1-1-2013

Positive Radial Solutions for P-Laplacian Singular Boundary Value Problems

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Positive radial solutions for p-Laplacian singular boundary value problems

By

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A Dissertation
Submitted to the Faculty of
Mississippi State University
in Partial Fulfillment of the Requirements
for the Degree of Doctor of Philosophy
in Mathematical Sciences
in the Department of Mathematics and Statistics

Mississippi State, Mississippi

August 2013

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Title of Study: Positive radial solutions for p-Laplacian singular boundary value problems

Pages of Study: 37

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In this dissertation, we study the existence and nonexistence of positive radial solutions for classes of quasilinear elliptic equations and systems in a ball with Dirichlet boundary conditions. Our nonlinearities are asymptotically p-linear at infinity and are allowed to be singular at zero with non-positone structure, which have not been considered in the literature.

In the one parameter single equation problem, we are able to show the existence of a positive radial solution with precise lower bound estimate for a certain range of the parameter.

We also extend the study to a class of asymptotically p-linear system with two parameters and in the presence of singularities. We establish the existence of a positive solution with a precise lower bound estimate when the product of the parameters is in a certain range.

Necessary and sufficient conditions for the existence of a positive solution are also obtained for both the single equation and system under additional assumptions. Our approach is based on the Schauder Fixed Point Theorem.

Key words: p-Laplacian, radial solutions, existence, nonexistence

ACKNOWLEDGEMENTS

First and foremost, I would like to thank my adviser Dr. Hai Dang for supervising this dissertation. I am very thankful and appreciative of his patience with introducing me to some new and exciting problems in partial differential equations. I have always been impressed by him as a teacher and his insight in mathematics. I will always be grateful that he was my adviser.

I would like to thank my committee members, Dr. Corlis Johnson, Dr. Robert Smith, Dr. Chuanxi Qian, and Dr. Xiangsheng Xu for their comments on this dissertation.

I am grateful for the faculty members at Mississippi State University (MSU) for providing me with a great learning experience. Also, I would like to thank my mentors Dr. Hyeona Lim, Mrs. Marjorie Crittenden, and Mrs. Lorraine Hughes for their professionalism and kind words. A special thanks to my department head Dr. Razzaghi and the associate department head Dr. Corlis Johnson for supporting my education as a graduate student here at MSU.

I am indebted to my parents Sandra Williams and Jessie Wheeler for their love and encouraging words. Lastly, I am content with my brothers, sisters, and friends who played a crucial role in my life. Thank God for putting the right people in my life, especially in the last year of my Ph.D program.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS iv

LIST OF SYMBOLS vi

CHAPTER

1. INTRODUCTION 1

1.1 History on the single equation 2
1.2 Statements of results for the single equation 5
1.3 History on the system of equations 8
1.4 Statements of results for the system 9

2. PRELIMINARIES 13

2.1 Preliminary Results 13

3. PROOF OF THEOREMS AND LEMMAS IN SECTION 1.2 16

3.1 Proof of Lemma 1 16
3.2 Proof of Lemma 2 18
3.3 Proof of Theorem 1 19
3.4 Proof of Theorem 2 22

4. PROOF OF THEOREMS AND LEMMAS IN SECTION 1.4 27

4.1 Proof of Theorem 3 27
4.2 Proof of Theorem 4 31

5. CONCLUSIONS 35

5.1 Further Plan 35

REFERENCES 36

LIST OF SYMBOLS

Ω : bounded domain in $\mathbb{R}^n$

$B(0,R) := \{x \in \mathbb{R}^n : \|x\| < R\}$

$\partial\Omega$: boundary of Ω

$\bar{\Omega}$: closure of Ω

$C[0,1] := \{u : [0,1] \to \mathbb{R} : u \text{ is continuous}\}$

$C^k[0,1] := \{u : [0,1] \to \mathbb{R} : u \text{ is k} - \text{times continuously differentiable}\}$

$C^{0,\alpha}(\bar{\Omega}) := \left\{u \in C(\bar{\Omega}) : \sup_{x\neq y} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha}} < \infty\right\} \Big(\alpha \in (0,1)$

$C^{1,\alpha}(\bar{\Omega}) := \{u \in C(\bar{\Omega}) : D^{\beta}u \in C^{0,\alpha}(\bar{\Omega})\ \forall\beta \text{ with } |\beta| = 1$

$\|u\|_{\infty} := \sup_{x\in[0,1]} |u(x)|$

$|u|_{1,\alpha} := \|u\|_{\infty} + \sup_{x\neq y} \frac{|\nabla u(x) - \nabla u(y)|}{|x-y|^{\alpha}}$

$L^q(\Omega) := \left\{u : \Omega \to \mathbb{R} : u \text{ is measurable and } \|u\|_q < \infty\right\} \Big(1 \leq q < \infty$

$L^q_{loc}(\Omega) := \{u : \Omega \to \mathbb{R} : u \text{ is measurable and } u \in L^q(K) \text{ for all compact set} K \subset \Omega\}$,
$1 \leq q < \infty$

$\|u\|_q := \left(\int_{\Omega} |u|^q\right)^{\frac{1}{q}}$

$W^{1,p}(\Omega) := \{u \in L^p(\Omega) : D^{\beta}u \in L^p(0,1)\ \forall\beta \leq 1$

$W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\}$

CHAPTER 1

INTRODUCTION

In this dissertation, we study the existence and nonexistence of positive radial solutions for classes of quasilinear elliptic equations and systems in a ball with Dirichlet boundary conditions. Our nonlinearities are asymptotically p-linear at infinity and are allowed to be singular at zero with non-positone structure, which have not been considered in the literature. Specifically , we consider the boundary value problems

$$\begin{cases} -\Delta_p u = h(u) + \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and

$$\begin{cases} -\Delta_p u_1 = h_1(u_2) + \mu_1 f_1(u_2) & \text{in } \Omega, \\ -\Delta_p u_2 = h_2(u_1) + \mu_2 f_2(u_1) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $p > 1, \Delta_p z :=\text{div}(|\nabla z|^{p-2}\nabla z)$, Ω is the open unit ball in $\mathbb{R}^n$, $h, f, h_i, f_i : (0,\infty) \to \mathbb{R}$ with f, f_i asymptotically p-linear at ∞, and λ, μ_i are positive parameters, $i = 1, 2$.

In the one parameter single equation problem (1.1), we are able to show the existence of a positive radial solution with precise lower bound estimate for a certain range of the parameter λ.

For the two-parameter system (1.2), we establish the existence of a positive solution with a precise lower bound estimate when the product of the parameters $\mu_1\mu_2$ is in a certain range. Necessary and sufficient conditions for the existence of a positive solution are also obtained for both the single equation (1.1) and system (1.2) under additional assumptions. Our results, to the best of our knowledge, are the first to deal with these classes of problems, which extended from previous work by Ambrosetti, Arcoya, and Buffoni [1], Ambrosetti and Hess [2], and Zhang [24].

Our plan is as follows: In Chapter 1, we introduce the history on the problems (1.1), and (1.2) and state our results. Chapter 2 includes some important properties of the p-Laplacian. We prove the results for the single equation in Chapter 3 and the results for the system in Chapter 4. I will make some concluding remarks in Chapter 5.

1.1 History on the single equation

The existence of a positive solution to (1.1) in the case $p = 2, h \equiv 0$, and $f : (0, \infty) \to \mathbb{R}$ is continuous and asymptotically linear was obtained by Ambrosetti and Hess [2] for $f(0) \geq 0$, and Ambrosetti et al. [1] for $f(0) < 0$. The results in [2, 1] were obtained via topological degree and bifurcation techniques. Let us briefly recall the literature concerning related singular problems. When $p = 2$, the problem

$$\begin{cases} -\Delta u = \frac{K(x)}{u^\beta} + \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\beta, q > 0, \lambda \geq 0$, arises in the study of nonNewtonian fluids and the theory of heat conduction in electrically conducting materials (see [9], [10]).

Such problems have been investigated by many authors. The case when $\lambda = 0$ and $K(x)$ is positive was studied by Crandall et al. [7], Del Pino [8], Fulks and Maybee [10], Gomes [11], and Lazer and McKenna [15]. The case when $\lambda \neq 0$ was studied in [5], [20], [21], [22], and [25]. In [25], assuming $K(x) \equiv -1$, Zhang proved the existence of a positive solution for (1.3) when $\beta, q \in (0,1)$, and λ is large enough. In [5], assuming $K(x) \equiv 1$, Coclite and Palmieri proved the existence of solutions $\forall \lambda \geq 0$ if $q \in (0,1)$, and, if $q \geq 1$, the existence of $\lambda^* > 0$ such that (1.3) has a solution for $\lambda \in [0, \lambda^*)$ and no solutions for $\lambda > \lambda^*$. Positive solutions of the general problem (1.3) were obtained by Shi and Yao [20] for $\beta, q \in (0,1)$, Stuart [21] for $\beta > 0, q < 1$, Sun et al. [22] for $\beta \in (0,1), 1 < q < \frac{n+2}{n-2}$. Note that, except for [5] where f is allowed to be linear, the above references concerning (1.4) dealt with the case when f is sublinear or superlinear at ∞.

Existence results for (1.1) when $p = 2, h \not\equiv 0$ and f is asymptotically linear and is allowed to be singular at 0 were established by the authors in [12], which extended a result by Zhang [24]. Note that the proof in [12] depends heavily on the linearity of the Laplace operator and cannot be applied for the general p-Laplacian (1.1). In this paper, we study the existence of positive radial solutions for (1.1) when Ω is the unit ball and f is asymptotically p-linear. Thus, we shall consider the ODE problem

$$\begin{cases} -(r^{n-1}A(u^{'}))^{'} = r^{n-1}(h(u) + \lambda f(u)) \qquad 0 < r < 1, \\ u^{'}(0) = 0, u(1) = 0, \end{cases} \tag{1.4}$$

where $A(z) = |z|^{p-2} z$. We shall prove in Theorem 1 below that (1.1) has a positive solution u_λ satisfying

$$u_\lambda \geq \left(\frac{\lambda \varepsilon_1}{\lambda_\infty - \lambda} \right)^{\frac{1}{p-1}} \phi_1 \qquad \text{in } (0,1),$$

when λ is sufficiently close to λ_∞ on the left, where $\lambda_\infty := \frac{\lambda_1}{m}$, ε_1 is a positive number, λ_1 is the first eigenvalue of $-\Delta_p u$ with Dirichlet boundary conditions and ϕ_1 is the associated positive eigenfunction with $\|\phi_1\|_\infty = 1$. In Theorem 2, we give conditions for (1.1) to have a positive solution for $\lambda < \lambda_\infty$, and no solution for $\lambda \geq \lambda_\infty$. In particular, our results when applied to the model case

$$\begin{cases} -\Delta_p u = \frac{a}{u^\beta} + \lambda \left(\frac{b}{u^\delta} + u^{p-1} e^{\frac{1}{u^\gamma + 1}} \right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a ball, $a, b \in \mathbb{R}$, $\beta, \delta \in (0,1)$, $\gamma \in (0, p-1]$, give the existence of a positive radial solution when λ is sufficiently close to λ_1 on the left, and, if $a, b > 0$, the existence of a positive radial solution if and only if $\lambda < \lambda_1$. Our approach is based on the Schauder Fixed Point Theorem. To be more precise, to prove Theorem 1, we look for solutions of (1.4) as fixed points of an associated compact operator in the set

$$\mathbf{K} = \{ \, v \in C[0,1] : c\phi_1 \leq v \leq M\phi_1 \text{ in } [0,1] \, \}$$

for suitable choices of positive numbers c and M.

To prove Theorem 2, we first show that for $\lambda < \lambda_\infty$ and suitable chosen $c, M > 0$, the modified problem

$$\begin{cases} -(r^{n-1}A(u'))' = r^{n-1}(h(\max(u, c\phi_1))) + \lambda f(\max(u, c\phi_1))) & 0 < r < 1, \\ u'(0) = 0, u(1) = 0, \end{cases}$$

has a solution u which is a fixed point of a compact operator on the set

$$\mathbf{C} = \{ \ v \in C[0,1] : v \leq M\phi_1 \text{ in } [0,1] \ \}$$

and then show that $u \geq c\phi_1$ in $(0,1)$. The non existence result when $\lambda \geq \lambda_\infty$ is proved with the aid of a strong comparison principle by Prashanth [18].

1.2 Statements of results for the single equation

We shall denote the norms in $L^p(0,1), C^1[0,1]$, and $C^{1,\alpha}[0,1]$, by $\|.\|_p$, $|.|_1$, and $|.|_{1,\alpha}$ respectively. We make the following assumptions:

$(A1)$ $f, h : (0, \infty) \to \mathbb{R}$ are continuous.

$(A2)$ There exists $\varepsilon_0, a, m > 0$ such that

$$\lim_{u \to \infty} \frac{f(u)}{u^{p-1}} = m$$

and

$$f(u) \geq mu^{p-1} + \varepsilon_0 \qquad \forall u > a.$$

$(A3)$ There exists $\alpha, \delta \in (0,1)$ and $k_0 > 0$ such that

$$|h(u)| \leq k_0 u^{-\alpha} \qquad \forall u > 0$$

and

$$\limsup_{u \to 0^+} u^\delta |f(u)| < \infty.$$

Note that from a result by Brock [4], ϕ_1 is radially symmetric and decreasing, and thus solves

$$\begin{cases} -(r^{n-1}A(\phi_1'))' = \lambda_1 r^{n-1}\phi_1^{p-1} & 0 < r < 1, \\ \phi_1'(0) = 0, \phi_1(1) = 0. \end{cases}$$

By a positive solution of (1.4), we mean a function $u \in C^1[0,1]$ with $u > 0$ on $[0,1)$ such that $r^{n-1}A(u')$ is differentiable on $(0,1)$ and satisfies (1.4).

Let $\lambda_\infty := \frac{\lambda_1}{m}$. Our main results are the following:

Theorem 1

Let $(A1)-(A3)$ hold. Then there exist a positive number ε such that for $\lambda \in (\lambda_\infty - \varepsilon, \lambda_\infty)$, problem (1.4) has a positive solution $u_\lambda \in C^{1,\beta}[0,1]$ for some $\beta \in (0,1)$. Furthermore,

$$u_\lambda \geq \left(\frac{\lambda\varepsilon_0}{2m(\lambda_\infty - \lambda)}\right)^{\frac{1}{p-1}} \phi_1 \qquad \text{in } (0,1).$$

Theorem 2

(i) Let $(A1), (A3)$ hold and $f, h \geq 0$. Suppose there exists $m \in (0,\infty)$ such that

$$\limsup_{u\to\infty} \frac{f(u)}{u^{p-1}} = m,$$

and

$$\liminf_{u\to 0^+} h(u) > 0.$$

Then for $\lambda < \lambda_\infty$, (1.4) has a positive solution $u_\lambda \in C^{1,\beta}[0,1]$ for some $\beta \in (0,1)$.

If, in addition,

$$f(u) \geq mu^{p-1} \qquad \forall u > 0,$$

then (1.4) *has no positive solutions for* $\lambda \geq \lambda_\infty$.

(ii) *Let* $(A1)$ *hold and suppose* $h \leq 0$ *and there exists* $m \in (0, \infty)$ *such that*

$$f(u) \leq mu^{p-1} \qquad \forall u > 0.$$

Then (1.4) *has no positive solutions for* $\lambda < \lambda_\infty$.

Remark 1. Theorem 2 (ii) shows that Theorem 1 may not be true if $\varepsilon_0 = 0$ in $(A2)$.

Lemma 1

Let $q > 1$. *Then there exists* $\beta \in (0, 1)$ *such that for each* $g \in L^q(0, 1)$, *the problem*

$$\begin{cases} -(r^{n-1}A(u'))' = r^{n-1}g, & 0 < r < 1, \\ u'(0) = 0, u(1) = 0 \end{cases} \tag{1.5}$$

has a unique solution $u = Kg \in C^{1,\beta}[0, 1]$. *Furthermore, there exists a constant* $C > 0$ *independent of* g *such that*

$$|u|_{1,\beta} \leq C \|g\|_q^{\frac{1}{p-1}} \tag{1.6}$$

and the operator $K : L^q(0, 1) \to C^1[0, 1]$ *is compact.*

Lemma 2

Let f *satisfy* $(A1) - (A3)$. *Then there exists a positive number* M_1 *such that*

$$f(u) \geq mu^{p-1} + \varepsilon_0 - M_1 u^{-\delta} \tag{1.7}$$

for all $u > 0$.

1.3 History on the system of equations

Let us briefly look at the literature on equations and systems with asymptotically p-linear nonlinearities. When $p = 2$, Peng and Yang [17] studied the nonsingular system

$$\begin{cases} -\Delta u + \mu\Delta v = g(x, v) & \text{in } \Omega, \\ -\Delta v + \lambda\Delta u = f(x, u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\lambda, \mu \geq 0$, g, f are asymptotically linear at ∞. They proved that (1.8) has a positive solution $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$ provided that

$$0 \leq \lambda\mu < 1 \quad \text{and} \quad \lambda_1 < \frac{\mathrm{m}\lambda + \mu\mathrm{l} + \sqrt{(\mathrm{m}\lambda - \mu\mathrm{l})^2 + 4\mathrm{ml}}}{2(1 - \lambda\mu)}, \tag{1.9}$$

where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions,

$$\lim_{t\to\infty} \frac{f(x,t)}{t} = l > 0, \lim_{t\to\infty} \frac{g(x,t)}{t} = m > 0$$

uniformly in $x \in \Omega$, f, g are positive and $\frac{f(x,t)}{t}$, $\frac{g(x,t)}{t}$ are nondecreasing in $t > 0$, among other assumptions. Related results for the case when $\lambda, \mu < 0$ and $\lambda\mu > 1$ were obtained in [16]. In the case of (1.8) with $\lambda = \mu = 0$ Hai [13] establish the existence of a positive solution (u, v) for (1.8) when $\lambda_1^2 > ml$. Note that when $\lambda = \mu = 0$, condition (1.9) becomes $\lambda_1^2 < ml$. The precise lower bound estimates on u, v, which show that $u, v \to \infty$ uniformly on compact subsets of Ω as $ml \to \lambda_1^2$ are also obtained in [13]. Related results on the singular equation case can be found in [1, 2] for the nonsingular case and in [6] for the singular case. We refer to [4] for the nonexistence of ground state solutions for a semilinear system of d equations, which, for $d = 2$, is similar the system considered in

[13]. In this paper, we are interested in extending the results in [13] to radial solutions for (1.2) when f_i are asymptotically p-linear at ∞ and are possibly singular at 0. Note that the proofs in [13] rely on the linearity of the Laplacian and do not carry over to the p-Laplacian with $p > 1$. Related results for the single equation can be found in [14] and the references therein. Our approach is based on the Schauder Fixed Point Theorem. In particular, our results when applied to the model case

$$\begin{cases} -\Delta_p u_1 = \frac{a_1}{u_2^{\beta}} + \lambda \left(\frac{b_1}{u_1^{\delta}} + u_1^{p-1} e^{\frac{1}{u_1^{\gamma}+1}} \right) & \text{in } \Omega, \\ -\Delta_p u_2 = \frac{a_2}{u_1^{\beta}} + \mu \left(\frac{b_2}{u_2^{\delta}} + u_2^{p-1} e^{\frac{1}{u_2^{\gamma}+1}} \right) & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.10}$$

where Ω is a ball, $a_i, b_i \in \mathbb{R}$, for $i = 1, 2$, $\beta, \delta \in (0, 1)$, $\gamma \in (0, p-1]$, give the existence of a positive radial solution when $\lambda\mu$ is sufficiently close to λ_1^2 on the left, and, if $a_i, b_i > 0$ for $i = 1, 2$,$p \geq 2$, the existence of a positive radial solution if and only if $\lambda < \lambda_1$.

1.4 Statements of results for the system

Since we are looking for radial solutions of (1.2), we shall consider the ODE system

$$\begin{cases} -(r^{n-1} A(u_1'))' = r^{n-1}(h_1(u_2) + \mu_1 f_1(u_2)) & 0 < r < 1, \\ -(r^{n-1} A(u_2'))' = r^{n-1}(h_2(u_1) + \mu_2 f_2(u_1)) & 0 < r < 1. \\ u_1'(0) = u_2'(0) = u_1(1) = u_2(1) = 0, \end{cases} \tag{1.11}$$

where $A(z) = |z|^{p-2}\, z$.

We shall make the following assumptions:

$(H1)$ $f_i, h_i : (0, \infty) \to \mathbb{R}$ are continuous for $i = 1, 2$, and there exist numbers $\alpha, \delta \in (0, 1)$ and $k_0 > 0$ such that

$$|h_i(t)| \leq k_0 t^{-\alpha}$$

for $t > 0$, and

$$\limsup_{t \to 0^+} t^{\delta} |f_i(t)| < \infty$$

for $i = 1, 2$.

$(H2)$ There exist positive numbers k, A, m_i, $i = 1, 2$, such that

$$\lim_{t \to \infty} \frac{f_i(t)}{t^{p-1}} = m_i$$

and

$$f_i(t) \geq m_i t^{p-1} + k$$

for $t \geq A, i = 1, 2$.

By a positive solution of (1.11), we mean a function $u = (u_1, u_2)$ with $u_i \in C^{1,\beta}[0, 1]$ for some $\beta \in (0, 1)$ and with $u_i > 0$ on $[0, 1)$ such that $r^{n-1}A(u_i^{'})$ is differentiable on $(0, 1)$ and satisfies (1.11).

Let $\mu_\infty := \frac{\lambda_1^2}{m_1 m_2}$. Our main results are the following:

Theorem 3

Let $(H1), (H2)$ hold. Then there exist positive numbers ε, α with $\alpha \in (0, 1)$ such that for $\mu_1\mu_2 \in (\mu_\infty - \varepsilon, \mu_\infty)$, the system (1.11) has a positive solution $u = (u_1, u_2)$ with

$$u_i \geq \frac{k\mu_1\mu_2}{4(\mu_\infty - \mu_1\mu_2)}\phi_1 \qquad \text{in } (0, 1)$$

for $i = 1, 2$.

Theorem 4

(i) Let $(H1)$ hold and let $f_i \geq 0, h_i > 0, i = 1, 2$. Suppose there exist constants $m_i > 0$ such that

$$\limsup_{t \to \infty} \frac{f_i(t)}{t^{p-1}} = m_i,$$

$$\liminf_{t\to 0^+} h_i(t) > 0,$$

and

$$\lim_{t\to\infty} t^{p-1} h_i^{\frac{\gamma}{p-1}}(t) = \infty$$

for $i = 1, 2$*, where* $\gamma = \max(\alpha, \delta)$*.*

Then system (1.11) *has a positive solution for* $\mu_1\mu_2 < \mu_\infty$*. If in addition,*

$$f_i(t) \geq m_i t^{p-1}$$

for $i = 1, 2, t > 0$*, then* (1.11) *has no positive solution for* $\mu_1\mu_2 \geq \mu_\infty$*.*

(ii) *Let* $(H1)$ *hold. Suppose* $h_i \leq 0$ *and there exist constants* $m_i > 0$ *such that*

$$f_i(t) \leq m_i t^{p-1}$$

for all $t > 0, i = 1, 2$*. Then* (1.11) *has no positive solutions for* $\mu_1\mu_2 < \mu_\infty$*.*

Remark 2. (i) Theorem 4 (ii) shows that Theorem 3 may not be true if $k = 0$ in $(H2)$.

(ii) From the proof of Theorem 4, we see that the conclusion of Theorem 4 holds when $h_i \equiv 0$ for $i = 1, 2$.

The proofs of Theorems 3 and 4 can be applied to the $n \times n$ system

$$\begin{cases} -(r^{n-1}A(u_i'))' = r^{n-1}(h_i(u_{i+1}) + \mu_i f_1(u_{i+1})) & 0 < r < 1, \\ u_i'(0) = u_i(1) = 0, \end{cases} \tag{1.12}$$

where $1 \leq i \leq n$ and $u_{n+1} \equiv u_1$. We state without proofs the following results:

Let $\mu_\infty := \frac{\lambda_1^n}{m_1 \ldots m_n}$.

Theorem 5

Let $(H1), (H2)$ hold with $i = 1, ..., n$. Then there exist positive numbers ε, α with $\alpha \in (0, 1)$ such that for $\mu_1, ..., \mu_n \in (\mu_\infty - \varepsilon, \mu_\infty)$, the system (1.12) has a positive solution $u = (u_1, ..., u_n)$ with

$$u_i \geq \frac{k\mu_1...\mu_n}{4(\mu_\infty - \mu_1...\mu_n)}\phi_1 \qquad \text{in } (0, 1)$$

for $i = 1, ..., n$.

Theorem 6

(i) Let $(H1)$ hold and let $f_i \geq 0, h_i > 0, i = 1, ..., n$. Suppose there exist constants $m_i > 0$ such that

$$\limsup_{t\to\infty} \frac{f_i(t)}{t^{p-1}} = m_i,$$

$$\liminf_{t\to 0^+} h_i(t) > 0,$$

and

$$\lim_{t\to\infty} t^{p-1} h_i^{\frac{\gamma}{p-1}}(t) = \infty$$

for $i = 1, ..., n$, where $\gamma = \max(\alpha, \delta)$. Then system (1.12) has a positive solution for $\mu_1...\mu_n < \mu_\infty$. If in addition,

$$f_i(t) \geq m_i t^{p-1}$$

for $i = 1, ..., n, t > 0$, then (1.12) has no positive solution for $\mu_1...\mu_n \geq \mu_\infty$.

(ii) Let $(H1)$ hold and suppose $h_i \leq 0$ and there exist constants $m_i > 0$ such that

$$f_i(t) \leq m_i t^{p-1}$$

for all $t > 0, i = 1, ..., n$. Then (1.12) has no positive solutions for $\mu_1...\mu_n < \mu_\infty$.

CHAPTER 2

PRELIMINARIES

In this chapter, we list some important results regarding the p-Laplacian operator such as the weak comparison principle, the strong maximum principle, the properties of the first eigenvalue, the strong comparison principle for radial solutions in a ball, and the radial symmetry of positive solutions in a ball.

2.1 Preliminary Results

Lemma 3 (Weak Comparison Principle, see [19])

Let $u_1, u_2 \in W^{1,p}(\Omega)$ *satisfy*

$$-\Delta_p u_1 \leq -\Delta_p u_2 \qquad \text{in } \Omega$$

in the weak sense i.e.,

$$\int_\Omega |\nabla u_1|^{p-2} \nabla u_1 \cdot \nabla \psi dx \leq \int_\Omega |\nabla u_2|^{p-2} \nabla u_2 \cdot \nabla \psi dx$$

for all non-negative $\psi \in W_0^{1,p}(\Omega)$ *with* $\psi \geq 0$.

Then the inequality

$$u_1 \leq u_2 \qquad \text{on } \partial\Omega$$

implies that

$$u_1 \leq u_2 \qquad \text{in } \Omega.$$

Lemma 4 (Strong Maximum Principle, see [23])

Let $u \in C^1(\bar{\Omega})$ *and satisfy*

$$\begin{cases} -\Delta_p u = f \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^2_{loc}(\Omega)$, $f \geq 0$ *in* Ω, $f \not\equiv 0$.

Then $u > 0 \quad$ *in* Ω *and*

$$\frac{\partial u}{\partial \nu} < 0 \quad \text{on } \partial\Omega,$$

where ν *denotes the unit exterior normal vector on* $\partial\Omega$.

Lemma 5 ([3])

The first eigenvalue λ_1 *of*

$$\begin{cases} -\Delta_p u = \lambda \left|u\right|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is positive, simple, and is the unique eigenvalue having a positive eigenfunction $\phi_1 \in C^1(\bar{\Omega})$.

The next two results are valid when Ω is a ball $B \equiv B(0, R)$.

Lemma 6 ([4])

Let $u \in C^1(\bar{B})$ *satisfy*

$$\begin{cases} -\Delta_p u = f(\left|x\right|, u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases}$$

where $f \in C([0,\infty) \times [0,\infty))$, $f(r,u)$ is nonincreasing in r. Suppose that $f(r,v) > 0$ for all $r \geq 0, v > 0$ and $u > 0$ in B. Then u is radially symmetric and radially decreasing i.e.

$$u = u(r), \qquad (r = |x|).$$

and

$$\frac{\partial u}{\partial r}(x) < 0 \quad \text{in } B \setminus \{0\}\,.$$

Lemma 7 (Strong Comparison Principle, see [18])

Let $b : \mathbb{R} \to \mathbb{R}$ be a nondecreasing continuous function and $\beta \in (0,1)$. Let $u_1, u_2 \in C^{1,\beta}(\bar{B})$ be two radial solutions of

$$-\Delta_p u_i = b(u_i) + f_i \quad \text{in } B,$$

where $f_i \in L^1_{loc}(B), i = 1,2$, with $f_1 \leq f_2$ in B and $f_1 \not\equiv f_2$. Then

$$u_1 \leq u_2 \quad \text{in } B \implies u_1 < u_2 \quad \text{in } B.$$

Furthermore, the following Hopf lemma holds: If $u_1(R) = u_2(R)$ then $u_1'(R) > u_2'(R)$.

CHAPTER 3

PROOF OF THEOREMS AND LEMMAS IN SECTION 1.2

3.1 Proof of Lemma 1

Let $g \in L^q(0,1)$ and let u be a solution of (1.5). By integrating, it follows that (1.5) has a unique solution u, given by

$$u(r) = \int_r^1 A^{-1}\left(\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}g(\tau)d\tau\right)ds \qquad \mathrm{r} \in [0,1], \tag{3.1}$$

where $A^{-1}(z) := |z|^{\tilde{p}-1}z, \tilde{p} := \frac{p}{p-1}$, and

$$(Ku)(r) := \int_r^1 A^{-1}\left(\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}g(\tau)d\tau\right)ds \qquad \mathrm{r} \in [0,1].$$

Note that, by Hölder's inequality,

$$\begin{aligned}\int_0^r \tau^{n-1}g(\tau)d\tau \ &\leq \left(\int_0^r \tau^{(n-1)q^*}g(\tau)d\tau\right)^{\frac{1}{q^*}}\left(\int_0^r |g(\tau)|^q\, d\tau\right)^{\frac{1}{q}} \\ &\leq \frac{r^{\frac{(n-1)q^*+1}{q^*}}}{((n-1)q^*+1)^{\frac{1}{q^*}}}\|g\|_q \leq r^{\frac{(n-1)q^*+1}{q^*}}\|g\|_q\,,\end{aligned} \tag{3.2}$$

and

$$\int_{r_1}^{r_2} |g(\tau)|\, d\tau \leq \|g\|_q\, |r_2 - r_1|^{\frac{1}{q^*}} \tag{3.3}$$

for $r_1 < r_2$, where $q^* := \frac{q}{q-1}$. Since

$$u^{'}(r) = -A^{-1}\left(\frac{1}{r^{n-1}}\int_0^r \tau^{n-1}g(\tau)d\tau\right) \tag{3.4}$$

it follows that

$$u'(r) \le A^{-1}\left(r^{\frac{1}{q^*}}\|g\|_q\right)$$

for $r \in (0,1)$, which implies $u' \in C[0,1]$.

We shall verify that u' is Hölder continuous on $[0,1]$. Let $r_1, r_2 \in (0,1)$ with $r_1 < r_2$. By the Mean Value Theorem and $(3.2) - (3.3)$,

$$\begin{aligned} \frac{1}{r_2^{n-1}}\int_0^{r_2}\tau^{n-1}g(\tau)d\tau - \frac{1}{r_1^{n-1}}\int_0^{r_1}\tau^{n-1}g(\tau)d\tau \quad &= \\ \int_{r_1}^{r_2}\left\{\,(1-n)\frac{1}{r^n}\int_0^{r}\tau^{n-1}g(\tau)d\tau + g(r)\,\right\}dr \\ \le \left[(n-1)\int_{r_1}^{r_2}\frac{1}{r^{1-\frac{1}{q^*}}} + |r_2-r_1|^{\frac{1}{q^*}}\right]\|g\|_q \\ \le C_0\,|r_2-r_1|^{\frac{1}{q^*}}\,\|g\|_q \end{aligned} \tag{3.5}$$

where $C_0 := (n-1)q^* + 1$. Here, we have used the inequality

$$r_2^{\frac{1}{q^*}} - r_1^{\frac{1}{q^*}} \le |r_2-r_1|^{\frac{1}{q^*}}.$$

Suppose $p > 2$. Then $\tilde{p} < 2$ and so A^{-1} is concave on $[0,\infty)$. This, together with the fact that A^{-1} is odd and increasing on $\mathbb{R}$, implies

$$A^{-1}(z_1) - A^{-1}(z_2) \le 2A^{-1}(|z_1-z_2|) = 2\,|z_1-z_2|^{\frac{1}{p-1}}$$

for all $z_1, z_2 \in \mathbb{R}$. Hence it follows from (3.4) and (3.5) that

$$u'(r_2) - u'(r_1) \le 2C_0^{\frac{1}{p-1}}\,|r_2-r_1|^{\frac{1}{q^*(p-1)}}\,\|g\|_q^{\frac{1}{p-1}}. \tag{3.6}$$

Suppose next that $p \le 2$. Then $\tilde{p} \ge 2$ and A^{-1} is differentiable on $\mathbb{R}$. Using (3.2), we get

$$\frac{1}{r^{n-1}}\int_0^{r}\tau^{n-1}g(\tau)d\tau \le r^{\frac{1}{q^*}}\,\|g\|_q \le \|g\|_q \quad \text{for } r \in (0,1) \tag{3.7}$$

By the Mean Value Theorem,

$$A^{-1}(z_1) - A^{-1}(z_2) \ \leq (\tilde{p}-1)\,|z_1 - z_2|\,|z|^{\tilde{p}-2},$$

where z is between z_1 and z_2. Hence it follows from (3.5) and (3.7) that

$$u^{'}(r_2) - u^{'}(r_1) \ \leq C_0(\tilde{p}-1)\,|r_2 - r_1|^{\frac{1}{q^*}}\,\|g\|_q^{\frac{1}{\tilde{p}-1}}. \tag{3.8}$$

Combining (3.6) and (3.8), we see that $u \in C^{1,\beta}[0,1]$, where $\beta = \frac{1}{max(p-1,1)}$, and satisfies (1.6).

Since $C^{1,\beta}[0,1]$ is compactly imbedded in $C^1[0,1]$, we need only to verify that K is continuous to conclude the proof. Let (g_n) be a sequence in $L^q(0,1)$ such that $g_n \to g$ in $L^q(0,1)$, and let $u_n = Kg$. Using (3.7), we obtain

$$\frac{1}{r^{n-1}}\int_0^r \tau^{n-1} g_n(\tau)d\tau \ \leq \|g_n\|_q,$$

and

$$\frac{1}{r^{n-1}}\int_0^r \tau^{n-1}(g_n(\tau) - g(\tau))d\tau \ \leq \|g_n(\tau) - g(\tau)\|_q,$$

for $r \in (0,1)$, which implies $\ u_n^{'} - u^{'} \ _\infty \to 0$ as $n \to \infty$, i.e., (u_n) converges to u in $C^1[0,1]$. This completes the proof of Lemma 1. ■

3.2 Proof of Lemma 2

In view of $(A2)$, we need only to show (1.7) for $u \leq a$. By $(A1)$ and $(A3)$, there exists $M_0 > 0$ such that

$$|f(u)| \leq M_0 u^{-\delta} \quad \text{for } r \in (0,1). \tag{3.9}$$

Let $M_1 > 0$ be such that

$$M_1 - M_0 = (ma^{p-1} + \varepsilon_0)a^{\delta}.$$

Then, for $u \leq a$,

$$(M_1 - M_0)u^{\delta} \geq (M_1 - M_0)a^{\delta} \geq ma^{p-1} + \varepsilon_0,$$

which implies

$$f(u) \geq -M_0 u^{-\delta} \geq mu^{p-1} + \varepsilon_0 - M_1 u^{-\delta},$$

and Lemma 2 follows. ■

3.3 Proof of Theorem 1

Let $\lambda < \lambda_\infty$ and

$$c = \left(\frac{\lambda \varepsilon_0}{2m(\lambda_\infty - \lambda)}\right)^{\frac{1}{p-1}}.$$

Let $M > c$ be a large number to be specified later and define

$$\mathbf{K} = \{\, v \in C[0,1] : c\phi_1 \leq v \leq M\phi_1 \text{ in } [0,1] \,\}\,.$$

For each $v \in \mathbf{K}$, let $u = Tv$ be the solution of

$$\begin{cases} -(r^{n-1}A(u'))' = r^{n-1}(h(v) + \lambda f(v)) & 0 < r < 1, \\ u'(0) = 0, u(1) = 0. \end{cases} \tag{3.10}$$

Let $q > 1$ be such that $q\max(\alpha, \delta) < 1$. We first verify that the mapping $L : \mathbf{K} \to L^q(0,1)$ defined by $Lv = h(v) + \lambda f(v)$ is bounded and continuous. By $(A1)$ and $(A3)$, there exists $k_M > 0$ such that

$$|f(z)| \leq k_M z^{-\delta}$$

for $z \in (0, M]$, which implies

$$|Lv| \leq k_0(c\phi_1)^{-\alpha} + \lambda k_M(c\phi_1)^{-\delta} \equiv \tilde{g} \in L^q(0,1)$$

for all $v \in \mathbf{K}$. Hence L is bounded and by the Lebesgue Dominated Convergence Theorem, L is continuous. Since $T = K \circ L$, where K is defined in Lemma 1, we see that $T : \mathbf{K} \to C^1[0,1]$ is a compact operator. We next show that $T : \mathbf{K} \to \mathbf{K}$ if λ is sufficiently close to λ_∞. Let $v \in \mathbf{K}$ and $u = Tv$. Using $(A3)$ and Lemma 2, we obtain

$$\begin{aligned}\int_0^s \tau^{n-1}(h(v) + \lambda f(v))d\tau &\geq \int_0^s \tau^{n-1}(-k_0 v^{-\alpha} + \lambda(m v^{p-1} + \varepsilon_0 - M_1 v^{-\delta}))d\tau \\ &\geq \int_0^s \tau^{n-1}\left(\left(-\frac{k_0}{(c\phi_1)^\alpha} - \frac{\lambda M_1}{(c\phi_1)^\delta} + \lambda m\left((c\phi_1)^{p-1} + \frac{\varepsilon_0}{m}\right)\right)\right) d\tau \\ &= \lambda m \int_0^s \tau^{n-1}\left((c\phi_1)^{p-1} + \frac{\varepsilon_0}{2m}\right) d\tau + \int_0^s \tau^{n-1}\left(\frac{\lambda\varepsilon_0}{2} - \frac{k_0}{(c\phi_1)^\alpha} - \frac{\lambda M_1}{(c\phi_1)^\delta}\right) d\tau. \quad (3.11)\end{aligned}$$

For $\gamma \in (0,1)$,

$$\begin{aligned}\lim_{s\to 0^+} s^{-n}\int_0^s \frac{\tau^{n-1}}{\phi_1^\gamma} d\tau &= \lim_{s\to 0^+} \frac{\int_0^s \frac{\tau^{n-1}}{\phi_1^\gamma} d\tau}{s^n} \\ &= \lim_{s\to 0^+} \frac{\frac{s^{n-1}}{\phi_1^\gamma(s)}}{n s^{n-1}} \\ &= \frac{1}{n}\lim_{s\to 0^+} \frac{s^{n-1}}{s^{n-1}\phi_1^\gamma(s)} \\ &= \frac{1}{n}\lim_{s\to 0^+} \frac{1}{\phi_1^\gamma(s)} \\ &= \frac{1}{n}\frac{1}{\phi_1^\gamma(0)} < \infty.\end{aligned}$$

Since ϕ_1 is continuous, $\phi_1(0)$ exists, and $\phi_1(s) \to \phi_1(0)$ as $s \to 0^+$.

Hence

$$s^{-n}\int_0^s \frac{\tau^{n-1}}{\phi_1^\gamma} d\tau$$

is bounded on $(0,1)$, there exists $\varepsilon > 0$ so that whenever $\lambda_\infty - \lambda < \varepsilon$,

$$\frac{k_0}{c^\alpha}\int_0^s \frac{\tau^{n-1}}{\phi_1^\alpha}d\tau + \frac{\lambda M_1}{c^\delta}\int_0^s \frac{\tau^{n-1}}{\phi_1^\delta}d\tau < \frac{\lambda\varepsilon_0}{2n}s^{-n}$$

for $s \in (0,1)$, which we shall assume. Hence

$$\int_0^s \tau^{n-1}\left(\frac{\lambda\varepsilon_0}{2} - \frac{k_0}{(c\phi_1)^\alpha} - \frac{\lambda M_1}{(c\phi_1)^\delta}\right)d\tau > 0 \tag{3.12}$$

for $s \in (0,1)$. Since $\|\phi_1\|_\infty = 1$, we obtain from the choice of c that

$$\lambda\left((c\phi_1)^{p-1} + \frac{\varepsilon_0}{2m}\right) \geq \lambda_\infty (c\phi_1)^{p-1}. \tag{3.13}$$

Combining $(3.11) - (3.13)$, we get

$$\begin{aligned}\int_0^s \tau^{n-1}(h(v)+\lambda f(v))d\tau &\geq \lambda m \int_0^s \tau^{n-1}\left((c\phi_1)^{p-1} + \frac{\varepsilon_0}{2m}\right)d\tau \\ &\geq \lambda_1 \int_0^s \tau^{n-1}(c\phi_1)^{p-1}d\tau\end{aligned}$$

which implies

$$\begin{aligned}u(r) &= \int_r^1 A^{-1}\left(\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}(h(v)+\lambda f(v))d\tau\right)ds \\ &\geq \int_r^1 A^{-1}\left(\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}(c\phi_1)^{p-1}d\tau\right)ds \\ &= c\phi_1(r),\end{aligned}$$

for $r \in (0,1)$. Next, choose $b > 1$ so that $\lambda b < \lambda_\infty$. By $(A2)$ and $(A3)$, there exists $d > 0$ such that

$$|f(v)| \leq bmv^{p-1} + dv^{-\delta} \tag{3.14}$$

for $v > 0$, which implies

$$\int_0^s \tau^{n-1}(h(v)+\lambda f(v))d\tau \leq \int_0^s \tau^{n-1}(k_0 v^{-\alpha} + \lambda(bmv^{p-1} + dv^{-\delta}))d\tau$$

$$\leq \lambda bmM^{p-1}\int_0^s \tau^{n-1}\phi_1^{p-1}d\tau + \int_0^s \tau^{n-1}\left\{k_0(c\phi_1)^{-\alpha} + \lambda d(c\phi_1)^{-\delta}\right\} d\tau \tag{3.15}$$

for $s \in (0,1)$. Choose $M > c$ large enough so that

$$(\lambda_1 - \lambda bm)M^{p-1}\int_0^s \tau^{n-1}\phi_1^{p-1}d\tau \geq \int_0^s \tau^{n-1}\left\{k_0(c\phi_1)^{-\alpha} + \lambda d(c\phi_1)^{-\delta}\right\} d\tau \tag{3.16}$$

for $s \in (0,1)$, which is possible since for $\gamma \in (0,1)$, there exist $k_\gamma > 0$ so that

$$\int_0^s \frac{\tau^{n-1}}{\phi_1^{\gamma}}d\tau \leq k_\gamma \int_0^s \tau^{n-1}\phi_1^{p-1}d\tau$$

for $s \in (0,1)$. Combining $(3.15)-(3.16)$, we get

$$\int_0^s \tau^{n-1}(h(v)+\lambda f(v))d\tau \leq \lambda_1 \int_0^s \tau^{n-1}(M\phi_1)^{p-1}d\tau,$$

which implies $u \leq M\phi_1$ in $(0,1)$. Thus $T : \mathbf{K} \to \mathbf{K}$ and by the Schauder Fixed Point Theorem, T has a fixed point u, which is a positive solution of (1.4). This completes the proof of Theorem 1. ■

3.4 Proof of Theorem 2

(i) Since

$$\liminf_{u\to 0^+} h(u) > 0,$$

there exists $c > 0$ so that

$$h(u) > \lambda_1 u^{p-1} \tag{3.17}$$

for $u \in (0,c]$. Let $\lambda < \lambda_\infty$ and choose $b > 1$ so that $\lambda b < \lambda_\infty$. Using

$$\limsup_{u\to\infty}\frac{f(u)}{u^{p-1}} = m$$

and $(A3)$, it follows that there exists $d > 0$ so that (3.14) holds. Choose $M > c$ large enough so that (3.16) holds and define

$$\mathbf{C} := \{\, v \in C[0,1] : v \le M\phi_1 \text{ in } [0,1] \,\}\,.$$

For each $v \in \mathbf{C}$, let $u = Tv$ be the solution of

$$\begin{cases} -(r^{n-1}A(u'))' = r^{n-1}(h(\max(v, c\phi_1))) + \lambda f(\max(v, c\phi_1))) & 0 < r < 1, \\ u'(0) = 0, u(1) = 0. \end{cases}$$

Then $T : \mathbf{C} \to C^1[0,1]$ is compact and, as in the proof of Theorem 2.1, we verify that $T : \mathbf{C} \to \mathbf{C}$.

Choose $M > c$ large enough so that

$$(\lambda_1 - \lambda bm)M^{p-1}\int_0^s \tau^{n-1}\phi_1^{p-1}d\tau \ge \int_0^s \tau^{n-1}\left\{k_0(c\phi_1)^{-\alpha} + d(c\phi_1)^{-\delta}\right\} d\tau.$$

Let

$$u(r) = \int_r^1 A^{-1}\left(\frac{1}{s^{n-1}}\int_0^s \tau^{n-1}(h(\max(v, c\phi_1)) + \lambda f(\max(v, c\phi_1)))d\tau\right)ds$$

and since

$$\begin{aligned} & h(\max(v, c\phi_1)) + \lambda f(\max(v, c\phi_1)) \\ \le\;& k_0(\max(v, c\phi_1))^{-\alpha} + \lambda\left(bm(\max(v, c\phi_1))^{p-1} + d(\max(v, c\phi_1))^{-\delta}\right) \\ \le\;& \frac{k_0}{(c\phi_1)^{\alpha}} + \lambda bm(M\phi_1)^{p-1} + \frac{d}{(c\phi_1)^{\delta}} \end{aligned}$$

where the last expression is in $L^q(0,1)$, it follows from Lemma 1 that $T(\mathbf{C})$ is bounded in $C^{1,\beta}[0,1]$, and

$$\int_0^s \tau^{n-1}\left(h(\max(v, c\phi_1)) + \lambda f(\max(v, c\phi_1))\right)d\tau$$

$$\leq \int_0^s \tau^{n-1} \left\{ k_0(\max(v, c\phi_1))^{-\alpha} + \lambda \left(bm(\max(v, c\phi_1))^{p-1} + d(\max(v, c\phi_1))^{-\delta} \right) \right\} d\tau$$

for some $s \in (0,1)$. Hence

$$\int_0^s \tau^{n-1} \left(h(\max(v, c\phi_1)) + \lambda f(\max(v, c\phi_1)) \right) d\tau \leq \lambda_1 \int_0^s \tau^{n-1}(M\phi_1)^{p-1} d\tau.$$

This implies that $u \leq M\phi_1$ in $(0,1)$. Therefore $T : \mathbf{C} \to \mathbf{C}$.

By the Schauder Fixed Point Theorem, T has a fixed point u in $\mathbf{C}$. Next, we claim that $u \geq c\phi_1$ in $(0,1)$. Indeed, suppose $u(r_0) < c\phi(r_0)$ for some $r_0 \in (0,1)$. Then there exists r_1, r_2 with $0 \leq r_1 < r_0 < r_2 \leq 1$ such that $u < c\phi_1$ in (r_1, r_2), $u(r_2) = c\phi_1(r_2)$, and either $u(r_1) = c\phi_1(r_1)$ or $u^{'}(r_1) = c\phi_1^{'}(r_1)$. Since $f \geq 0$,

$$-(r^{n-1}A(u^{'}))^{'} \geq r^{n-1}h(\max(u, c\phi_1)) = r^{n-1}h(c\phi_1)) \qquad \text{in}(r_1, r_2)$$

On the other hand,

$$-(r^{n-1}A(c\phi_1^{'}))^{'} = \lambda_1 r^{n-1}(c\phi_1)^{p-1} \qquad \text{in } (r_1, r_2).$$

Hence, in view of (3.17)

$$-(r^{n-1}(A(u^{'}) - A(c\phi_1^{'})))^{'} \geq r^{n-1}(h(c\phi_1) - \lambda_1(c\phi_1)) > 0 \qquad \text{in } (r_1, r_2). \qquad (3.18)$$

Multiplying (3.18) by $(u - c\phi_1)$ and integrating on (r_1, r_2) gives

$$\int_{r_1}^{r_2} r^{n-1}(A(u^{'}) - A(c\phi_1^{'}))(u^{'} - c\phi_1^{'})dr < 0.$$

Since $(A(z_1) - A(z_2))(z_1 - z_2) \geq 0$ for all $z_1, z_2 \in \mathbb{R}$, we have a contradiction. This proves the claim and hence u is a positive solution of (1.4). Next, suppose that

$$f(u) \geq mu^{p-1}$$

for all $u > 0$. Let $\lambda \geq \lambda_\infty$ and let u be a positive solution of (1.4). Since

$$-(r^{n-1}A(u^{'}))^{'} = r^{n-1}(h(u) + \lambda f(u)) \geq 0,$$

and since

$$\liminf_{u \to 0^+} h(u) > 0,$$

it follows from Lemma A.3 in [19] that $u^{'}(1) < 0$. Let $\delta_0 > 0$ be such that $u \geq \delta_0 \phi_1$ in $[0, 1)$. Let

$$\delta = \sup \{\delta_0 > 0 : u \geq \delta_0 \phi_1 \text{ in } [0, 1)\}.$$

Then $\delta \in (0, \infty)$ and $u \geq \delta\phi_1$ in $[0, 1)$. Since

$$-(r^{n-1}A(\delta\phi_1^{'}))^{'} = \lambda_1 r^{n-1}(\delta\phi_1)^{p-1}$$

in $(0, 1)$ and

$$h(u) + \lambda f(u) \geq \lambda m u^{p-1} \geq \lambda_\infty m(\delta\phi_1)^{p-1} = \lambda_1(\delta\phi_1)^{p-1}$$

in $(0, 1)$ with $h(u) + \lambda f(u) \not\equiv \lambda_1(\delta\phi_1)^{p-1}$, it follows from The Strong Comparison Principle in [18] that $u > \lambda\phi_1$ in $[0, 1)$ and $u^{'}(1) < \delta\phi_1^{'}$. This implies the existence of $\tilde{\delta} > \delta$ such that $u \geq \tilde{\delta}\phi_1$ in $[0, 1)$, which is a contradiction with the choice of δ. Hence (1.4) does not have any positive solutions for $\lambda \geq \lambda_\infty$.

(ii) Let $\lambda < \lambda_\infty$ and u be a positive solution of (1.4) and

$$B = \inf \{B_0 > 0 : u \leq B_0\phi_1 \text{ in } (0, 1)\}.$$

Then $u \leq B\phi_1$ in $(0, 1)$ and $B > 0$. Thus

$$-(r^{n-1}A(u^{'}))^{'} = r^{n-1}(h(u) + \lambda f(u)) \leq \lambda r^{n-1} m u^{p-1} \leq \lambda r^{n-1} m(B\phi_1)^{p-1}$$

in $(0,1)$.

Since

$$-\left(r^{n-1}A\left(\left(\frac{\lambda}{\lambda_\infty}\right)^{\frac{1}{p-1}}B\phi_1'\right)\right)' = \lambda r^{n-1}m(B\phi_1)^{p-1}$$

in $(0,1)$, it follows from the weak comparison principle $(see[19])$ that

$$u \le \left(\frac{\lambda}{\lambda_\infty}\right)^{\frac{1}{p-1}}B\phi_1$$

in $(0,1)$, a contradiction. Hence (1.4) has no positive solutions for $\lambda < \lambda_\infty$, which completes the proof of Theorem 2. ■

CHAPTER 4

PROOF OF THEOREMS AND LEMMAS IN SECTION 1.4

By writing

$$\mu_i f_i(u_j) = \tilde{\mu}_i \tilde{f}_i(u_j),$$

where $\tilde{\mu}_i = m_i\mu_i$ and $\tilde{f}_i = \frac{f_i}{m_i}$, we assume without loss of generality that $m_i = 1, i = 1, 2$.

4.1 Proof of Theorem 3

Let $\mu_1\mu_2 \in (\lambda_1^2 - \varepsilon, \lambda_1^2)$ where $0 < \varepsilon < \frac{\lambda_1^2}{2}$ is a small number to be prescribed later. Let

$$c_i = \left(\frac{\mu_1\mu_2 k/2 - k\lambda_1^2/8 + 3\mu_i\lambda_1 k/8}{\lambda_1^2 - \mu_1\mu_2}\right)^{\frac{1}{p-1}}$$

and

$$M_i = \left(\frac{\mu_1\mu_2 k/2 + \mu_i m + \lambda_1(\mu_i k/2 + 1)}{\lambda_1^2 - m^2\mu_1\mu_2}\right)^{\frac{1}{p-1}}$$

where $\varepsilon_0 = k\lambda_1/8$ and $m > 1$ is such that $\lambda_1^2 > m^2\mu_1\mu_2, i = 1, 2$. Note that c_i and M_i satisfy the systems

$$\mu_i(c_j^{p-1} + k/2) - k\lambda_1/8 = \lambda_1 c_i^{p-1} \tag{4.1}$$

and

$$\mu_i(mM_j^{p-1} + k/2) + 1 = \lambda_1 M_i^{p-1} \tag{4.2}$$

respectively, for $i, j \in 1, 2, i \neq j$. Clearly, $c_i \leq M_i$ for $i = 1, 2$. Since $k\lambda_1^2/8 \leq \mu_1\mu_2 k/4$, it follows that

$$c_i \geq \left(\frac{k\mu_1\mu_2}{4(\mu_\infty - \mu_1\mu_2)} \right)^{\frac{1}{p-1}} \tag{4.3}$$

for $i = 1, 2$. In particular, $c_i \to \infty$ as $\mu_1\mu_2 \to \lambda_1^2$. Define

$$\mathbf{K} = \{ (v_1, v_2) \in C[0,1] \times C[0,1] : c_i\phi_1 \leq u_1 \leq M_i\phi_1 \quad \text{in } [0,1], i = 1, 2 \}.$$

For each $(v_1, v_2) \in \mathbf{K}$, let $(u_1, u_2) = T(v_1, v_2)$ be the solution of

$$\begin{cases} -(r^{n-1}A(u_1'))' = r^{n-1}(h_1(v_2) + \mu_1 f_1(v_2)) & 0 < r < 1, \\ -(r^{n-1}A(u_2'))' = r^{n-1}(h_2(v_1) + \mu_2 f_2(v_1)) & 0 < r < 1. \\ u_1'(0) = u_2'(0) = u_1(1) = u_2(1) = 0, \end{cases}$$

Note that

$$u_i(t) = \int_t^1 A^{-1} \left(\frac{1}{s^{n-1}} \int_0^s \tau^{n-1}(h_i(v_j) + \mu_i f_i(v_j))d\tau \right) ds \tag{4.4}$$

where $i \neq j$.

Let $q > 1$ be such that $q \max(\alpha, \delta) < 1$. By $(H1)$, for each $M > 0$, there exists a constant $k_M > 0$ such that

$$|f_i(t)| \leq k_M t^{-\delta}$$

for $t \in (0, M]$, which implies

$$|h_i(v_j) + \mu_i f_i(v_j)| \leq k_0(c_j\phi_1)^{-\alpha} + \mu_i k_M (c_j\phi_1)^{-\delta} \in L^q(0,1) \tag{4.5}$$

for $i \neq j$. Hence $T : \mathbf{K} \to L^q(0,1) \times L^q(0,1)$ is continuous by Lebesgue Dominated Convergence Theorem. Also, in view of (4.5) and Lemma 3.1 in [12], we see that there exists $\beta \in (0,1)$ depending only on p and q such that $u_i \in C^{1,\beta}[0,1]$. Consequently, $T : \mathbf{K} \to C[0,1] \times C[0,1]$ is a compact operator. We next show that $T : \mathbf{K} \to \mathbf{K}$ if $\mu_1\mu_2$ is sufficiently close to λ_1^2.

Let $(v_1, v_2) \in \mathbf{K}$ and $(u_1, u_2) = T(v_1, v_2)$. Note that, by $(H1)$ and $(H2)$, there exists a constant $B > 0$ such that

$$f_i(t) \geq t^{p-1} + k - \frac{B}{t^\delta}$$

for all $t > 0$. Hence

$$\begin{aligned}
&\int_0^s \tau^{n-1}(h_i(v_j) + \mu_i f_i(v_j))d\tau \\
&\geq \int_0^s \tau^{n-1}(-k_0 v_j^{-\alpha} + \mu_i(v_j^{p-1} + k - Bv_j^{-\delta}))d\tau \\
&\geq \int_0^s \tau^{n-1}\left(-\frac{k_0}{(c_j\phi_1)^\alpha} - \frac{\mu_i B}{(c_j\phi_1)^\delta} + \mu_i((c_j\phi_1)^{p-1} + k\right)d\tau
\end{aligned} \tag{4.6}$$

Choose ε small enough so that

$$\frac{k_0}{c_j^\alpha}\int_0^s \frac{\tau^{n-1}}{\phi_1^\alpha}d\tau \leq \min\left(\frac{k\lambda_1}{8}, 1\right)\int_0^s \tau^{n-1}\phi_1^{p-1}d\tau \tag{4.7}$$

and

$$\frac{B}{c_j^\alpha}\int_0^s \frac{\tau^{n-1}}{\phi_1^\delta}d\tau \leq \frac{ks^n}{2n}$$

for $j = 1, 2$. Then it follows from (4.1) and (4.6) that

$$\begin{aligned}
\int_0^s \tau^{n-1}(h_i(v_j) + \mu_i f_i(v_j))d\tau &\geq \int_0^s \tau^{n-1}\left[\mu_i((c_j\phi_1)^{p-1} + k/2) - \frac{k\lambda_1}{8}\phi_1^{p-1}\right]d\tau. \\
&\geq \int_0^s \tau^{n-1}\left[\mu_i(c_j + k/2) - \frac{k\lambda_1}{8}\right]\phi_1^{p-1}d\tau \\
&= \lambda_1 \int_0^s \tau^{n-1}(c_i\phi_1)^{p-1}d\tau
\end{aligned}$$

for $i, j = 1, 2, i \neq j$, which, together with (4.4) and the fact that

$$-(r^{n-1}A(\phi_1'))' = \lambda_1 r^{n-1}\phi_1^{p-1} \qquad 0 < r < 1,$$

implies

$$u_i \geq c_i\phi_1 \quad \text{in} \quad (0,1)$$

for $i = 1, 2.$

Next, choose a constant $D > 0$ such that

$$f_i(t) \leq mt^{p-1} + \frac{D}{t^\delta} \tag{4.8}$$

for $t > 0, i = 1, 2.$ Then

$$\begin{aligned} \int_0^s \tau^{N-1}(h_i(v_j) + \mu_i f_i(v_j))d\tau &\leq \int_0^s \tau^{n-1}(k_0 v_j^{-\alpha} + \mu_i(mv_j^{p-1} + Dv_j^{-\delta}))d\tau \\ &\leq \int_0^s \tau^{n-1}\left(\frac{k_0}{(c_j\phi_1)^\alpha} + \frac{\mu_i D}{(c_j\phi_1)^\delta} + \mu_i m(M_j\phi_1)^{p-1}\right)d\tau. \end{aligned} \tag{4.9}$$

Choose ε small enough so that

$$\frac{D}{c_j^\delta}\int_0^s \frac{\tau^{n-1}}{\phi_1^\delta}d\tau \leq \frac{k}{2}\int_0^s \tau^{n-1}\phi_1^{p-1}d\tau$$

for $j = 1, 2$. Then it follows from (4.2), (4.7), and (4.9) that

$$\begin{aligned} \int_0^s \tau^{n-1}(h_i(v_j) + \mu_i f_i(v_j))d\tau &\leq \int_0^s \tau^{n-1}\left[\mu_i(mM_j^{p-1} + k/2) + 1\right]d\tau \\ &= \lambda_1 \int_0^s \tau^{n-1}(M_i\phi_1)^{p-1}d\tau \end{aligned}$$

for $i, j = 1, 2, i \neq j$, which implies

$$u_i \leq M_i\phi_1 \quad \text{in} \quad (0,1)$$

for $i = 1, 2$. Hence $T : \mathbf{K} \to \mathbf{K}$ and by the Schauder Fixed Point Theorem, T has a fixed point (u_1, u_2) in $\mathbf{K}$ satisfying (4.3). This completes the proof of Theorem 3. ■

4.2 Proof of Theorem 4

(i) Let $\mu_1, \mu_2 > 0$ satisfy $\mu_1\mu_2 < \lambda_1^2$. Choose constants $\sigma \in (0,1), m > 1$ so that

$$\lambda_1^2\sigma^2 > m^2\mu_1\mu_2.$$

Note that (4.8) holds in view of $(H1)$ and the fact the $\limsup_{t\to\infty} \frac{f_i(t)}{t^{p-1}} = 1$.

Let $g_i(t) = \inf_{0<s\le t} h_i(s)$. Then g_i is nonincreasing, positive, bounded on $(0,\infty)$, and it can be verified that

$$\lim_{t\to\infty} t^{p-1} g_i^{\frac{\gamma}{p-1}} = \infty \tag{4.10}$$

for $i = 1,2$. Define

$$\tilde{M}_i = \left(\frac{K(\lambda_1\sigma + \mu_i m)}{\lambda_1^2\sigma^2 - m^2\mu_1\mu_2}\right)^{\frac{1}{p-1}} \qquad \tilde{c}_i = \left(\frac{g_i(M_j)}{\lambda_1}\right)^{\frac{1}{p-1}}$$

for $i = 1,2$, where K is a large constant to be determined later. Then $c_i \le M_i$ for $i = 1,2$, and $\tilde{M}_i$ satisfy the system

$$\mu_i m \tilde{M}_j^{\,p-1} + K = \lambda_1\sigma\tilde{M}_i^{\,p-1} \tag{4.11}$$

for $i,j = 1,2, i \ne j$. Define

$$\mathbf{C} = \Big\{(v_1,v_2) \in C[0,1]\times C[0,1] : \tilde{c}_i\phi_1 \le v_i \le \tilde{M}_i\phi_1 \quad \text{in} \quad [0,1], i = 1,2\Big\}$$

and let T be the operator defined in the proof of Theorem 4. We claim that $T : \mathbf{C} \to \mathbf{C}$. To this end, let $(v_1,v_2) \in \mathbf{C}$ and $(u_1,u_2) = T(v_1,v_2)$. Since

$$-(r^{n-1}A(u_i^{'}))' = r^{n-1}(h_i(v_j) + \mu_i f_i(v_j)) \ge \mu_i r^{N-1} g_i(v_j)$$

$$\ge r^{n-1} g_i(\tilde{M}_j\phi_1) \ge r^{n-1} g_i(\tilde{M}_j)\phi_1 \quad \text{in} \quad (0,1),$$

it follows from the weak comparison principle that

$$u_i \geq \left(\frac{g_i(\tilde{M}_j)}{\lambda_1} \right)^{\frac{1}{p-1}} \phi_1 = \tilde{c}_i \phi_1 \quad \text{in} \quad (0,1)$$

for $i, j = 1, 2, i \neq j$. Next, in view of (4.8) and $(H1)$,

$$\int_0^s \tau^{n-1}(h_i(v_j) + \mu_i f_i(v_j))d\tau \leq \int_0^s \tau^{n-1} \left(\frac{k_0}{(c_j\phi_1)^\alpha} + \frac{\mu_i D}{(c_j\phi_1)^\delta} + \mu_i m(\tilde{M}_j\phi_1)^{p-1} \right) d\tau$$

$$\leq \frac{k_0}{g_j^{\frac{\alpha}{p-1}} \tilde{M}_i} \int_0^s \frac{\tau^{n-1}}{\phi_1^\alpha} d\tau + \frac{\mu_i D}{g_j^{\frac{\delta}{p-1}} \tilde{M}_i} \int_0^s \frac{\tau^{n-1}}{\phi_1^\delta} d\tau + \int_0^s \tau^{n-1}(\mu_i m \tilde{M}_j^{p-1} + K)\phi_1^{p-1} d\tau. \quad (4.12)$$

Since g_i are bounded, it follows from (4.10) that

$$\lim_{t\to\infty} t^{p-1} g_i^{\frac{\alpha}{p-1}} = \infty, \quad \lim_{t\to\infty} t^{p-1} g_i^{\frac{\delta}{p-1}} = \infty.$$

Hence, for $K >> 1$,

$$\frac{k_0}{g_j^{\frac{\alpha}{p-1}} \tilde{M}_i} \int_0^s \frac{\tau^{n-1}}{\phi_1^\alpha} d\tau + \frac{\mu_i D}{g_j^{\frac{\delta}{p-1}} \tilde{M}_i} \int_0^s \frac{\tau^{n-1}}{\phi_1^\delta} d\tau$$

$$\lambda_1(1-\sigma)\tilde{M}_i^{p-1} \int_0^s \tau^{n-1}\phi_1^{p-1} d\tau. \quad (4.13)$$

Combining $(4.11) - (4.13)$, we obtain,

$$\int_0^s \tau^{n-1}(h_i(v_j) + \mu_i f_i(v_j))d\tau \leq \int_0^s \tau^{n-1} \left(\mu_i m \tilde{M}_j^{p-1} + K + \lambda_1(1-\sigma)\tilde{M}_i^{p-1} \right) \phi_1^{p-1} d\tau$$

$$= \lambda_1 \int_0^s \tau^{n-1} \left(\tilde{M}_i \phi_1 \right)^{p-1} d\tau,$$

which implies

$$u_i \leq \tilde{M}_i \phi_1 \quad \text{in} \quad (0,1)$$

for $i = 1, 2$. Thus $T : \mathbf{C} \to \mathbf{C}$, as claimed. Suppose

$$f_i(t) \geq t^{p-1}$$

for $t > 0, i = 1, 2$. Let $\mu_1, \mu_2 > 0$ be such that $\mu_1\mu_2 \geq \lambda_1^2$ and let (u_1, u_2) be a positive solution of (1.11). Since $\liminf_{t\to 0^+} h_i(t) > 0$, it follows from Lemma A.3 in [19] that $u_i'(t) < 0$ for $i = 1, 2$.

Let δ_0 be the largest positive number so that $u_1 \geq \delta_0\phi_1$ in $(0, 1)$. Then

$$-(r^{n-1}A(u_2'))' \geq \mu_2 r^{n-1}u_1^{p-1} \geq \mu_2 r^{n-1}(\delta_0\phi_1)^{p-1} \quad \text{in} \quad (0, 1),$$

and the Weak Comparison Principle implies

$$u_2 \geq \left(\frac{\mu_2}{\lambda_1}\right)^{\frac{1}{p-1}} \delta_0\phi_1 \quad \text{in} \quad (0, 1).$$

Hence

$$-(r^{n-1}A(u_1'))' \geq \mu_1 r^{n-1}u_2^{p-1} \geq (\mu_1\mu_2/\lambda_1)r^{n-1}\delta_0^{p-1} \quad \text{in} \quad (0, 1).$$

Since $-(r^{n-1}A(u_1'))' \not\equiv \mu_1 r^{n-1}(\mu_2/\lambda_1)\delta_0^{p-1}$, it follows from the Strong Comparison Principle in [18] that

$$u_1 > \left(\frac{\mu_1\mu_2}{\lambda_1^2}\right)^{\frac{1}{p-1}} \delta_0\phi_1 \quad \text{in} \quad (0, 1)$$

and $u_1'(1) < \left(\frac{\mu_1\mu_2}{\lambda_1^2}\right)^{\frac{1}{p-1}} \delta_0\phi_1'(1)$. Consequently, there exists a constant $\delta_1 > \left(\frac{\mu_1\mu_2}{\lambda_1^2}\right)^{\frac{1}{p-1}} \delta_0$ such that

$$u_1 \geq \delta_1\phi_1 \quad \text{in} \quad (0, 1),$$

a contradiction with the maximality of δ_0.

(ii) Suppose $h_i \leq 0$ and

$$f_i(t) \leq t^{p-1}$$

for $t > 0, i = 1, 2$. Let $\mu_1, \mu_2 > 0$ be such that $\mu_1\mu_2 < \lambda_1^2$ and let (u_1, u_2) be a positive solution of (1.11).

Let B be the smallest number so that $u_1 \le B\phi_1$ in $(0,1)$. Note that $B > 0$. Thus

$$-(r^{n-1}A(u_2'))' \le \mu_2 r^{n-1} u_1^{p-1} \le \mu_2 r^{n-1}(B\phi_1)^{p-1} \quad \text{in} \quad (0,1),$$

and therefore

$$u_2 \le \left(\frac{\mu_2 B}{\lambda_1}\right)^{\frac{1}{p-1}} \phi_1 \quad \text{in} \quad (0,1).$$

Hence

$$-(r^{n-1}A(u_1'))' \le \mu_1 r^{n-1} u_2^{p-1} \le r^{n-1}\left(\frac{\mu_1\mu_2 B}{\lambda_1}\right)\phi_1^{p-1} \quad \text{in} \quad (0,1),$$

which implies

$$u_1 \le \left(\frac{\mu_1\mu_2 B}{\lambda_1^2}\right)^{\frac{1}{p-1}} \phi_1 \quad \text{in} \quad (0,1)$$

which contradicts the maximality of B. This completes the proof of Theorem 4. ∎

CHAPTER 5

CONCLUSIONS

We discussed the existence and nonexistence of positive radial solutions for classes of boundary value problems for the p-Laplacian equations and systems with asymptotically p-linear nonlinearities.

A precise lower bound estimate for the solution is obtained, which shows the existence of a number λ_∞ such that the solution goes to infinity when the parameter goes to λ_∞ in the single equation case, or the product of the two parameters goes to λ_∞ in the system case. Our approach depends on the integral formula for the solutions together with the comparison principle and the Schauder Fixed Point Theorem.

5.1 Further Plan

- Since our approach depends heavily on the explicit integral formula for the solution, it can not apply to the case of the general domain. We would like to extend the study in this thesis to include a general domain.
- We are also interested in studying the case when the nonlinearities are p-superlinear at infinity and are possibly singular at zero, which is challenging and has not been considered in the literature.

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