb{Z}_r \leq \mathrm{Aut}(\Gamma)$. Then Γ is isomorphic to both a (p, qr)- and a (q, pr)-metacirculant color digraph, and the result follows.

If $\mathcal{D}$ consists of p blocks of size qr. By Lemma 22, we may assume without loss of generality that $x \geq 2$. If a Sylow q-subgroup of $\mathrm{fix}_{H/\mathcal{B}}(\mathcal{C}/\mathcal{B})$ has order at least q^2, then the result follows by Lemma 23. Otherwise, as $\langle\tau\rangle/\mathcal{B}$ is a regular cyclic subgroup, by Lemma 19, after an appropriate conjugate by $\delta \in H$, we may assume that $H/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$. By Lemma 24, we then have that $\gamma(i, j) = (ai, \alpha_i j + b_i)$, where $a \in \mathbb{Z}_{pq}^*$, $\alpha_i \in \mathbb{Z}_r^*$, and $b_i \in \mathbb{Z}_r$. By Lemma 25, $\alpha_i = \alpha_{i+p}$ for every $i \in \mathbb{Z}_{pq}$, and by Lemma 26, we may assume that $b_i = 0$ for all $i \in \mathbb{Z}_{pq}$. In view of Lemma 27, we may assume without loss of generality that $a = 1$. By Lemma 28, we may assume that $\alpha_i = \alpha_0$ for all $i \in \mathbb{Z}_{pq}$. We then conclude that $\gamma^{-1}\langle\rho, \tau\rangle\gamma = \langle\rho, \tau\rangle$, and the result follows by Lemma 17.

If $\mathcal{D}$ consists of q blocks of size pr, then note that the only result used in settling the previous cases that relied on the fact that $p < q < r$ was Lemma 21. Thus if it can be established in this case that $H \leq \mathbb{Z}_q \wr (\mathrm{AGL}(1, p) \wr \mathrm{AGL}(1, r))$ (even if γ is replaced by $\gamma\delta$ for some $\delta \in H^{(2)}$), then the result will follow by arguments in the previous paragraph. Note that $H/\mathcal{B}$ admits a complete block system consisting of p blocks of size q induced by $\mathcal{C}$, and that $H/\mathcal{B}$ admits a complete block system of q blocks of size p induced by $\mathcal{D}$. By Lemma 20 we have that $H/\mathcal{B}$ is permutation isomorphic to $S_q \times S_p$ as $\langle\tau\rangle/\mathcal{B}$ is a regular cyclic subgroup, and by Lemma 10, there exists $\delta \in H$ such that $\langle\langle\rho,\tau\rangle, \delta^{-1}\gamma^{-1}\langle\rho,\tau\rangle\gamma\delta\rangle/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$. Replacing γ with $\gamma\delta$, we assume without loss of generality that $H/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$. Then $H \leq \mathbb{Z}_{pq} \wr \mathrm{AGL}(1,r) \leq \mathbb{Z}_q \wr (\mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,r))$, and the result follows in this case.

If $\mathcal{D}$ consists of 1 block of size pqr, then as $H \leq \mathbb{Z}_p \wr (\mathrm{AGL}(1,q) \wr \mathrm{AGL}(1,r))$, $|H/\mathcal{B}|$ divides $|\mathbb{Z}_p \wr \mathrm{AGL}(1,q)| = p \cdot [q(q-1)]^p$. As $r > q > p$, r does not divide $|H/\mathcal{B}|$ and so a Sylow p-subgroup of H is contained in $\mathrm{fix}_H(\mathcal{B})$. As there is only one equivalence class of $\equiv$, we conclude that $\langle\rho\rangle$ is a Sylow r-subgroup of $\mathrm{fix}_H(\mathcal{B})$. Hence $\gamma^{-1}\langle\rho\rangle\gamma = \rho$.

If a Sylow q-subgroup of $\mathrm{fix}_{H/\mathcal{B}}(\mathcal{C}')$ has order q, then by Lemma 19, there exists $\delta \in H$ such that $\delta^{-1}\gamma^{-1}\langle\tau\rangle\gamma\delta/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$. Replacing γ with $\gamma\delta$, we assume without loss of generality that $\gamma^{-1}\langle\tau\rangle\delta/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$. By Lemma 24, we may assume that $\gamma\delta(i,j) = (ai, \alpha_i j + b_i)$, where $a \in \mathbb{Z}_{pq}^*$, $\alpha_i \in \mathbb{Z}_r^*$, and $b_i \in \mathbb{Z}_r$. Additionally, as there is only one equivalence class of $\equiv$, it must be the case that $\alpha_i = \alpha_j$ for every $i,j \in \mathbb{Z}_{pq}$. Setting $\beta = \alpha_0$, we then have that $\gamma(i,j) = (ai, \beta j + b_i)$. By Lemma 24, $\gamma^{-1}\tau\gamma(i,j) = (i + a^{-1}, \alpha j + \alpha\beta^{-1}b_i - \beta^{-1}b_{i+a^{-1}})$. Then $\tau^{-a^{-1}}\gamma^{-1}\tau\gamma(i,j) = (i, \alpha^{1-a^{-1}}j + \alpha^{-a^{-1}}(\alpha\beta^{-1}b_i - \beta^{-1}b_{i+a^{-1}}))$. If $a = 1$, then either $\gamma^{-1}\tau\gamma = \tau$, or $\tau^{-1}\gamma^{-1}\tau\gamma$ has order r, in which case $\tau^{-1}\gamma^{-1}\tau\gamma \in \langle\rho\rangle$. Then $\gamma^{-1}\tau\gamma \in \langle\tau,\rho\rangle$, and so $\gamma^{-1}\langle\rho,\tau\rangle\gamma = \langle\rho,\tau\rangle$, and Γ is a CI-color digraph of $(\langle\rho,\tau\rangle, \langle\tau^p\rangle)$ by Lemma 17.

If $a \neq 1$, then there exists a prime $\ell | pq$ such that $a \not\equiv 1 \pmod{\ell}$. Then $1 - a^{-1} \not\equiv 0 \pmod{\ell}$, and so there exists a positive integer m relatively prime to pq such that $m(1 - a^{-1}) \equiv -1 \pmod{\ell}$. Then $\tau(\tau^{-a^{-1}}\gamma^{-1}\tau\gamma)^m(i,j) = (i+1, \alpha^{1+m(1-a^{-1})}j + c_i)$ for appropriate $c_i \in \mathbb{Z}_r$ (we remark that at this time we are not concerned with the exact values of the c_i). Set $z = p^x$ if $\ell = q$ and $z = q^y$ if $\ell = p$. Then $\alpha^{1+m(1-a^{-1})} \equiv 1 \pmod{\ell}$, and so $[\tau(\tau^{-a^{-1}}\gamma^{-1}\tau\gamma)^m]^z(i,j) = (i+z, j+d_i)$ for appropriate $d_i \in \mathbb{Z}_r$, and $z \not\equiv 0 \pmod{\ell}$ while $z \equiv 0 \pmod{pq/\ell}$. Set $\rho_1 = [\tau(\tau^{-a^{-1}}\gamma^{-1}\tau\gamma)^m]^z$.

For the remainder of the analysis of this subcase, it will be convenient to label V with $\mathbb{Z}_{pq/\ell} \times \mathbb{Z}_\ell \times \mathbb{Z}_r$ in such a way that $\tau(i,j,k) = (i+1, j+1, \alpha k)$ and $\rho(i,j,k) = (i,j,k+1)$. As $z \equiv 0 \pmod{pq/\ell}$ but $z \not\equiv 1 \pmod{\ell}$, if $c \equiv z \not\equiv 0 \pmod{\ell}$, then $\rho_1(i,j,k) = (i, j+c, k+e_{i,j})$ for appropriate $e_{i,j} \in \mathbb{Z}_r$. Replacing ρ_1 with an appropriate power of itself, we may assume without loss of generality that $c = 1$. By Lemma 29, Γ is isomorphic to a (p, qr)-metacirculant digraph. It now only remains to consider the case where a Sylow q-subgroup of $\mathrm{fix}_{H/\mathcal{B}}(\mathcal{C}')$ has order at least q^2.

If a Sylow q-subgroup of $\mathrm{fix}_{H/\mathcal{B}}(\mathcal{C}')$ has order at least q^2, then let $\omega \in H$ such that $\omega/\mathcal{B}$ has order q but $\omega/\mathcal{B} \notin \langle\tau\rangle/\mathcal{B}$. Similar to the end of the previous case, it will be convenient to view V as $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ in such a way that $\tau(i,j,k) = (i+1, j+1, \alpha k)$ and $\rho(i,j,k) = (i,j,k+1)$. As $H/\mathcal{B} \leq \mathbb{Z}_p \wr \mathrm{AGL}(1,q)$ and $q > p$, we have that a Sylow q-subgroup of $H/\mathcal{B}$ is $1_{S_p} \wr \mathbb{Z}_q$. As $H \leq \mathbb{Z}_p \wr (\mathrm{AGL}(1,q) \wr \mathrm{AGL}(1,r))$, we have that $\omega(i,j,k) = (i, j+b_i, \beta_{i,j}k + c_{i,j})$, where $b_i \in \mathbb{Z}_q$, $\beta_{i,j} \in \mathbb{Z}_r^*$, and each $c_{i,j} \in \mathbb{Z}_r$. As a Sylow r-subgroup of $\mathrm{fix}_H(\mathcal{B})$ has order r, we have that $\beta_{i,j} = \beta_{i',j'}$ for every $i, i' \in \mathbb{Z}_p$ and $j, j' \in \mathbb{Z}_q$. We set $\beta = \beta_{0,0}$. We also observe that it cannot be the case that each $b_i = b_{i'}$,

$i, i' \in \mathbb{Z}_p$, as if that does occur, then $\omega/\mathcal{B} \in \langle\tau\rangle/\mathcal{B}$. By raising ω to an appropriate power relatively prime to q, we can and do assume that ω has order a power of q. Then β has order a power of q as well. We now show that there exists $\phi \in H$ such that $\phi/\mathcal{B}$ has order q and $\phi(i,j,k) = (i, j + c_i, k + d_{i,j})$, $c_i \in \mathbb{Z}_p$ and each $d_{i,j} \in \mathbb{Z}_r$.

Note that $\tau^{p^x}(i,j,k) = (i, j + p^x, \alpha^{p^x}k)$, $p^x \not\equiv 0 \pmod q$, and α^{p^x} has order q^y. Then $\langle \alpha^{p^x}, \beta\rangle$ is cyclic, and as both α^{p^x} and β have order a power of q, $\langle \alpha^{p^x}, \beta\rangle$ has order a power of q, and is generated by either α^{p^x} or β. If $\langle \alpha^{p^x}, \beta\rangle$ is generated by α^{p^x}, then let m be a power of q such that $\alpha^{mp^x} = \beta$. We then set $\phi = \tau^{-mp^x}\omega$. If $\langle \alpha^{p^x}, \beta\rangle$ is generated by β, then we let n be a power of q such that $\beta^n = \alpha^{p^x}$. We then set $\phi = \tau^{p^x}\omega^{-n}$. In either case, straightforward computations show that ϕ has the required properties.

Now observe that by raising ϕ to an appropriate power relatively prime to q, we may additionally assume that ϕ has order q. It then follows that $\sum_{j=0}^{q-1} d_{i,j} \equiv 0 \pmod r$ as $\phi^q(i,j,) = (i, j, k + \sum_{j'=0}^{p-1} d_{i,j'})$. Let $K = \{\psi \in H : \psi(i,j,k) = (i, j + e_i, k + f_{i,j}) : e_i \in \mathbb{Z}_q, f_{i,j} \in \mathbb{Z}_r \text{ and } \sum_{j=0}^{q-1} f_{i,j} \equiv 0 \pmod r\}$. It is straightforward to verify that K is a subgroup of H, and every nontrivial element of K has order q. Let $L = \langle \tau, \rho, K\rangle$. Then L is transitive, and clearly $\rho^{-1}\psi\rho \in K$ for every $\psi \in K$. Note that

$$\begin{aligned}
\tau^{-1}\psi\tau(i,j,k) &= \tau^{-1}\psi(i+1, j+1, \alpha k) && (3.4.1)\\
&= \tau^{-1}(i+1, j+1+e_{i+1}, \alpha k + f_{i+1,j+1}) && (3.4.2)\\
&= (i, j + e_{i+1}, k + \alpha^{-1} f_{i+1,j+1}). && (3.4.3)
\end{aligned}$$

As $\sum_{j=0}^{p-1} f_{i,j} \equiv 0 \pmod r$, we have that $\alpha^{-1}\sum_{j=0}^{p-1} f_{i+1,j+1} \equiv 0 \pmod r$. We conclude that $\tau^{-1}\psi\tau \in K$ and so $K \triangleleft L$.

As $K \triangleleft L$, L admits a complete block system $\mathcal{E}$ formed by the orbits of K. As K is a q-group, the orbits of K have order q, and $\mathcal{E}$ is a complete block system of $L^{(2)} \leq \mathrm{Aut}(\Gamma)$ [41, Theorem 4.11] (observe that [41] is contained in the more readily accessible [**?**]). Define $\equiv'$ on $\mathcal{E}$ by $E_1 \equiv' E_2$ if and only if whenever $\psi \in \mathrm{fix}_L(\mathcal{E})$ then $\psi|_{E_1}$ is a q-cycle if and only if $\psi|_{E_2}$ is a q-cycle. Applying Lemma 18, we may conclude that the map $\rho_1 : V \to V$ by $\rho_1(i,j,k) = (i, j+1, k+g_{i,j})$ is contained in $L^{(2)} \leq \mathrm{Aut}(\Gamma)$, for appropriate $g_{i,j} \in \mathbb{Z}_r$ (note that $\sum_{i=0}^{p-1} g_{i,j} \equiv 0 \pmod r$). Then Γ is isomorphic to a (p, qr)-metacirculant digraph by Lemma 29. □

Lemma 21

Let $p < q < r$. If $\gamma \in S_V$, then there exists $\delta \in \langle\langle \rho, \tau, \rangle, \gamma^{-1}\langle \rho, \tau\rangle\gamma\rangle$ such that

$$H' = \langle\langle \rho, \tau\rangle, \delta^{-1}\gamma^{-1}\langle \rho, \tau\rangle\gamma\delta\rangle \leq \mathbb{Z}_p \wr (\mathrm{AGL}(1,q) \wr \mathrm{AGL}(1,r)).$$

Furthermore, H' admits complete block systems $\mathcal{B} \prec \mathcal{C}$ such that $\mathcal{B}$ consists of pq blocks of size r formed by the orbits of $\langle \rho \rangle$ and $\mathcal{C}$ consists of p blocks of size qr formed by the orbits of $\langle \tau^p, \rho\rangle$.

PROOF. By [18, Lemma 6], there exists $\delta_1 \in H$ such that $H_1 = \langle\langle \rho, \tau\rangle, \delta_1^{-1}\gamma^{-1}\langle \rho, \tau\rangle\gamma\delta_1\rangle$ admits a complete block system $\mathcal{B}$ of pq blocks of size r formed by the orbits of $\langle \rho \rangle$. By the same result, there exists $\delta_2 \in H_1$ such that $H_2/\mathcal{B} = \langle\langle \rho, \tau\rangle, \delta_2^{-1}\delta_1^{-1}\gamma^{-1}\langle \rho, \tau\rangle\gamma\delta_1\delta_2\rangle/\mathcal{B}$ admits a complete block system $\mathcal{C}'$ consisting of p blocks of size q formed by the orbits of $\langle \tau^p\rangle/\mathcal{B}$. Hence H_2 admits a complete block system $\mathcal{C}$ consisting of p blocks of size

qr formed by the orbits of $\langle\tau^p,\rho\rangle$. Replacing $\gamma\delta_1\delta_2$ with γ we assume without loss of generality that H admits $\mathcal{B}$ and $\mathcal{C}$ as complete block systems.

Now observe that $\gamma^{-1}\langle\rho\rangle\gamma/\mathcal{B} \leq S_{pq}$ and $|\gamma^{-1}\langle\rho\rangle\gamma/\mathcal{B}|$ divides $r \geq pq+1$. Thus $|\gamma^{-1}\langle\rho\rangle\gamma/\mathcal{B}| = 1$ so $\gamma^{-1}\langle\rho\rangle\gamma \leq \mathrm{fix}_H(\mathcal{B})$. Now, $\langle\rho\rangle$ and $\gamma^{-1}\langle\rho\rangle\gamma$ are contained in Sylow r-subgroups P_1 and P_2 of $\mathrm{fix}_H(\mathcal{B})$ and of course, P_1 and P_2 are conjugate in $\mathrm{fix}_H(\mathcal{B})$ by $\delta_3 \in \mathrm{fix}_H(\mathcal{B})$. We may thus assume without loss of generality that $P_1 = P_2$ so that both ρ and $\gamma^{-1}\rho\gamma$ are contained in $P_1 \leq \langle\rho|_B : B \in \mathcal{B}\rangle$ (see [35, pg. 10]). Now, let $B \in \mathcal{B}$. Then $\mathrm{fix}_H(\mathcal{B})|_B \triangleleft \mathrm{Stab}_H(B)|_B$ and is permutation isomorphic to a subgroup of S_r. Furthermore, $\langle\rho\rangle|_B$ is a Sylow r-subgroup of S_r and $\mathrm{fix}_H(\mathcal{B})|_B$ contains a normal Sylow r-subgroup, namely $\langle\rho\rangle|_B$, which is characteristic. Thus $\langle\rho\rangle|_B \triangleleft \mathrm{Stab}_H(\mathcal{B})|_B$. By [13, Exercise 3.5.1], $\mathrm{Stab}_H(\mathcal{B})|_B$ is permutation isomorphic to a subgroup of $\mathrm{AGL}(1,r)$. Hence by the Embedding Theorem [29, Theorem 2.6], H is permutation isomorphic to a subgroup of $S_{pq} \wr \mathrm{AGL}(1,r)$.

Similarly, both $\langle\tau^p\rangle/\mathcal{B}$ and $\gamma^{-1}\langle\tau^p\rangle\gamma/\mathcal{B}$ are contained in Sylow q-subgroups of $H/\mathcal{B}$. As a Sylow q-subgroup of S_{pq} is $\langle\tau^p/\mathcal{B}|_{C'} : C' \in \mathcal{C}'\rangle$ (again see [35, pg. 10]), by a Sylow Theorem $\langle\tau^p\rangle/\mathcal{B}$ and $\gamma^{-1}\langle\tau^p\rangle\gamma/\mathcal{B}$ are conjugate by $\delta_4 \in H$. We may thus assume that both $\langle\tau^p\rangle/\mathcal{B}$ and $\gamma^{-1}\langle\tau^p\rangle\gamma/\mathcal{B}$ are contained in $\langle\tau^p/\mathcal{B}|_{C'} : C' \in \mathcal{C}'\rangle$, and that $H/\mathcal{B} \leq S_p \wr \mathrm{AGL}(1,q) = \mathrm{AGL}(1,p) \wr \mathrm{AGL}(1,q)$. Finally, as $\langle\tau\rangle/\mathcal{C}$ and $\gamma^{-1}\langle\tau\rangle\gamma/\mathcal{C}$ are Sylow p-subgroups of $H/\mathcal{B}$, by a Sylow Theorem there exists $\delta_5 \in H$ such that $\delta_5^{-1}\gamma^{-1}\langle\tau\rangle\gamma\delta_5/\mathcal{C} = \langle\tau\rangle/\mathcal{C}$. Setting $\delta = \delta_1\delta_2\delta_3\delta_4\delta_5$, we have that $\langle\langle\rho,\tau\rangle, \delta^{-1}\gamma^{-1}\langle\rho,\tau\rangle\gamma\delta\rangle \leq \mathbb{Z}_p \wr (\mathrm{AGL}(1,q) \wr \mathrm{AGL}(1,r))$. □

Henceforth we will use $\mathcal{B}$ and $\mathcal{C}$ to indicate the complete block systems of $\langle \rho, \tau \rangle$ formed by the orbits of $\langle \rho \rangle$ and $\langle \rho, \tau^p \rangle$ respectively. We also let $\mathcal{B} = \{B_i : i \in \mathbb{Z}_{pq}\}$, where $(i, 0) \in B_i$, $i \in \mathbb{Z}_{qp}$ and $\mathcal{C} = \{C_i : i \in \mathbb{Z}_r\}$ where $(i, 0) \in C_i$, $i \in \mathbb{Z}_p$.

Lemma 22

Let Γ be a (pq, r)-metacirculant color digraph with $|\alpha| = pq^y$, $y \geq 1$. Suppose that $\langle \rho|_C : C \in \mathcal{C} \rangle \leq \mathrm{Aut}(\Gamma)$. Then Γ is isomorphic to a (q, pr)-metacirculant color digraph.

PROOF. Let $a \in \mathbb{Z}_r$ such that $|a| = p$, and $\alpha = ab$, where $b \in \mathbb{Z}_r$ has order q^y. Let $\rho' = \Pi_{i=0}^{p-1} \rho^{a^i}|_{C_i}$. For ease of computation, it will be convenient to relabel V as $\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r$ so that $\tau(i, j, k) = (i+1, j+1, \alpha k)$ and $\rho(i, j, k) = (i, j, k+1)$. Then $\rho'(i, j, k) = (i, j, k+a^i)$, and $\tau^{q^y}(i, j, k) = (i + q^y, j, a^{q^y} k)$. Then

$$\begin{aligned}
\tau^{-q^y} \rho' \tau^{q^y}(i, j, k) &= \tau^{-1} \rho'(i + q^y, j, a^{q^y} k) && (3.4.4)\\
&= \tau^{-1}(i + q^y, j, a^{q^y} k + a^{i+q^y}) && (3.4.5)\\
&= (i, j, k + a^{-q^y} a^{i+q^y}) && (3.4.6)\\
&= (i, j, k + a^i) && (3.4.7)\\
&= \rho'(i, j, k) && (3.4.8)
\end{aligned}$$

Thus $\rho' \tau^{q^y} = \tau^{q^y} \rho'$. As ρ' is semiregular of order r and τ^{q^y} is semiregular of order p, $\rho' \tau^{q^y}$ is semiregular of order pr. Then $\langle \tau^p, \rho' \tau^{q^y} \rangle$ is transitive. Also note that

$\tau^{-p} \rho' \tau^{q^y} \tau^p = \tau^{-p} \rho' \tau^p \tau^{q^y}$,

and

$$\begin{aligned}\tau^{-p}\rho'\tau^p(i,j,k) &= \tau^{-p}\rho'(i,j+p,a^pk) && (3.4.9)\\ &= \tau^p(i,j+p,a^pk+a^i) && (3.4.10)\\ &= (i,j,k+a^{-p}a^i) && (3.4.11)\end{aligned}$$

Let $c \in \mathbb{Z}_{pr}$ such that $c \equiv a^{-p} \pmod r$ and $c \equiv 1 \pmod p$. Then $\tau^{-p}\rho'\tau^{q^y}\tau^p = (\rho'\tau^{q^y})^c$ and so $\langle \tau^p, \rho'\tau^{q^y}\rangle$ is a (q, pr)-metacyclic group, and the result follows by Theorem 7 $\square$

Lemma 23

Let Γ be a (pq, r)-metacirculant color digraph, and $\gamma \in S_V$ such that $H = \langle\langle\rho,\tau\rangle, \gamma^{-1}\langle\rho,\tau\rangle\gamma\rangle \leq \mathrm{Aut}(\Gamma)$ admits $\mathcal{B}$ and $\mathcal{C}$. If a Sylow r-subgroup of $\mathrm{fix}_{H^{(2)}}(\mathcal{B}) = \langle \rho|_C : C \in \mathcal{C}\rangle$, and a Sylow q-subgroup of $\mathrm{fix}_{H/\mathcal{B}}(\mathcal{C}/\mathcal{B})$ has order at least q^2, then Γ is isomorphic to the wreath product of a circulant color digraph of order p and a (q, r)-metacirculant color digraph of order qr. Hence Γ is isomorphic to (q, pr)-metacirculant color digraph.

Proof. We will show that if $v \in V$ and $C \in \mathcal{C}$ such that $v \notin C$, then $\mathrm{Stab}_H(v)$ is transitive on C. This will then imply the result as then v is adjacent to every element of C or to no elements of C, and as $\langle \tau^p, \rho\rangle$ is transitive on every block of $\mathcal{C}$, we will then have that every element of the block of $\mathcal{C}$ that contains v is adjacent to every element of C or to no elements of C.

As stabilizers of points are conjugate [13, Corollary 1.4A (i)], we may assume without loss of generality that $v = (0, 0)$. Let $C_i \in \mathcal{C}$, $i \neq 0$. As a Sylow q-subgroup Q of $\mathrm{fix}_{H/\mathcal{B}}(\mathcal{C}/\mathcal{B})$ has order at least q^2, there exists $1 \neq \gamma \in \mathrm{Stab}_Q(C_0/\mathcal{B})$. Let s be the smallest positive integer such that $(\gamma/\mathcal{B})|_{C_{si}/\mathcal{B}} \neq 1$ while $(\gamma/\mathcal{B})|_{C_{(s-1)i}/\mathcal{B}} = 1$. Such an s exists as i generates $\mathbb{Z}_p$. Then $\tau^{-(s-1)i}\gamma\tau^{(s-1)i}|_{C_0}$ fixes each block of $\mathcal{B}$ contained in C_0, and as ρ is transitive on each block of $\mathcal{B}$ but $\rho/\mathcal{B} = 1$, there exists $t \in \mathbb{Z}$ such that $\tau^{-(s-1)i}\gamma\tau^{(s-1)i}\rho^t(0,0) = (0,0)$. Also $\tau^{-(s-1)i}\gamma\tau^{(s-1)i}\rho^t$ is transitive on the blocks of $\mathcal{B}$ contained in C_i, and as $\rho|_{C_i} \in H^{(2)}$ and is transitive on each block of $\mathcal{B}$ contained in C_i, $\langle\tau^{-(s-1)i}\gamma\tau^{(s-1)i}\rho^t, \rho|_{C_i}\rangle$ fixes $(0,0)$ and is transitive on C_i. Then $\Gamma = \Gamma_1 \wr \Gamma_2$ where Γ_1 is circulant of order p and Γ_2 is a metacirculant of order q and $\mathbb{Z}_p \times \mathbb{Z}_q \ltimes \mathbb{Z}_r \cong \mathbb{Z}_q \ltimes (\mathbb{Z}_p \times \mathbb{Z}_r) \cong \mathbb{Z}_q \ltimes \mathbb{Z}_{pr} \leq \mathrm{Aut}(\Gamma)$. □

Lemma 24

Let $H = \langle\langle\rho,\tau\rangle, \gamma^{-1}\langle\rho,\tau\rangle\gamma\rangle$ admit $\mathcal{B}$. If $\mathrm{fix}_H(\mathcal{B})$ has a unique Sylow r-subgroup, $r > pq$, and $H/\mathcal{B}$ is cyclic, then

1. $\gamma(i,j) = (ai, \alpha_i j + b_i)$, where $a \in \mathbb{Z}_{pq}^*$, $\alpha_i \in \mathbb{Z}_r^*$, and $b_i \in \mathbb{Z}_r$.
2. $\gamma^{-1}\tau\gamma(i,j) = (i + a^{-1}, \alpha\alpha_{i+a^{-1}}^{-1}\alpha_i j + \alpha\alpha_{i+a^{-1}}^{-1}b_i - \alpha_{i+a^{-1}}^{-1}b_{i+a^{-1}})$.

PROOF. As $\mathcal{B}$ is a complete block system of $\langle\rho,\tau\rangle$ formed by the orbits of $\langle\rho\rangle$, $\gamma^{-1}\langle\rho\rangle\gamma$ is a complete block system of $\gamma^{-1}\langle\rho,\tau\rangle\gamma$ formed by the orbits of $\gamma^{-1}\langle\rho\rangle\gamma$. Again, as $\mathcal{B}$ is a complete block system of H and $r > pq$, $\gamma^{-1}\langle\rho\rangle\gamma/\mathcal{B} = 1$ and $\gamma^{-1}\langle\rho\rangle\gamma \leq \mathrm{fix}_H(\mathcal{B})$. Thus the orbits of $\gamma^{-1}\langle\rho\rangle\gamma$ are the same as the orbits of $\langle\rho\rangle$, $\gamma(\mathcal{B}) = \mathcal{B}$, and $\mathcal{B}$ is a complete block system of $\langle H,\gamma\rangle$. By the Embedding Theorem [29, Theorem 2.6], $\gamma \in S_{pq} \wr S_r$, with say $\gamma(i,j) = (\sigma(i), \omega_i(j))$ for $\sigma \in S_{pq}$, $\omega_i \in S_r$. As $H/\mathcal{B}$ is cyclic, $H/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$ and σ normalizes $(\mathbb{Z}_{pq})_L$. As $N_{S_{pq}}((\mathbb{Z}_{pq})_L) = \mathrm{Aut}(\mathbb{Z}_{pq}) \cdot (\mathbb{Z}_{pq})_L$ [13, Corollary 4.2B], $\sigma(i) = ai + b$, $a \in {\mathbb{Z}^*}_{pq}$, $b \in \mathbb{Z}_{pq}$. As $\mathrm{fix}_H(\mathcal{B})|_B$ has a unique Sylow r-subgroup, $\gamma^{-1}\rho\gamma|_B \in \langle\rho\rangle|_B$ and so, each ω_i normalizes $(\mathbb{Z}_p)_L$. As $N_{S_{pq}}((\mathbb{Z}_r)_L) = \mathrm{Aut}(\mathbb{Z}_r) \cdot (\mathbb{Z}_r)_L$ [13, Corollary

4.2B], we have that $\omega_i(j) = \alpha_i j + b_i$, $\alpha_i \in \mathbb{Z}_r^*$, $b_i \in \mathbb{Z}_r$. As $\tau \in \mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma')$, replacing γ with $\tau^{-b}\gamma$ we may assume without loss of generality that $b = 0$. Hence $\gamma(i,j) = (ai, \alpha_i j + b_i)$ and $\gamma^{-1}(i,j) = (a^{-1}i, \alpha^{-1}_{a^{-1}i} j - \alpha^{-1}_{a^{-1}i} b_{a^{-1}i})$. Thus

$$\begin{aligned}\gamma^{-1}\tau\gamma(i,j) &= \gamma^{-1}\tau(ai, \alpha_i j + b_i)\\ &= \gamma^{-1}(ai+1, \alpha\alpha_i j + \alpha b_i)\\ &= (i + a^{-1}, \alpha\alpha^{-1}_{i+a^{-1}}\alpha_i j + \alpha\alpha^{-1}_{i+a^{-1}} b_i - \alpha^{-1}_{i+a^{-1}} b_{i+a^{-1}}). \quad (3.4.12)\end{aligned}$$

□

Lemma 25

Suppose that $\gamma : V \to V$ by $\gamma(i,j) = (ai, \alpha_i j + b_i)$, where $a \in \mathbb{Z}_{pq}^*$, $\alpha_i \in \mathbb{Z}_r^*$, and $b_i \in \mathbb{Z}_r$. Let $H = \langle\langle\rho,\tau\rangle, \gamma^{-1}\langle\rho,\tau\rangle\gamma\rangle$, so that H admits $\mathcal{B}$. If a Sylow r-subgroup of $\mathrm{fix}_{H^{(2)}}(\mathcal{B})$ is $\langle\rho|_C : C \in \mathcal{C}\rangle$, then $\alpha_i = \alpha_{i+p}$ for every $i \in \mathbb{Z}_{pq}$.

Proof. As $\mathcal{C}/\mathcal{B}$ is a complete block system of $\langle\rho,\tau\rangle/\mathcal{B} = \langle\tau\rangle/\mathcal{B}$, and $\langle\rho,\tau\rangle/\mathcal{B}$ is a regular cyclic subgroup by [41, Exercise 6.5], the blocks of $\langle\rho,\tau\rangle/\mathcal{B}$ of size q are simply cosets of the unique subgroup of $\mathbb{Z}_{pq}$ of order q. Thus

$$C_j = \bigcup_{i=0}^{q-1} B_{(j+ip)}$$

for $j = 0, \ldots, p-1$. Note that $\gamma^{-1}(i,j) = (a^{-1}i, \alpha^{-1}_{a^{-1}i} j - \alpha^{-1}_{a^{-1}i} b_{a^{-1}i})$ as in Lemma 24. Then

$$\begin{aligned}
\gamma^{-1}\rho\gamma(i,j) &= \gamma^{-1}\rho(ai,\alpha_i j+b_i) && (3.4.13)\\
&= \gamma^{-1}(ai,\alpha_i j+b_i+1) && (3.4.14)\\
&= (a^{-1}(ai),\alpha^{-1}_{a^{-1}ai}(\alpha_i j+b_i+1)-\alpha^{-1}_{a^{-1}ai}b_{a^{-1}i}) && (3.4.15)\\
&= (i,j+\alpha_i^{-1}). && (3.4.16)
\end{aligned}$$

Also observe that $\gamma^{-1}\rho\gamma \in \mathrm{fix}_H(\mathcal{B})$, and $\gamma^{-1}\rho\gamma|_B$ is an r-cycle for every $B \in \mathcal{B}$. Similarly, for each $i \in \mathbb{Z}_{pq}$, we have that $\rho^{-\alpha_i^{-1}}\gamma^{-1}\rho\gamma \in \mathrm{fix}_H(\mathcal{B})$, and $\rho^{-\alpha_i^{-1}}\gamma^{-1}\rho\gamma|_B$ is either an r-cycle or the identity for every $i \in \mathbb{Z}_{pq}$. As $\rho^{-\alpha_i^{-1}}\gamma^{-1}\rho\gamma|_{B_i} = 1$ and a Sylow r-subgroup of $\mathrm{fix}_H(\mathcal{B})$ is $\langle \rho|_C \colon C \in \mathcal{C}\rangle$, $\rho^{-\alpha_i^{-1}}\gamma^{-1}\rho\gamma|_{B_{j+ip}} = 1$. As $C_j = \bigcup_{i=0}^{q-1} B_{(j+ip)}$, we conclude that $\alpha_i = \alpha_{p+i}$ for all $i \in \mathbb{Z}_{pq}$. □

Lemma 26

Let $\gamma(i,j) = (ai,\alpha_i j + b_i)$, where $a \in \mathbb{Z}_{pq}^*$, $\alpha_i \in \mathbb{Z}_r^*$ such that $\alpha_i = \alpha_{i+p}$ for every $i \in \mathbb{Z}_{pq}$, and $b_i \in \mathbb{Z}_r$. Let $H = \langle\langle\rho,\tau\rangle,\gamma^{-1}\langle\rho,\tau\rangle\gamma\rangle$, so that H admits $\mathcal{B}$. If a Sylow r-subgroup of $\mathrm{fix}_{H^{(2)}}(\mathcal{B})$ is $\langle\rho|_C : C \in \mathcal{C}\rangle$, and $x \geq 2$, then there exists $\delta \in H^{(2)}$ such that $\gamma\delta(i,j) = (ai,\alpha_i j)$.

PROOF. Fix $C \in \mathcal{C}$, and consider $K = \mathrm{fix}_H(\mathcal{B})|_C$. Let $B \in \mathcal{B}$ such that $B \subset C$, and define $\pi : K \to S_B$, by $\pi(h) = h|_B$. As a normal subgroup of a primitive group is transitive [41, Theorem 8.8] and a transitive group of prime degree r contains an r-cycle, it must be the case that $\mathrm{Ker}(\pi) = 1$. Otherwise, $\mathrm{Ker}(\pi)|_{B'} \neq 1$ and so contains an element of order r for some $B' \neq B$. But then $B' \not\equiv B$, a contradiction. Hence $K|_B \cong K|_{B'}$ for every $B, B' \in \mathcal{B}$ such that $B, B' \subset C$. As $K|_B \leq \mathrm{AGL}(1,p)$ for every $B \in \mathcal{B}$ as

$H \leq \mathbb{Z}_p \wr (\mathrm{AGL}(1,q) \wr \mathrm{AGL}(1,r)$ and K acts faithfully on each $B \in \mathcal{B}$ such that $B \subset C$, we have that $|K| = \ell r$, $\ell|(r-1)$. As $\tau^{pq} \neq 1$, we have that $\ell > 1$. As $\mathrm{AGL}(1,r)$ is solvable and $|\mathrm{AGL}(1,r)| = (r-1)r$, K is solvable so that any two subgroups of $K|_B$ of order $\ell|(r-1)$ are conjugate in $K|_B$. We conclude by [13, Lemma 1.6B] and the comments following that reference that the actions of $K|_B$ and $K|_{B'}$ are equivalent for every $B, B' \in \mathcal{B}$ with $B, B' \subset C$.

Now define an equivalence relation $\approx$ on C by $c \approx c'$ if and only if $\mathrm{Stab}_K(c) = \mathrm{Stab}_K(c')$, $c, c' \in C$. As conjugation by elements of $\mathrm{fix}_H(\mathcal{C})|_C$ permutes stabilizers of points in $\mathrm{fix}_H(\mathcal{B})|_C$, we see that the equivalence classes of $\approx$ are blocks of $\mathrm{fix}_H(\mathcal{B})|_C$. As $K|_B$ is equivalent to $K|_{B'}$ for every $B, B' \in \mathcal{B}$, each equivalence class of $\approx$ contains the same number of points, and as a transitive subgroup of $\mathrm{AGL}(1,r)$ that is not cyclic has stabilizer which fixes exactly one point, there are r equivalence classes $\approx$ each containing q points. That is, each equivalence class of $\approx$ contains one point from each block of $\mathcal{B}$ contained in C. In fact, an equivalence class of $\approx$ consists of those elements of C whose first coordinates is congruent modulo p and whose second coordinates are congruent modulo r.

Now let $h \in \mathrm{fix}_H(\mathcal{B})$, so that $h(i,j) = (i, \beta_i j + c_i)$, $\beta_i \in \mathbb{Z}_r^*$, $c_i \in \mathbb{Z}_r$, and if $i \equiv i' \pmod p$, then $\beta_i \equiv \beta_{i'} \pmod r$. As $\langle \rho|_B : B \in \mathcal{B}\rangle \leq \mathrm{Aut}(\Gamma)$, there exists $g \in \langle \rho|_B : B \in \mathcal{B}\rangle$ such that hg fixes some point of $(u, 0)$, where $u \in \mathbb{Z}_p$. As an equivalence class of $\approx$ consists of those elements of C whose first coordinates is congruent modulo p and whose second coordinates are congruent modulo r, we have that $hg(u,j) = (u, \beta_u j)$. We conclude that $h(i,j) = (i, \beta_i j + c_i)$, and if $i \equiv i' \pmod p$, then $c_i \equiv c_{i'} \pmod r$.

Consider

$$\gamma^{-1}\tau^{pq}\gamma(i,j) = \gamma^{-1}\tau^{pq}(ai,\alpha_i j+b_i) \quad (3.4.17)$$

$$= \gamma^{-1}(ai,\alpha^{pq}\alpha_i j+\alpha^{pq}b_i) \quad (3.4.18)$$

$$= (i,\alpha^{pq}j+\alpha_i^{-1}\alpha^{pq}b_i-\alpha_i^{-1}b_i) \quad (3.4.19)$$

As $\gamma^{-1}\tau^{pq}\gamma \in \mathrm{fix}_H(\mathcal{B})$, for $i \equiv i' \pmod p$, we have that $\alpha_i^{-1}\alpha^{pq}b_i - \alpha_i^{-1}b_i \equiv \alpha_{i'}^{-1}\alpha^{pq}b_{i'} - \alpha_{i'}^{-1}b_{i'} \pmod r$, and as $\alpha^{pq} \neq 1$, we conclude that $b_i \equiv b_{i'} \pmod r$. Then $\delta : V \to V$ by $\delta(i,j) = (i, j-\alpha_i^{-1}b_i)$ is in $\langle \rho|_C : C \in \mathcal{C}\rangle$ and $\gamma\delta(i,j) = (ai,\alpha_i j)$. □

Lemma 27

Let Γ be a (pq,r)-metacirculant digraph with $|\alpha| = p^x q^y$, $x \geq 2, y \geq 1$. Let G be the largest subgroup of $\mathrm{Aut}(\Gamma)$ that admits $\mathcal{B}$ as a complete block system and assume that a Sylow r-subgroup of $\mathrm{fix}_G(\mathcal{B})$ is $\langle \rho|_C\rangle$, $C \in \mathcal{C}$. Let $\gamma : V \to V$ by $\gamma(i,j) = (ai,\alpha_i j)$, $a \in \mathbb{Z}_p^*$, and $\alpha_i \in \mathbb{Z}_r^*$. If $\gamma^{-1}\langle\rho,\tau\rangle\gamma \leq \mathrm{Aut}(\Gamma)$ and Γ is not isomorphic to a (p,qr)-metacirculant or a (q,pr)-metacirculant, then $a = 1$.

PROOF. Towards a contradiction, assume that $a \neq 1$. Then $a \not\equiv 1 \pmod \ell$ for some prime $\ell|(pq)$. Let $\omega = \tau^{-a^{-1}}\gamma^{-1}\tau\gamma$, so that

$$\omega(i,j) = (i,\alpha^{1-a^{-1}}\alpha^{-1}_{i+a^{-1}}\alpha_i j).$$

Let $z = x$ if $\ell = p$ and $z = y$ if $\ell = q$. Set $\alpha = \beta_1\beta_2$, where $\beta_1 \in \mathbb{Z}_r$ has order ℓ^z and $\beta_2 \in \mathbb{Z}_r$ has order relatively prime to ℓ. Suppose that for some i', $\langle \alpha^{1-a^{-1}}\alpha^{-1}_{i'+a^{-1}}\alpha_{i'}\rangle \geq \langle\beta_1\rangle$. Let $t \in \mathbb{Z}$ such that $(\alpha^{1-a^{-1}}\alpha^{-1}_{i'+a^{-1}}\alpha_{i'})^t = \beta_1^{-1}$. Then $\omega \in \mathrm{fix}_H(\mathcal{B})$, and as $\mathrm{fix}_H(\mathcal{B})|_C \in$

$\text{Aut}(\Gamma)$ for every $C \in \mathcal{C}$, we have that $\delta : V \to V$ by $\delta(i,j) = (i, \alpha^{1-a^{-1}}\alpha^{-1}_{i'+a^{-1}}\alpha_{i'}j)$ is contained in $\text{Aut}(\Gamma)$. Then $\tau\delta^{-t}(i,j) = (i+1, \beta j)$, and β has multiplicative order b relatively prime to ℓ. Then $\tau^{-\ell}(\rho(\tau\delta^{-t})^b)\tau^{\ell} = (\rho\tau^b\delta^{-t})^c$ for some c, and $\langle\rho(\tau\delta^{-t})^b\rangle$ is semiregular of order ℓr. Thus in this case Γ is isomorphic to a $(pq/\ell, \ell r)$-metacirculant digraph as claimed.

Otherwise, set $\alpha^{-1}_{i+a^{-1}}\alpha_i = \iota_i\psi_i$, where ι_i has order a power of ℓ and ψ_i has order relatively prime to ℓ. Replace Γ with $\Delta(\Gamma)$, where $\Delta : V \to V$ by $\Delta(i,j) = (i, \alpha_0 j)$. With this replacement, we assume that $\alpha_0 = 1$ and the result will follow provided we obtain a contradiction. As $\langle\alpha^{1-a^{-1}}\alpha^{-1}_{a^{-1}}\rangle$ does not contain $\langle\beta_1\rangle$ and $\mathbb{Z}_r^*$ is cyclic, $\beta_1^{1-a^{-1}}\iota^{-1}_{a^{-1}} = \beta_1^{m_1\ell}$ for some $m_1 \in \mathbb{Z}$, and so $\iota_{a^{-1}} = \beta_1^{1-a^{-1}-m_1\ell}$. Note that $\tau^{pq}(i,j) = (i, \alpha^{pq}j)$ and $\beta_1^{\ell} \in \langle\alpha^{pq}\rangle$. As $\delta|_C \in \text{Aut}(\Gamma)$ for every $C \in \mathcal{C}$ and $\delta \in \text{fix}_H(\mathcal{B})$, as $\tau^{pq} \in \text{fix}_H(\mathcal{B})$, we conclude that the map ω defined by $(i,j) \to (i,j)$ if $i \neq a^{-1}$ and $(a^{-1}, j) \to (a^{-1}, \beta^{-m_1\ell}j)$ is contained in $\text{Aut}(\Gamma)$. Replacing γ with $\gamma\omega$, we may assume without loss of generality that $m_1 = 0$. As $\langle\alpha^{1-a^{-1}}\alpha^{-1}_{2a^{-1}}\alpha_{a^{-1}}\rangle$ does not contain $\langle\beta_1\rangle$, we conclude that $\iota_{2a^{-1}} = \beta_1^{2-2a^{-1}-m_2\ell}$ for some $m_2 \in \mathbb{Z}$. Arguing as above, we may assume without loss of generality that $m_2 = 0$. Continuing this argument inductively, we may assume that $\iota_{ia^{-1}} = \beta_1^{i(1-a^{-1})}$. As $\alpha_i = \alpha_{i+p}$ for all $i \in \mathbb{Z}_{pq}$, we have that $\beta^{p(1-a^{-1})} = \beta^0 = 1$, and so $p(1-a^{-1}) \equiv 0 \pmod{\ell^z}$. If $\ell = p$, then as $x \geq 2$ it must be the case that $p|(1-a^{-1})$. However, then $a \equiv 1 \pmod p$ and $\ell \neq p$. If $\ell = q$, then $q|(1-a^{-1})$ and so $a \equiv 1 \pmod q$, a contradiction as well. □

Lemma 28

Let $\gamma(i,j) = (i, \alpha_i j)$, where $\alpha_i \in \mathbb{Z}_r^*$ such that $\alpha_i = \alpha_{i+p}$ for every $i \in \mathbb{Z}_{pq}$. Let $H = \langle\langle\rho,\tau\rangle, \gamma^{-1}\langle\rho,\tau\rangle\gamma\rangle$, so that H admits $\mathcal{B}$. If a Sylow r-subgroup of $\text{fix}_{H^{(2)}}(\mathcal{B})$ is

$\langle\rho|_C : C \in \mathcal{C}\rangle$, where $\mathcal{B} \prec \mathcal{C}$ consists of p blocks of size qr formed by the orbits of $\langle\rho, \tau^p\rangle$, and $x \geq 2$, then there exists $\delta \in H^{(2)}$ such that $\gamma\delta(i,j) = (i, \alpha_0 j)$.

PROOF. As in Lemma 27, let $\omega = \tau^{-1}\gamma^{-1}\tau\gamma$, so that $\omega(i,j) = (i, \alpha_{i+1}^{-1}\alpha_i j)$. Let $\bar{\omega}_0 = \omega|_{C_0}$. Then for $i \equiv 0 \pmod p$, $\bar{\omega}_0(i,j) = (i, \alpha_1^{-1}\alpha_0 j)$, $\bar{\omega}_0^{-1}(i,j) = (i, \alpha_1\alpha_0^{-1}j)$, and $\bar{\omega}_0(i,j) = (i,j)$ if $i \not\equiv 0 \pmod p$. Hence for $i \equiv 0 \pmod p$, $\bar{\omega}_0\gamma^{-1}\tau\gamma\bar{\omega}_0^{-1}(i,j) = (i+1, \alpha j)$. Also, as $\bar{\omega}_0\gamma^{-1}\tau\gamma\bar{\omega}_0^{-1}(i-1,j) = (i, \alpha\alpha_1^{-1}\alpha_{-1}j)$, setting $\bar{\omega}_1 = \tau^{-1}\bar{\omega}_0\gamma^{-1}\tau\gamma\bar{\omega}_0^{-1}|_{C_{-1}}$, we can see that $\bar{\omega}_1(i-1,j) = (i-1, \alpha_1^{-1}\alpha_{-1}j)$, and $\bar{\omega}_1^{-1}(i-1,j) = (i-1, \alpha_1\alpha_{-1}^{-1}j)$, for $i \equiv 0 \pmod p$. Hence for $i \equiv 0 \pmod p$, $\bar{\omega}_1\bar{\omega}_0\gamma^{-1}\tau\gamma\bar{\omega}_0^{-1}\bar{\omega}_1^{-1}(i-1,j) = (i, \alpha j)$. Note that

$$\bar{\omega}_1\bar{\omega}_0\gamma^{-1}\tau\gamma\bar{\omega}_0^{-1}\bar{\omega}_1^{-1}(i-2,j) = (i-1, \alpha\alpha_1^{-1}\alpha_{i-2}j). \tag{3.4.20}$$

Continuing the above process, we find $\bar{\omega}_0, \bar{\omega}_1, \ldots, \bar{\omega}_{p-2}$ such that

$$\bar{\omega}_\ell = \tau^{-1}\bar{\omega}_{\ell-1}\cdots\bar{\omega}_0\gamma^{-1}\tau\gamma\bar{\omega}_0^{-1}\cdots\bar{\omega}_{\ell-1}^{-1}|_{C_{-\ell}}, \tag{3.4.21}$$

$$\bar{\omega}_\ell(-\ell,j) = (-\ell, \alpha_1^{-1}\alpha_{-\ell}j). \tag{3.4.22}$$

Setting $\delta_\ell = \bar{\omega}_\ell\cdots\bar{\omega}_0$, we see that

$$\delta_\ell\gamma^{-1}\tau\gamma\delta_\ell^{-1}(-i,j) = (-(i-1), \alpha j) \tag{3.4.23}$$

for $i = 0, \ldots, \ell - 1$,

$$\begin{aligned}
\delta_\ell \gamma^{-1} \tau \gamma \delta_\ell^{-1}(-\ell, j) &= \delta_\ell \gamma^{-1} \tau \gamma(-\ell, \alpha_1^{-1}\alpha_{-\ell} j) && (3.4.24)\\
&= \delta_\ell(-(\ell-1), \alpha\alpha_{-(\ell-1)}^{-1}\alpha_1 j) && (3.4.25)\\
&= (-(\ell-1), \alpha j), && (3.4.26)
\end{aligned}$$

and

$$\begin{aligned}
\delta_\ell \gamma^{-1} \tau \gamma \delta_\ell^{-1}(-(\ell+1), j) &= \delta_\ell \gamma^{-1} \tau \gamma(-(\ell+1), j) && (3.4.27)\\
&= \delta_\ell(-\ell, \alpha\alpha_{-\ell}^{-1}\alpha_{-(\ell+1)} j) && (3.4.28)\\
&= (-\ell, \alpha\alpha_1^{-1}\alpha_{-(\ell+1)}). && (3.4.29)
\end{aligned}$$

Note that if $\ell = p - 2$, then

$$\begin{aligned}
\delta_\ell \gamma^{-1} \tau \gamma \delta_\ell^{-1}(\ell, j) &= (-\ell, \alpha\alpha_1^{-1}\alpha_{-(\ell+1)}) && (3.4.30)\\
&= (2, \alpha j) && (3.4.31)
\end{aligned}$$

By Lemma 18, each $\bar{\omega}_\ell \in \mathrm{Aut}(\Gamma)$, and so $\delta = \delta_{p-2}^{-1} \in \mathrm{Aut}(\Gamma)$. Then $\delta^{-1}\gamma^{-1}\tau\gamma\delta = \tau$. Replacing γ with $\gamma\delta$, we may assume without loss of generality that $\gamma^{-1}\tau\gamma = \tau$. Then $\alpha_i = \alpha_j$ for all $i, j \in \mathbb{Z}_p$, and $\gamma(i, j) = (i, \alpha_0 j)$. □

Lemma 29

Let Γ be a (pq, r)-metacirculant digraph with $|\alpha| = p^x q^y$, $x, y \geq 1$, and $p < q < r$. Suppose that $\ell | pq$ is prime and V has been labeled with elements of $\mathbb{Z}_{pq/\ell} \times \mathbb{Z}_\ell \times \mathbb{Z}_r$ in such a way that $\tau(i,j,k) = (i+1, j+1, \alpha k)$ and $\rho(i,j,k) = (i,j,k+1)$. If there exists $\rho_1 \in \mathrm{Aut}(\Gamma)$ such that $\rho_1(i,j,k) = (i, j+1, k+e_{i,j})$, $e_{i,j} \in \mathbb{Z}_r$, and a Sylow r-subgroup of $\mathrm{fix}_H(\mathcal{B})$ has order r, then Γ is isomorphic to a $(pq/\ell, \ell r)$-metacirculant digraph for $\ell = p$ or q.

PROOF. Set $u = p^x$ if $\ell = p$ and $u = q^y$ if $\ell = q$. Let $v \in \mathbb{Z}$ such that $uv \equiv 1 \pmod{pq/\ell}$, and $\tau_1 = \tau^{uv}$. As $uv \equiv 0 \pmod{\ell}$ but $uv \equiv 1 \pmod{pq/\ell}$, we have that $\tau_1(i,j,k) = (i+1, j, \alpha^{uv}k)$. Observe that $\rho_1^\ell \in \mathrm{fix}_H(\mathcal{B})$, and has order a divisor of r. We conclude that $\langle \rho_1^\ell \rangle = \rho^d$ for some $d \in \mathbb{Z}_r$. Then $\tau_1^{-1}\rho\tau_1 \in \langle \rho \rangle$, and $\rho_1^{-1}\tau_1^{-1}\rho_1\tau_1 \in \mathrm{fix}_H(\mathcal{B})$ and has order r. As a Sylow r-subgroup of $\mathrm{fix}_H(\mathcal{B})$ has order r, we conclude that $\rho_1^{-1}\tau_1^{-1}\rho_1\tau_1 \in \langle \rho \rangle$, and so $\tau_1^{-1}\rho_1\tau = \rho_1\rho^f$ for some constant f. Finally, as

$$\rho_1\rho(i,j,k) = \rho_1(i,j,k+1) = (i, j+1, k+1+e_{i,j}) \tag{3.4.32}$$

$$= \rho(i, j+1, k+e_{i,j}) = \rho_1\rho(i,j,k), \tag{3.4.33}$$

ρ_1 commutes with ρ. As ρ_1 is semiregular of order ℓ and ρ is semiregular of order r, we have that $\tau_1^{-1}(\rho_1\rho)\tau_1 = (\rho_1\rho)^g$ for some g, and so Γ is isomorphic to a $(pq/\ell, \ell r)$-metacirculant digraph. □

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