A class of immersed finite element methods for Stokes interface problems

Derrick T. Jones
dtj1992@gmail.com

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A class of immersed finite element methods for Stokes interface problems

By

Derrick T. Jones

Approved by:

Mohsen Razzaghi (Major Professor)
Xu Zhang (Director of Dissertation)
Hai Dang
Amanda Diegel
Seongjai Kim
Mohammad Sepehrifar (Graduate Coordinator)
Rick Travis (Dean, College of Arts & Sciences)

A Dissertation
Submitted to the Faculty of
Mississippi State University
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for the Degree of Doctor of Philosophy
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In this dissertation, we explore applications of partial differential equations with discontinuous coefficients. We consider the nonconforming immersed finite element methods (IFE) for modeling and simulating these partial differential equations.

A one-dimensional second-order parabolic initial-boundary value problem with discontinuous coefficients is studied. We propose an extension of the immersed finite element method to a high-order immersed finite element method for solving one-dimensional parabolic interface problems. In addition, we introduce a nonconforming immersed finite element method to solve the two-dimensional parabolic problem with a moving interface. In the nonconforming IFE framework, the degrees of freedom are determined by the average integral value over the element edges. The continuity of the nonconforming IFE framework is in the weak sense in comparison the continuity of the conforming IFE framework. Numerical experiments are provided to demonstrate the features and the robustness of these methods.
We introduce a class of lowest-order nonconforming immersed finite element methods for solving two-dimensional Stokes interface problem. On triangular meshes, the Crouzeix-Raviart element is used for velocity approximation, and piecewise constant for pressure. On rectangular meshes, the Rannacher-Turek rotated $Q_1-Q_0$ finite element is used. We also consider a new mixed immersed finite element method for the Stokes interface problem on an unfitted mesh. The proposed IFE space uses conforming linear elements for one velocity component and nonconforming linear elements for the other component. The new vector-valued IFE functions are constructed to approximate the interface jump conditions. Basic properties including the unisolvency and the partition of unity of these new IFE methods are discussed. Numerical approximations are observed to converge optimally.

Lastly, we apply each class of the new immersed finite element methods to solve the unsteady Stokes interface problem. Based on the new IFE spaces, semi-discrete and full-discrete schemes are developed for solving the unsteady Stokes equations with a stationary or a moving interface. A comparison of the degrees of freedom and number of elements are presented for each method. Numerical experiments are provided to demonstrate the features of these methods.

Key words: Stokes Interface Problem, Immersed Finite Element, Moving Interface Problem, Parabolic Interface Problem, Nonconforming Rotated $Q_1$, Crouziex-Raviart Finite Element, Mixed Finite Element
DEDICATION

This work is dedicated to:

My loving wife Renee’ and our expectant child.

Totianna Jones, Edell Buckley, and James Faulkner Sr., may you rest well.

My mother, Maggie Jones and family.

My mentors Drs. Stanford & Andrea Johnson and the late Dr. Constance Bland.
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For I know the plans I have for you, declares the Lord, plans to prosper you and not to harm you, plans to give you hope and a future - Jeremiah 29:11. I would first like to thank my Lord and Savior Jesus Christ for his everlasting love, guidance, and purpose He has on my life. I thank You for leading me through this journey and sustaining me. I love you.

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CHAPTER I
INTRODUCTION

Numerical mathematics has made significant progress in studying the complexities of differential equations. Differential equations and systems of differential equations may result in large-scale problems due to its application in nature. Through the advent and use of computers and programming languages, numerical mathematics is used to solve the world’s most complex, large-scale problems. Many branches of numerical mathematics were developed and studied in recent years. These branches of mathematics include numerical linear algebra, numerical ordinary differential equations, and numerical partial differential equations, as well as many others. A host of numerical methods and solvers have been developed to analyze concepts derived from these areas. Notable numerical methods used today are the finite difference method, the finite volume method, the spectral method, and the finite element method, to name a few. Mathematical advancements in these areas have led to faster, more accurate solutions. However, there are still many challenges in numerical mathematics. Numerical solvers, when applied to certain problems, may be computationally expensive or may result in instabilities and inaccuracies. In many problems, instabilities and inaccuracies arise when solving differential equations with a sharp transition or an interface. In recent years, there has been a significant effort made to mitigate issues of instabilities and
inaccuracies around the interface. The next section will discuss numerical methods used to solve interface problems.

### 1.1 A Survey of Numerical Methods for Solving Interface Problems

Many physical problems involve multiple materials and are often modeled by partial differential equations with a material interface. Examples of physical problems with a material interface include wave propagation, which can be described by the Helmholtz equation and the wave equation, topology optimization and solid mechanics, which can be described by the elasticity interface problem, and turbulent flow for liquids and gases, which can be described by the Navier-Stokes equations, to name a few. The solutions to these interface problems often exhibit kinks, discontinuities, singularities, and other non-smooth behaviors. Generally, there are mainly two types of numerical methods used to solve interface problems: interface-fitted mesh methods and non-interface-fitted mesh methods.

Conventional numerical methods such as the finite element, finite difference, and finite volume methods often require the solution meshes to align with material interfaces, otherwise the approximation may not be accurate if not worse [9, 10]. These meshes are generally known as interface-fitted meshes or body-fitted meshes. Interface-fitted meshes are constructed to align with the interface such that each element is distinctly on one side of the interface or the other, see Figure 1.1. The interface-fitted mesh method has been used to solve elliptic and parabolic problems in [93, 21, 14, 90]. Dynamic problems with a moving interface are often computationally expensive since constant re-meshing is required as the interface evolves in time. In addition, the number and the location of the global degrees of freedom for a time-dependent moving interface problem
are usually different at each time level. This issue will result in a complicated global and local matrix assembly procedure which is an important feature of the finite element method. Therefore, it is desirable to construct a numerical method that relies on a non-interface-fitted Cartesian mesh to solve interface problems, see Figure 1.1 for an illustration of an interface-fitted mesh and a non-interface-fitted mesh.

There is a growing interest in solving interface problems on Cartesian meshes. Methods to solve interface problems on a Cartesian mesh in the finite difference framework includes the immersed interface method [61, 12, 59, 62, 64], the ghost fluid method [81, 54, 25, 78], and the immersed boundary method [83, 86, 84], to name a few. A few methods to solve interface problems in the finite element framework includes the Discontinuous Galerkin method [38, 2], the penalty finite element [11, 9], and the unfitted finite element [39] as well as others.

In this dissertation, we will focus on a special class of the extended finite element method called the immersed finite element method (IFEM) for solving partial differential equations with an interface. Since the IFE framework is closely related to the research presented in this dissertation, the next Section will be devoted to a brief introduction of the method.

1.2 A Survey of Previous Work on the Immersed Finite Element Method

The immersed finite element method is an extension of the finite element method and has been shown to accurately solve PDE interface problems on a mesh that is independent of the interface location. The immersed finite element method was first introduced to solve a one-dimensional elliptic interface problem involving discontinuities in the coefficients of the partial differential equation [63]. The idea is to construct finite element basis functions that adapt and satisfy the jump
conditions across the interface. The immersed finite element method has clear distinctions in comparison to the standard finite element method when solving an interface problem. The IFE method can use non-interface-fitted meshes for interface problems, even problems with a moving interface. The standard finite element method with an interface-fitted mesh must continuously restructure its mesh when there is movement within the domain. This may result in expensive computational time, especially for two-dimensional and three-dimensional moving interface problems. Since the IFE method is constructed on a mesh independent of the interface location, the interface can cut the interior of elements within the domain. The use of standard FE in this way may fail to produce optimal convergence. In addition, the number and the location of the global degrees of freedom for the IFE method remain unchanged throughout the whole simulation, even with moving interface problems. The standard finite element functions are utilized in the immersed finite element method on non-interface elements while a special IFE function is used on the interface element.

Over the past twenty years since the immersed finite element idea was introduced [63], there have been significant advances in the area. Several higher-order IFE methods have been introduced [16, 68, 5] for one-dimensional interface problems. The IFE method has also been used to solve two-dimensional elliptic interface problems [42, 67, 66, 43, 30, 31, 36, 1] as well as three-dimensional interface problems [53, 91, 32, 37]. Other closely related work in IFE has been studied in the last decade includes [44, 45, 46, 99, 6, 41] and the references therein. In recent years, the immersed finite element method has been extended to two-dimensional time-independent problems [72, 76, 47]. Time-dependent numerical schemes such as the Backward-Euler scheme and the Crank-Nicolson scheme using IFE algorithms are proposed to solve parabolic problems with a static interface. Error estimation of these time-dependent IFE schemes has been studied in
Moving interface problems of the parabolic type have also been proposed using the idea of the immersed finite element method in [70, 69, 47]. Due to the advantages of the IFE method, no mesh restructuring is needed when the interface moves throughout the domain. Numerical approximations for moving interface problems are examined and have been shown to produce optimal convergence in [76, 95].

Increasingly, the IFE extension to systems of PDEs have been studied, i.e [34, 13]. Lin and Zhang proposed a nonconforming rotated-$Q_1$ IFE method to solve the planar elasticity interface problem [75, 97]. Fundamental properties are investigated for the new nonconforming IFE space as well as several numerical experiments to validate the efficiency and effectiveness of the space. A partially penalized immersed finite element (PPIFE) method for elasticity interface problems is studied in [33, 77]. The PPIFE method was employed to stabilize the discontinuity in the global IFE functions across the interface. The PPIFE method was shown to converge optimally in both the $L^2$ norm and energy norm. An error analysis is provided for the PPIFE method. Liu and Chen in [77] propose a conforming-nonconforming $P_1$ PPIFE method to solve the planar elasticity problem. The conforming-nonconforming $P_1$ IFE space produced optimal approximation errors in the $L^2$ norm and the $H^1$ semi-norm.

1.3 Application of the Immersed Finite Element Method and the Motivation to Study

In this work, our motivation is to implement and to analyze the nonconforming IFE method for the parabolic interface problem and the Stokes interface problem. Traditionally, the standard (or conforming) finite element method degrees of freedom are determined by the nodal values on the vertices. The nonconforming IFE framework proposed in this work determines the degrees of
From left: the domain of an interface problem, an interface-fitted mesh, and an interface-unfitted mesh.

freedom by the average integral value over the element edges. The continuity of the nonconforming IFE framework is in the weak sense as opposed to the continuity of the conforming IFE framework. Both conforming and nonconforming IFE can be implemented on the same mesh, therefore nonconforming FE and nonconforming IFE solvers can be used to solve interface problems more efficiently.

The simplest nonconforming finite element for triangular meshes was first introduced by Crouziex and Raviart in 1973 and is known as the nonconforming $P_1$ or $CR$ element [22]. On a rectangular mesh, the simplest nonconforming finite element was introduced by Rannacher and Turek and is known as the rotated-$Q_1$ element [85]. Nonconforming finite elements have been used to solve elliptic problems [15, 56, 74], and elasticity problems [55, 24, 98, 97], as well as others. The nonconforming immersed finite element method has improved performance in comparison to
the conforming immersed finite element method. The conforming IFE has challenges due to the discontinuity of IFE functions across the interface edges, which causes suboptimal global convergence. This discontinuity is largely due to unfavorable interface locations or large jump coefficients [97, 71]. New methods such as the partial penalized IFE (PPIFE) method are introduced to mitigate the discontinuity across the interface edge with the addition of stabilization terms. However, these added terms require extra edge structure to facilitate data communication between two neighboring elements, especially for time-dependent moving interface problems. The nonconforming IFE method is simpler and does not require the added partial penalty stabilization terms while maintaining optimal accuracy. A challenge of the nonconforming IFE is to adequately perform error analysis due to its nonconformity. See [66] for a theoretical analysis on the nonconforming interpolation error.

1.4 Outline of the Dissertation

In this dissertation, we construct and analyze a nonconforming immersed finite element method for parabolic equations with a moving interface. We also construct and analyze several new nonconforming IFE spaces for the Stokes interface problem. Lastly, we apply the new IFE spaces to both the unsteady Stokes problem with a moving interface. The rest of the dissertation will be organized as follows.

In Chapter 2, a basic illustration of the immersed finite element method is discussed for a one-dimensional parabolic interface problem. Derivation of the one-dimensional IFE structure, the semi-discretization, and fully discrete schemes used to solve the parabolic interface problem, is discussed. In addition, we extend the one-dimensional parabolic interface problem to a two-
dimensional parabolic interface problem. We develop a numerical scheme using nonconforming rotated-$Q_1$ elements on Cartesian meshes to solve the two-dimensional parabolic interface problem with a static and a moving interface. Numerical results are provided to demonstrate the performance of the schemes.

In Chapter 3, we develop a class of nonconforming IFE spaces for the Stokes interface problem. Two vector-valued IFE spaces are constructed for the Stokes interface problem using the $CR-P_0$ and the rotated-$Q_1-Q_0$ finite element pairs. Fundamental properties such as the partition of unity, consistency with FE functions, and uniqueness and existence are proven for each finite element pair. Numerical examples are given to demonstrate the robustness of this method.

In Chapter 4, we construct a conforming-nonconforming IFE space for the Stokes interface problem. Similar to the fundamental properties from the $CR-P_0$ and rotated-$Q_1-Q_0$ finite element pairs are proven: the partition of unity, consistency with FE functions, and uniqueness and existence. Numerical results are reported to demonstrate the performance of the new IFE space.

In Chapter 5, we apply the new immersed finite element spaces to the time-dependent Stokes problem with a moving interface. Both the semi-discrete and the fully discrete schemes are developed. We compare the efficiency of the rotated-$Q_1-Q_0$ IFE space and the $CR-P_0$ IFE space to the conforming-nonconforming IFE space by their total degrees of freedom and total number of elements. Numerical results are provided at the end of the chapter.

In Chapter 6, we briefly discuss our future research direction.
CHAPTER II
IMMERSED FINITE ELEMENT METHODS FOR PARABOLIC INTERFACE PROBLEMS

In this chapter, we recall the basic framework of the immersed finite element method (IFEM), which was first introduced by Li in 1998 [63]. The immersed finite element method is proposed to solve a one-dimensional second-order boundary value problem with discontinuous coefficients. The idea of the IFE method is to construct finite element basis functions on an interface-independent mesh such that the interface jump conditions are satisfied. The IFE method can be implemented on non-interface-fitted meshes, therefore the interface can cut the interior of an element. In this chapter, we briefly demonstrate the approach of the immersed finite element method. A high order extension of IFE method is introduced for solving one-dimensional parabolic interface problems [49].

In addition, we introduce a nonconforming immersed finite element method to solve the two-dimensional parabolic problem with a moving interface. The nonconforming immersed finite element space is derived using a rotated-$Q_1$ element. The nonconforming IFE framework proposed in this work determines the degrees of freedom by the average integral value over the element edges. The continuity of the nonconforming IFE framework is in the weak sense as opposed to the continuity of the conforming IFE framework. To show the robustness of the nonconforming immersed finite element method, we present a parabolic moving interface case. The standard FE
method used in this way must continuously restructure its mesh when there is movement within the domain. This results in significant computational cost. Using the immersed finite element method, the number and the location of the global degrees of freedom are unchanged throughout the whole simulation.

The semi-discrete and the full-discrete schemes for the one-dimensional and two-dimensional parabolic problems are detailed. Time marching schemes, including Backward-Euler and Crank-Nicolson methods, are implemented to fully discretize the system. Numerical examples are provided to test the performance of our numerical schemes.

2.1 One-Dimensional Parabolic Interface Problem

Consider the following one-dimensional parabolic interface problem:

\[ u_t - (\beta u_x)_x = f(t, x), \quad x \in \Omega, \ t \in (0, T], \]  
\[ u(t, x) = g(t, x), \quad x \in \partial \Omega, \ t \in (0, T], \]  
\[ u(0, x) = u_0(x), \quad x \in \Omega. \]

Assume that \( \Omega \) is an open interval separated by an interface point \( \alpha \). The interface point divides \( \Omega \) into two sub-domains \( \Omega^+ \) and \( \Omega^- \) such that \( \Omega = \Omega^+ \cup \Omega^- \cup \{\alpha\} \). We assume that there is only one type of material in each sub-domain. This means that the coefficient function \( \beta \) is continuous within each sub-domain but may be discontinuous across the interface:

\[ \beta(t, x) = \begin{cases} 
\beta^-(t, x), & x \in \Omega^-; \\
\beta^+(t, x), & x \in \Omega^+. 
\end{cases} \]

The solution \( u \) is assumed to satisfy the following interface jump conditions:

\[ [u]_{\alpha} = 0, \]

\[ 10 \]
\[ [\beta u_x] \big|_\alpha = 0, \quad (2.6) \]

where the jump operator is defined as \( [u] \big|_\alpha = u^+(\alpha) - u^-(\alpha) \). The standard finite element method can accurately solve an interface problem if the mesh is fitted to the interface. However, if the constructed mesh neglects the location of the interface, the standard finite element method may fail to hold an optimal order of convergence. The main idea of the immersed finite element method is to modify the finite element basis functions to satisfy the interface jump conditions, i.e. to construct a piecewise polynomial that satisfies the jump conditions (2.5) and (2.6) across the interface.

Without loss of generality, let \( T_h \) be a partition of the domain \( \Omega \). \( T_h \) is divided into two collections of elements: a collection of interface elements \( T_h^i \) and a collection of non-interface elements \( T_h^n \). Hence, \( T_h = T_h^i \cup T_h^n \). Consider the reference interface element \( T = [0, 1] \) which contains an interface point \( \alpha \in (0, 1) \). The interface element \( T \) is subdivided into two subintervals: \( T^- = (0, \alpha) \) and \( T^+ = (\alpha, 1) \). The local IFE basis functions on the interface element are non-smooth which results in two piecewise polynomials being produced on \( T^- \) and \( T^+ \), respectively. The local IFE basis functions are constructed by the following procedure. Define the two local piecewise linear IFE basis functions, \( \phi_1 \) and \( \phi_2 \), such that

\[
\phi_j(x) = \begin{cases} 
    a_j^- + b_j^- x, & \text{in } T^-, \quad j = 1, 2. \\
    a_j^+ + b_j^+ x, & \text{in } T^+. 
\end{cases}
\]

(2.7)

For \( \phi_1 \) and \( \phi_2 \) respectively, enforce following the Lagrange conditions and the interface jump conditions (2.5)-(2.6):

\[
\begin{align*}
\phi_1(0) &= 1, \quad \phi_1(1) = 0, \quad [\phi_1(x)] \big|_\alpha = 0, \quad [\beta \phi_1'(x)] \big|_\alpha = 0, \\
\phi_2(0) &= 0, \quad \phi_2(1) = 1, \quad [\phi_2(x)] \big|_\alpha = 0, \quad [\beta \phi_2'(x)] \big|_\alpha = 0.
\end{align*}
\]

(2.8)
The two linear systems produce a unique solution for the coefficients $a_1^+, a_2^+, b_1^+, b_2^+$. Illustrated in Figure 2.1 are the local IFE basis functions $\phi_1$ and $\phi_2$ on the reference element with $\beta^- = 1$, $\beta^+ = 5$, and $\alpha = 1/3$. Globally these IFE basis functions are continuous, therefore they are in $H^1(\Omega)$. Hence, using a Galerkin scheme which utilizes these IFE basis functions is considered a conforming finite element method. Let $S_h = \text{span}\{\phi_1, \phi_2, \cdots, \phi_K\} \subset H^1(\Omega)$ denote the global IFE space, where K is the number of local nodes from all elements of $T_h$.

The IFE method has been extended to higher-order approximations in [87], in [4] for Lagrange type IFE basis functions, and in [17] for orthogonal IFE basis functions. Numerical results are given to show their approximation capabilities. Plots for some quadratic IFE basis functions are shown in Figure 2.2.

Figure 2.1

IFE local linear basis functions $\phi_1(x)$ (left) and $\phi_2(x)$ (right) on reference element $[0, 1]$ with $\beta^- = 1$, $\beta^+ = 5$, and $\alpha = 1/3$. 
IFE local quadratic basis functions on the element \((0, 1)\) with \(\beta^- = 1\), \(\beta^+ = 5\), and \(\alpha = 1/3\).

### 2.2 Two-Dimensional Parabolic Interface Problem

Consider the two-dimensional parabolic interface problem:

\[
\frac{\partial u}{\partial t} - \nabla \cdot (\beta \nabla u) = f(x, t), \quad x = (x, y) \in \Omega^+ \cup \Omega^-, \ t \in (0, T],
\]

\[
u = 0, \quad x \in \partial \Omega, \quad t \in (0, T],
\]

\[
u = u_0(x), \quad x \in \overline{\Omega}, \quad t = 0.
\]

Here, the diffusion coefficient \(\beta\) is assumed to be time independent and, without loss of generality, a piecewise constant function over \(\Omega\), \(i.e.,\)

\[
\beta(x) = \begin{cases} 
\beta^-, & x \in \Omega^-, \\
\beta^+, & x \in \Omega^+,
\end{cases}
\]

and \(\min\{\beta^-, \beta^+\} > 0\). Across the interface curve \(\Gamma\), we assume that the solution and the normal component of the flux are continuous for any time \(t \in [0, T]\), \(i.e.,\)

\[
[u]_{\Gamma} = 0,
\]

\[
[\beta \nabla u \cdot \mathbf{n}]_{\Gamma} = 0.
\]
Here \( [v]_{\Gamma} = (v|_{\Omega^+})|_{\Gamma} - (v|_{\Omega^-})|_{\Gamma} \) denotes the jump across the interface \( \Gamma \), and \( \mathbf{n} \) is the unit normal vector to the interface \( \Gamma \).

In this section, a nonconforming rotated-\( Q_1 \) immersed finite element space is proposed to solve the two-dimensional parabolic interface problem with a moving interface. The nonconforming IFE approach used in this paper has distinct advantages in comparison to conforming IFE approaches when solving interface problems. A distinct difference between conforming FE and nonconforming FE is the way the continuity is imposed. Conforming FE imposes continuity traditionally through the nodal values at the mesh points; however, the nonconforming FE weakly enforces continuity through the mean value over the element’s edge. The IFE space produced from the conforming FE spaces are usually nonconforming since the IFE functions are discontinuous across the interface. Therefore, with large interface jumps and unfavorable interface configurations, the conforming IFE space may produce suboptimal results. To mitigate this issue, partial penalized immersed finite element (PPIFE) methods have been used to address the discontinuity of conforming IFE space [71]. The PPIFE method on conforming IFE spaces have been shown to converge optimally [71]. The nonconforming IFE configuration uses no stabilizing terms which makes the derivation and the computation more efficient while producing accurate results.

### 2.3 Nonconforming Immersed Finite Element Space

In this subsection, we review the nonconforming IFE space introduced in [74]. Suppose \( \Omega \) is a rectangular domain. Let \( R_h \) be a Cartesian mesh in \( \Omega \), where \( h > 0 \). Let \( R \) be an element in \( R_h \) such that \( R = \square A_1 A_2 A_3 A_4 \). The vertices and edges of \( R \) are defined to be

\[
A_1 = (x_0, y_0), \ A_2 = (x_0 + h_x, y_0), \ A_3 = (x_0 + h_x, y_0 + h_y), \ A_4 = (x_0, y_0 + h_y)
\]  

(2.15)
and

\[ e_1 = \overline{A_1A_2}, \quad e_2 = \overline{A_2A_3}, \quad e_3 = \overline{A_3A_4}, \quad e_4 = \overline{A_4A_1}, \quad \]  

(2.16)

respectively. Without loss of generality, define \( R^I_h \) to be the collection of interface elements in \( R_h \).

Suppose the interface curve \( \Gamma \) intersects \( R \) at two different points, D and E, where the line segment \( DE \) divides the element \( R \) into two sub-elements \( R^+ \) and \( R^- \). An interface type is defined by the location of the interface points D and E. Type I interface elements are classified as elements in which the two interface points are located on adjacent edges. Alternatively, Type II elements are classified as elements where the interface points are located on opposite edges. Refer to Figure 2.3 for a pictorial illustration.

Type II interface element will be used to illustrate the derivation of the local IFE functions. Without loss of generality, define the interface points located on the interface element to be \( D = (x_0 + dh_x, y_0) \) and \( E = (x_0 + eh_x, y_0 + h_y) \), where \( d, e \in (0, 1) \). The local IFE space is an extension of the local FE space. A piecewise rotated-\( Q_1 \) polynomial for a local IFE basis function, defined by \( \phi_R \), is as followed:

\[
\phi_R(x, y) = \begin{cases} 
  c_1^+ + c_2^+ \left( \frac{x-x_0}{h_x} \right) + c_3^+ \left( \frac{y-y_0}{h_y} \right) + c_4^+ \left( \frac{x-x_0}{h_x} \right)^2 - \left( \frac{y-y_0}{h_y} \right)^2 & \in R^+ \\
  c_1^- + c_2^- \left( \frac{x-x_0}{h_x} \right) + c_3^- \left( \frac{y-y_0}{h_y} \right) + c_4^- \left( \frac{x-x_0}{h_x} \right)^2 - \left( \frac{y-y_0}{h_y} \right)^2 & \in R^- 
\end{cases} 
\]  

(2.17)

Define \( P^I_h(R) \) to be \( \text{Span}\{\phi_{j,R} : j = 1, 2, 3, 4\} \), where \( P^I_h(R) \) is the local rotated-\( Q_1 \) IFE space on the interface element \( R \). Each variable, \( c_j^- \) or \( c_j^+ \), is determined by the mean values over the edges:

\[
\frac{1}{|e_k|} \int_{e_k} \phi_{j,R} ds = \delta_{j,k}, \quad j, k = 1, 2, 3, 4, 
\]  

(2.18)
where $|e_k|$ is the length of the respective edges of $R$ and $\delta_{j,k}$ is the Kronecker delta function. The following interface jump conditions are imposed to determine the values of each coefficient:

$$[\phi_R(D)] = 0, \quad [\phi_R(E)] = 0, \quad \text{(2.19)}$$

$$c_4^+ = c_4^-, \quad \text{(2.20)}$$

$$\int_{DE} [\beta \nabla \phi_R \cdot \mathbf{n}_{DE}] ds = 0. \quad \text{(2.21)}$$

Each of these conditions, (2.18) - (2.21), produces eight constraints which are used to construct an $8 \times 8$ linear system, $M\mathbf{c}_j = \mathbf{b}_j$, where $\mathbf{c}_j = (c_1^-, c_4^-, c_1^+, \ldots, c_4^+)^T$ and $\mathbf{b}_j = (b_{j1}, \ldots, b_{j4}, 0, \ldots, 0)^T$. The values in vector $\mathbf{b}_j$ are determined by the Kronecker delta function. Overall, this procedure produces the $j^{th}$ vector used to calculate the nonconforming rotated-$Q_1$ IFE basis function $\phi_{j,R}$. For further details see [74].
2.4 Numerical Schemes

In this section, we present the IFE method of lines algorithm for solving parabolic interface problems. The method of lines technique is useful since the spatial discretization and the temporal discretization are done separately. The semi-discretization scheme is presented for each, the one-dimensional and two-dimensional parabolic interface problem. Time marching schemes, including Backward-Euler and Crank-Nicolson methods, are implemented to fully discretize the systems.

2.4.1 Semi-Discretization for One-dimensional Parabolic Interface Problem

We first consider the spatial discretization for model problem (2.1). To derive its weak formulation, multiply (2.1) by any test function \( v(x) \in H^1_0(\Omega) \). Applying integration by parts and the jump conditions, we have

\[
\int_{\Omega} u_t(t,x)v(x)dx + \int_{\Omega^-} \beta^- u_x(t,x)v_x(x)dx + \int_{\Omega^+} \beta^+ u_x(t,x)v_x(x)dx = \int_{\Omega} f(t,x)v(x)dx. \quad (2.22)
\]

Note that the interface jump condition (2.6) must hold to obtain the weak formulation of the interface problem. Define the IFE trial and test function spaces as

\[
U_h = \{ v \in S_h : v|_{\partial \Omega} = g(t, \cdot) \}, \quad \text{and} \quad V_h = \{ v \in S_h : v|_{\partial \Omega} = 0 \}. \quad (2.23)
\]

The semi-discrete problem for a fixed time \( t \) is: find \( u_h(t, \cdot) \in U_h \) such that

\[
\int_{\Omega} \partial_t u_h(t,x)v_h(x)dx + \int_{\Omega} \beta(t,x)u_{hx}(t,x)v_{hx}(x)dx = \int_{\Omega} f(t,x)v_h(x)dx, \quad \forall v_h \in V_h, \quad (2.24)
\]

where

\[
u_h(t,x) = \sum_{j=1}^{N} u_j(t)\phi_j(x). \quad (2.25)\]
Then the weak formulation (2.24) is equivalent to
\[
\int_\Omega \sum_{j=1}^N u_j'(t) \phi_j(x) \phi_i(x) dx + \int_\Omega \beta(t, x) \sum_{j=1}^N u_j(t) \phi_j'(x) \phi_i(x) dx = \int_\Omega f(t, x) \phi_i(x) dx, \tag{2.26}
\]
\[\forall \phi_i \in V_h. \]

This can be equivalently written in the matrix-vector form:
\[
M \mathbf{u}'(t) + S(t) \mathbf{u}(t) = \mathbf{f}(t). \tag{2.27}
\]

The notations in (2.27) are specified as follows
- $M$ is the mass matrix and $M_{ij} = \int_\Omega \phi_j(x) \phi_i(x) dx$.
- $S(t)$ is the stiffness matrix with $S_{ij}(t) = \int_\Omega \beta(t, x) \phi_j'(x) \phi_i'(x) dx$. Note that the stiffness matrix $S(t)$ is time dependent since the coefficient function $\beta$ is time dependent.
- $\mathbf{f}(t)$ is the source vector with $f_i(t) = \int_\Omega f(t, x) \phi_i(x) dx$.
- $\mathbf{u}(t) = [u_1(t), u_2(t), \cdots, u_N(t)]^T$ is the unknown vector.

### 2.4.2 Semi-Discretization for Two-dimensional Parabolic Interface Problem

Given the model problem (2.9), let's consider the weak form
\[
\int_\Omega u_t v dx dy + \int_\Omega \beta \nabla u \cdot \nabla v dx dy = \int_\Omega f v dx dy, \quad \forall v \in H^1_0(\Omega). \tag{2.28}
\]

The IFE trial and test function spaces are defined as:
\[
\bar{U}_h = \{ v \in P_h : v|_{\partial \Omega} = 0 \}, \quad \text{and} \quad \bar{V}_h = \{ v \in P_h : v|_{\partial \Omega} = 0 \}. \tag{2.29}
\]

Find $u_h(t, \cdot) \in \bar{U}_h$ such that
\[
\int_\Omega \partial_t u_h v_h dx dy + \int_\Omega \beta \nabla u_h \cdot \nabla v_h dx dy = \int_\Omega f v_h dx dy, \quad \forall v_h \in \bar{V}_h \tag{2.30}
\]
where

\[ u_h(t, x, y) = \sum_{j=1}^{N} u_j(t) \phi_j(x, y). \]  

(2.31)

Thus, the semi-discrete formulation is:

\[
\int_{\Omega} \sum_{j=1}^{N} u_j'(t) \phi_j(x, y) \phi_i(x, y) dx dy + \int_{\Omega} \beta \sum_{j=1}^{N} u_j(t) \nabla \phi_j(x, y) \cdot \nabla \phi_i(x, y) dx dy \\
= \int_{\Omega} f(t, x, y) \phi_i(x, y) dx dy, \quad \forall \phi_i \in V_h,
\]

(2.32)

which is equivalent to the weak formulation. The semi-discrete form can be written in the matrix-vector form:

\[
\tilde{M} \dot{u}'(t) + \tilde{S}(t) u(t) = \tilde{f}(t).
\]

(2.33)

The notations in (2.33) are specified as followed:

- \( \tilde{M} \) is the mass matrix and \( \tilde{M} = \int_{\Omega} \phi_j(x, y) \phi_i(x, y) dx dy \).
- \( \tilde{S}(t) \) is the stiffness matrix where \( \tilde{S}(t) = \tilde{S}_{xx} + \tilde{S}_{yy} \) with
  \[
  \tilde{S}_{xx} = \int_{\Omega} \beta \frac{\partial}{\partial x} \phi_j(x, y) \frac{\partial}{\partial x} \phi_i(x, y) dx dy, \text{ and}
  \]
  \[
  \tilde{S}_{yy} = \int_{\Omega} \beta \frac{\partial}{\partial y} \phi_j(x, y) \frac{\partial}{\partial y} \phi_i(x, y) dx dy.
  \]
- \( \tilde{f}(t) \) is the source vector with \( \tilde{f}(t) = \int_{\Omega} f(t, x, y) \phi_i(x, y) dx dy \).
- \( \tilde{u}(t) = [\tilde{u}_1(t), \tilde{u}_2(t), \ldots, \tilde{u}_N(t)]^T \) is the unknown vector.

### 2.4.3 Full-Discretization

We further discretize the ODE systems (2.27) and (2.33) by some well-known difference schemes such as the forward-Euler method, the Backward-Euler method, and the Crank-Nicolson method. We derive a general framework that incorporates each method by introducing a parameter \( \theta \in [0, 1] \).
Consider a uniform partition of the time domain, \(0 = t_0 < t_1 < \cdots < t_K = T\), where \(t_n = n\Delta t\) and \(\Delta t = T/K\). Evaluate equation (2.27) at \(t = t_n + \theta\Delta t\) and use the following finite-difference approximations:

\[
\mathbf{u}'(t_n + \theta\Delta t) \approx \frac{\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n)}{\Delta t},
\]

(2.34)

\[
S(t_n + \theta\Delta t)\mathbf{u}(t_n + \theta\Delta t) \approx (1 - \theta)S(t_n)\mathbf{u}(t_n) + \theta S(t_{n+1})\mathbf{u}(t_{n+1}),
\]

(2.35)

and

\[
f(t_n + \theta\Delta t) \approx (1 - \theta)f(t_n) + \theta f(t_{n+1}).
\]

(2.36)

Our fully-discretized scheme becomes

\[
(M + \theta\Delta t S^{n+1}) \mathbf{u}^{n+1} = \left( M - (1 - \theta)\Delta t S^n \right) \mathbf{u}^n + (1 - \theta)\Delta t \mathbf{f}^n + \theta \Delta t \mathbf{f}^{n+1}.
\]

(2.37)

Similarly, the two-dimensional fully-discretized scheme is:

\[
(\tilde{M} + \theta\Delta t \tilde{S}^{n+1}) \tilde{\mathbf{u}}^{n+1} = \left( \tilde{M} - (1 - \theta)\Delta t \tilde{S}^n \right) \tilde{\mathbf{u}}^n + (1 - \theta)\Delta t \tilde{\mathbf{f}}^n + \theta \Delta t \tilde{\mathbf{f}}^{n+1}.
\]

(2.38)

For \(\theta = 0, \frac{1}{2}\) and 1, equation (2.37) and (2.38) becomes the forward-Euler, the Crank-Nicolson, and the Backward-Euler methods, respectively. To start the time marching scheme (2.37) and (2.38), the initial vectors \(\mathbf{u}_0\) and \(\tilde{\mathbf{u}}_0\) can take the interpolation of the initial functions from (2.1) and (2.9), respectively. In particular, we can choose

\[
\mathbf{u}_0 = [u_0(x_1), u_0(x_2), \cdots, u_0(x_N)]^T \text{ and } \tilde{\mathbf{u}}_0 = [\tilde{u}_0(x_1), \tilde{u}_0(x_2), \cdots, \tilde{u}_0(x_N)]^T.
\]

The forward-Euler method is notoriously unstable as it creates heavy oscillations in general. Therefore, we only consider Backward-Euler and Crank-Nicolson for this analysis. The Backward-Euler (BE) method and the Crank-Nicolson (CN) method are known to be very stable. From this derivation, we can expect the convergence rates in the temporal discretization to be \(O(\Delta t)\)
and $O(\Delta t^2)$ for BE and CN, respectively. Combining with the optimal convergence for spatial discretization from Section 2.4.1, we look to achieve the overall convergence rate listed in Table 2.1. In this table, $k$ denotes the polynomial degree for IFEM spaces.

Table 2.1

<table>
<thead>
<tr>
<th></th>
<th>$L^\infty$ Norm</th>
<th>$L^2$ Norm</th>
<th>$H^1$ Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>BE</td>
<td>$O(h^{k+1} + \Delta t)$</td>
<td>$O(h^{k+1} + \Delta t)$</td>
<td>$O(h^k + \Delta t)$</td>
</tr>
<tr>
<td>CN</td>
<td>$O(h^{k+1} + \Delta t^2)$</td>
<td>$O(h^{k+1} + \Delta t^2)$</td>
<td>$O(h^k + \Delta t^2)$</td>
</tr>
</tbody>
</table>

### 2.5 Numerical Results For One-Dimensional Parabolic Interface Problems

In this section, we present numerical examples to test the performance of our schemes. For each test, we use both linear and quadratic IFEMs for spatial discretization on a family of uniform meshes $\{T_h\}$, where each mesh size is $h = \frac{1}{N}$. The temporal discretization uses the Backward-Euler and Crank-Nicolson methods on a uniform time partition with step size $\Delta t$ equal to the spatial mesh size $h$. 

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2.5.1 Example 2.1: Interface Problem with Piecewise Constant Coefficient $\beta$

In this example, we consider the model problem (2.1) on the domain $\Omega = (-1, 1)$ and an interface point at $\alpha = \frac{2}{3}$. We assume the coefficient function to be piecewise constant as follows:

$$
\beta(t, x) := \begin{cases} 
1 & \text{if } x \in (0, \alpha), \\
10 & \text{if } x \in (\alpha, 1).
\end{cases}
$$

(2.39)

The exact equation of this problem is:

$$
\begin{align*}
\frac{\partial u}{\partial t}(t, x) &= \begin{cases} 
\cos(x)e^{2t}, & \text{if } x \in (0, \alpha), \\
\frac{1}{10}\cos(x)e^{2t} + \frac{9}{10}\cos(\alpha)e^{2t} & \text{if } x \in (\alpha, 1).
\end{cases}
\end{align*}
$$

(2.40)

One can easily verify that the solution (2.40) satisfies the jump conditions (2.5) and (2.6). We report the error of the numerical solution at the final time level in $L^\infty$, $L^2$, and $H^1$ norms. Numerical results for linear and quadratic IFEM with the Backward-Euler scheme are reported in Table 2.2 and Table 2.3, respectively. Numerical results for linear and quadratic IFEM with the Crank-Nicolson scheme are reported in Table 2.4 and Table 2.5, respectively. We observe from these tables that the convergence rates in all three norms match the expected convergence rates in Table 2.1.
Table 2.2

Errors and Convergence Rate for Linear IFEM-BE Approximation for Example 2.1.

<table>
<thead>
<tr>
<th>1/h</th>
<th>$L^\infty$ Norm</th>
<th>$L^\infty$ Order</th>
<th>$L^2$ Norm</th>
<th>$L^2$ Order</th>
<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
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<td>10</td>
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<tr>
<td>80</td>
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<td>0.9912</td>
<td>6.0301e-03</td>
<td>0.9737</td>
<td>3.2148e-02</td>
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<td>160</td>
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<td>3.0420e-03</td>
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<td>1.6096e-02</td>
<td>0.9980</td>
</tr>
</tbody>
</table>

Table 2.3

Errors and Convergence Rate for Quadratic IFEM-BE Approximation for Example 2.1.

<table>
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<tr>
<th>1/h</th>
<th>$L^\infty$ Norm</th>
<th>$L^\infty$ Order</th>
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<th>$L^2$ Order</th>
<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7.4509e-02</td>
<td>4.6375e-02</td>
<td>1.8966e-01</td>
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<td></td>
<td></td>
</tr>
<tr>
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<td>2.3855e-02</td>
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<td>9.7560e-02</td>
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<td>2.4917e-02</td>
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</tr>
<tr>
<td>160</td>
<td>4.9110e-03</td>
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<td>3.0576e-03</td>
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</tbody>
</table>
Table 2.4

<table>
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<th>$L^2$ Order</th>
<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
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<td>1.5913e-01</td>
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<td>1.9994</td>
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<tr>
<td>160</td>
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<td>1.9994</td>
<td>1.0122e-02</td>
<td>0.9993</td>
</tr>
</tbody>
</table>

2.5.2 Example 2.2: Parabolic Interface Problem With Non-Constant Coefficient $\beta$

In this example, we consider the interface problem with a large contrast non-constant coefficient function as defined below:

$$\beta(t, x) := \begin{cases} 
e^t(x + 1)^2, & x \in (0, \alpha), \\ 100\ne^t(x + 2)^2, & x \in (\alpha, 1), \end{cases}$$

(2.41)

where the interface $\alpha = 5/6$ in this example. The exact solution is

$$u(t, x) := \begin{cases} 
e^{2t}(x + 2)^3, & x \in (0, \alpha), \\ \ne^{2t}(c_1 + c_2(x - \alpha) + c_3(x - \alpha)^2 + d(x)), & x \in (\alpha, 1), \end{cases}$$

(2.42)
Table 2.5

Errors and Convergence Rate for Quadratic IFEM-CN Approximation for Example 2.1.

<table>
<thead>
<tr>
<th>1/h</th>
<th>$L^\infty$ Norm</th>
<th>$L^\infty$ Order</th>
<th>$L^2$ Norm</th>
<th>$L^2$ Order</th>
<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
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</thead>
<tbody>
<tr>
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<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
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<td>1.6836e-03</td>
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</tr>
<tr>
<td>40</td>
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<td>1.0226e-04</td>
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<td>4.2097e-04</td>
<td>1.9997</td>
<td></td>
</tr>
<tr>
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</tr>
<tr>
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<td>2.6338e-05</td>
<td>2.0002</td>
<td></td>
</tr>
</tbody>
</table>

where,

\[
\begin{align*}
  c_1 &= \frac{(2+\alpha)^2 - \left( (3+\alpha)(6+\alpha) - 6(2+\alpha) \log(2+\alpha) \right)}{100}, \\
  c_2 &= \frac{3(1-2\alpha+2(5+2\alpha) \log(2+\alpha))}{100}, \\
  c_3 &= \frac{6(1+(2+\alpha) \log(2+\alpha))}{(2+\alpha)100}, \\
  d(x) &= \frac{((3+x)(x(6+x) - 6(2+x) \log(2+x)))}{100}.
\end{align*}
\]  

(2.43)

Tables 2.6 and 2.7 show the error and convergence rates for the Backward-Euler solution with linear and quadratic IFEM functions, respectively. Tables 2.8 and 2.9 report the errors for the Crank-Nicolson method using linear and quadratic IFEM functions. Again, these results are consistent with the anticipated convergence order in Table 2.1. This example shows the robustness of our numerical schemes concerning the non-constant coefficient functions and high-jump circumstances.
### Table 2.6

Errors and Convergence Rate for Linear IFEM-BE Approximation for Example 2.2.

<table>
<thead>
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<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
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<tbody>
<tr>
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<td>2.7536e-01</td>
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<td>2.8711e-00</td>
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</tr>
<tr>
<td>20</td>
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<td>0.9200</td>
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<td>1.1490</td>
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<td>0.9809</td>
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<td>7.3514e-01</td>
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<tr>
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### Table 2.7

Errors and Convergence Rate for Quadratic IFEM-BE Approximation for Example 2.2.

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<th>$L^\infty$ Order</th>
<th>$L^2$ Norm</th>
<th>$L^2$ Order</th>
<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
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</thead>
<tbody>
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<td>2.0869e-01</td>
<td></td>
<td>8.0338e-01</td>
<td></td>
</tr>
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</tr>
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</tbody>
</table>
### Table 2.8

Errors and Convergence Rate for Linear IFEM-CN Approximation for Example 2.2.

<table>
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<tr>
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<th>$L^\infty$ Order</th>
<th>$L^2$ Norm</th>
<th>$L^2$ Order</th>
<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
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<td>1.3989e-00</td>
<td>0.9914</td>
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<td>9.5152e-03</td>
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<td>5.9479e-04</td>
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<td>1.7707e-01</td>
<td>0.9968</td>
</tr>
</tbody>
</table>

### Table 2.9

Errors and Convergence Rate for Quadratic IFEM-CN Approximation for Example 2.2.

<table>
<thead>
<tr>
<th>1/h</th>
<th>$L^\infty$ Norm</th>
<th>$L^\infty$ Order</th>
<th>$L^2$ Norm</th>
<th>$L^2$ Order</th>
<th>$H^1$ Norm</th>
<th>$H^1$ Order</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1.1819e-01</td>
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<td>3.0281e-01</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
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<td>1.9958</td>
<td>1.9853e-02</td>
<td>1.9948</td>
<td>7.6012e-02</td>
<td>1.9941</td>
</tr>
<tr>
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<td>3.1056e-04</td>
<td>1.9999</td>
<td>1.1892e-03</td>
<td>1.9999</td>
</tr>
</tbody>
</table>
2.6 Numerical Examples for Two-Dimensional Parabolic Problems

We present numerical examples to test the performance of the nonconforming $Q_1$-IFE method for parabolic interface problems. For each test, we use a collection of cartesian meshes $\{R_h\}$, where each mesh consists of $N \times N$ congruent rectangles. The temporal discretization uses the Backward-Euler, and the Crank-Nicolson methods on a uniform time partition with step size $\Delta t$. In our numerical test, we choose $\Delta t = 2h$. The IFE approximation errors are reported in the $L^\infty$, $L^2$, and $H^1$ norm for the Backward-Euler and the Crank-Nicolson methods.

2.6.1 Example 2.3: Circular Interface

Let $\Omega = (-1, 1)^2$ with an enclosed circular interface at the origin with radius $r_0 = \pi/6$. Thus $\Omega$ can be separated into two regions $\Omega^-$ and $\Omega^+$, such that:

$$
\Omega^- = \{(x, y) \in \Omega : x^2 + y^2 < r_0^2\},
\Omega^+ = \{(x, y) \in \Omega : x^2 + y^2 > r_0^2\}.
$$

Choose the exact solution to be:

$$
u(t, x, y) = \begin{cases}
\frac{r^a}{\beta_-} e^{mt}, & \text{if } r < r_0 \\
\left(\frac{r^a}{\beta_+} + \left(\frac{1}{\beta_-} - \frac{1}{\beta_+}\right)r_0^a\right) e^{mt}, & \text{if } r > r_0,
\end{cases}
$$

(2.45)

where $a = 5$, $m = 1$, and $r = \sqrt{x^2 + y^2}$. The numerical solution for the nonconforming IFE with the circular interface is given above. Tables 2.10 and 2.11 report the low jump cases, $(\beta^-, \beta^+) = (1, 2)$, while Tables 2.12 and 2.13 report high jump cases, $(\beta^-, \beta^+) = (1000, 1)$, for the Backward-Euler and Crank-Nicolson methods. Data confirms that the Crank-Nicolson and Backward-Euler methods converges optimally in the $L^\infty$, $L^2$, and $H^1$ norm for both the low jump and the high jump cases.
Table 2.10

IFE solutions for Example 2.3 with $\beta^- = 1$ and $\beta^+ = 2$ using Backward-Euler.

| N  | $|| \cdot ||_{L^\infty}$          | Rate    | $|| \cdot ||_{L^2}$          | Rate    | $|| \cdot ||_{H^1}$          | Rate    |
|----|----------------------------------|---------|--------------------------------|---------|--------------------------------|---------|
| 8  | 7.7829e-02                       | 1.6264e-01 | 2.6999e+00                      |
| 16 | 2.2305e-02 1.8029                | 3.9791e-02 2.0312 | 1.3658e+00 0.9832              |
| 32 | 5.8868e-03 1.9218                | 9.9916e-03 1.9937 | 6.8397e-01 0.9977              |
| 64 | 2.7809e-03 1.0819                | 2.9905e-03 1.7403 | 3.4249e-01 0.9979              |
| 128| 1.4511e-03 0.9384                | 1.2123e-03 1.3027 | 1.7141e-01 0.9986              |
| 256| 7.5776e-04 0.9374                | 5.9463e-04 1.0276 | 8.5726e-02 0.9996              |
| 512| 3.8777e-04 0.9665                | 3.0304e-04 0.9725 | 4.2867e-02 0.9998              |

Table 2.11

IFE solutions for Example 2.3 with $\beta^- = 1$ and $\beta^+ = 2$ using Crank-Nicolson.

| N  | $|| \cdot ||_{L^\infty}$          | Rate    | $|| \cdot ||_{L^2}$          | Rate    | $|| \cdot ||_{H^1}$          | Rate    |
|----|----------------------------------|---------|--------------------------------|---------|--------------------------------|---------|
| 8  | 6.1018e-02                       | 1.6081e-01 | 2.6919e+00                      |
| 16 | 2.0751e-02 1.5561                | 4.1088e-02 1.9685 | 1.3645e+00 0.9802              |
| 32 | 5.4698e-03 1.9236                | 1.0256e-02 2.0023 | 6.8368e-01 0.9970              |
| 64 | 1.4932e-03 1.8731                | 2.5794e-03 1.9913 | 3.4238e-01 0.9977              |
| 128| 3.7282e-04 2.0018                | 6.4642e-04 1.9965 | 1.7136e-01 0.9985              |
| 256| 9.6381e-05 1.9517                | 1.6225e-04 1.9943 | 8.5704e-02 0.9996              |
| 512| 2.4266e-05 1.9898                | 4.0542e-05 2.0007 | 4.2857e-02 0.9998              |
Table 2.12

IFE solutions for Example 2.3 with $\beta^- = 1,000$ and $\beta^+ = 1$ using Backward-Euler.

<table>
<thead>
<tr>
<th>N</th>
<th>$| \cdot |_{L^\infty}$</th>
<th>Rate</th>
<th>$| \cdot |_{L^2}$</th>
<th>Rate</th>
<th>$| \cdot |_{H^1}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.4210e-01</td>
<td></td>
<td>4.0303e-01</td>
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<td>6.3287e+00</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>3.8368e-02</td>
<td>1.8890</td>
<td>1.0866e-01</td>
<td>1.8911</td>
<td>3.2266e+00</td>
<td>0.9719</td>
</tr>
<tr>
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<td>3.1129e-02</td>
<td>1.8035</td>
<td>1.6210e+00</td>
<td>0.9931</td>
</tr>
<tr>
<td>64</td>
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<td>1.4133</td>
<td>9.8747e-03</td>
<td>1.6565</td>
<td>8.1152e-01</td>
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<tr>
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<tr>
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</tbody>
</table>

Table 2.13

IFE solutions for Example 2.3 with $\beta^- = 1,000$ and $\beta^+ = 1$ using Crank-Nicolson.

<table>
<thead>
<tr>
<th>N</th>
<th>$| \cdot |_{L^\infty}$</th>
<th>Rate</th>
<th>$| \cdot |_{L^2}$</th>
<th>Rate</th>
<th>$| \cdot |_{H^1}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
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<td>2.5520e-01</td>
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<td>4.6699e+00</td>
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</tr>
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<td>2.9642e-01</td>
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<td>1.4816e-01</td>
<td>1.0004</td>
</tr>
</tbody>
</table>
2.6.2 Example 2.4: Petal-like Moving Interface

Let $\Omega = (-1, 1)^2$ with an enclosed petal-like interface at the origin. The interface is defined by the function

$$\Gamma(t, x, y) = \left(x^2 + y^2\right)^2 \left(1 + 0.4 \sin \left(n \tan^{-1} \left(\frac{y}{x}\right)\right)\right) - h \left(1 + \left(\frac{19}{2}\right) \sin(t)\right), \quad (2.46)$$

where $h = 0.3$ and $n = 6$. Thus $\Omega$ can be separated into two regions $\Omega^-(t)$ and $\Omega^+(t)$, such that

$$\Omega^-(t) = \{(x, y) \in \Omega : \Gamma(t, x, y) < 0\} \text{ and } \Omega^+(t) = \{(x, y) \in \Omega : \Gamma(t, x, y) > 0\}. \quad (2.47)$$

Choose the exact solution to be

$$u(t, x, y) = \begin{cases} 
\frac{1}{\beta} \Gamma(t, x, y) e^{mt}, & \text{if } (x, y) \in \Omega^-(t) \\
\frac{1}{\beta^*} \Gamma(t, x, y) e^{mt}, & \text{if } (x, y) \in \Omega^+(t),
\end{cases} \quad (2.48)$$

where $m = 1$. 
### Table 2.14

IFE solutions for Example 2.4 with $\beta^{-} = 1$ and $\beta^{+} = 2$ using Backward-Euler.

<table>
<thead>
<tr>
<th>N</th>
<th>$| \cdot |_{L^{\infty}}$ Rate</th>
<th>$| \cdot |_{L^{2}}$ Rate</th>
<th>$| \cdot |_{H^{1}}$ Rate</th>
</tr>
</thead>
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<td>8</td>
<td>1.1349e-01  2.2422e-01</td>
<td>3.1084e+00</td>
<td></td>
</tr>
<tr>
<td>16</td>
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### Table 2.15

IFE solutions for Example 2.4 with $\beta^{-} = 1$ and $\beta^{+} = 2$ using Crank-Nicolson.

<table>
<thead>
<tr>
<th>N</th>
<th>$| \cdot |_{L^{\infty}}$ Rate</th>
<th>$| \cdot |_{L^{2}}$ Rate</th>
<th>$| \cdot |_{H^{1}}$ Rate</th>
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</tr>
</tbody>
</table>
Table 2.16

IFE solutions for Example 2.4 with $\beta^- = 1,000$ and $\beta^+ = 1$ using Backward-Euler.

<table>
<thead>
<tr>
<th>N</th>
<th>$|\cdot|_{L^\infty}$ Rate</th>
<th>$|\cdot|_{L^2}$ Rate</th>
<th>$|\cdot|_{H^1}$ Rate</th>
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</thead>
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</tr>
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</table>

Table 2.17

IFE solutions for Example 2.4 with $\beta^- = 1,000$ and $\beta^+ = 1$ using Crank-Nicolson.

<table>
<thead>
<tr>
<th>N</th>
<th>$|\cdot|_{L^\infty}$ Rate</th>
<th>$|\cdot|_{L^2}$ Rate</th>
<th>$|\cdot|_{H^1}$ Rate</th>
</tr>
</thead>
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<td>512</td>
<td>6.4710e-04 1.2657</td>
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</table>
CHAPTER III
NONCONFORMING IMMERSED FINITE ELEMENT METHODS FOR STOKES INTERFACE PROBLEMS

In this chapter, we introduce the two-dimensional Stokes interface problem along with its applications. A survey of the literature is given to discuss various numerical methods used for solving the Stokes problem. We introduce a class of lowest-order nonconforming immersed finite element (IFE) methods for solving the two-dimensional Stokes interface problem. The proposed methods do not require the solution mesh to align with the fluid interface and can use either triangular or rectangular meshes. On triangular meshes, the Crouzeix-Raviart element is used for velocity approximation and a piecewise constant element for pressure approximation. On rectangular meshes, the Rannacher-Turek rotated-$Q_1$-$Q_0$ finite element is used. The new vector-valued IFE functions are constructed to approximate the interface jump conditions. Basic properties including the unisolvency and the partition of unity of these new IFE functions are discussed. Approximation capabilities of the new IFE spaces for the Stokes interface problem are examined through a series of numerical examples. Numerical approximations in the $L^2$-norm and the broken $H^1$-norm for the velocity and the $L^2$-norm for the pressure are observed to converge optimally.
3.1 Statement of the Problem

We are interested in steady-state fluid flow problems consisting of two immiscible fluids separated by an interface. Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain separated by a smooth interface $\Gamma$. The interface $\Gamma$ separates $\Omega$ into two disjoint subdomains $\Omega^+$ and $\Omega^-$ such that $\tilde{\Omega} = \tilde{\Omega}^+ \cup \tilde{\Omega}^-$ and $\Omega^+ \cap \Omega^- = 0$. Each subdomain is occupied by a fluid. See Figure 3.1 for an illustration of the domain. Consider the governing incompressible Stokes equations:

\begin{align*}
-\nabla \cdot S(\mathbf{u}, p) &= f \quad \text{in} \quad \Omega^+ \cup \Omega^-, \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\
\mathbf{u} &= 0 \quad \text{in} \quad \partial \Omega,
\end{align*}

Figure 3.1

Domain of an interface problem.
where \( \mathbf{u} \) and \( p \) denote the velocity and the pressure, respectively. \( S(\mathbf{u}, p) \) is the stress tensor defined as

\[
S(\mathbf{u}, p) = 2\mu \varepsilon(\mathbf{u}) - p \mathbf{I},
\]

(3.4)

where \( \varepsilon(\mathbf{u}) = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2 \) is the strain tensor, and \( \mathbf{I} \) is the identity tensor. The viscosity function \( \mu(x) \) is assumed to have a finite jump across the interface \( \Gamma \). For simplicity, we assume that \( \mu(x) \) is a piecewise constant function

\[
\mu(x) = \begin{cases} 
\mu^- & \text{in } \Omega^-, \\
\mu^+ & \text{in } \Omega^+,
\end{cases}
\]

(3.5)

where \( \mu^\pm \) are positive constants and \( x = (x, y) \). Across the fluid interface \( \Gamma \), the solution is assumed to satisfy the following velocity and stress jump conditions:

\[
[u]_\Gamma = 0,
\]

(3.6)

\[
[S(\mathbf{u}, p) \mathbf{n}]_\Gamma = 0,
\]

(3.7)

where the jump \( [v(x)]_\Gamma := v^+(x)|_\Gamma - v^-(x)|_\Gamma \), and \( \mathbf{n} \) denotes the unit normal vector to the interface \( \Gamma \) pointing from \( \Omega^- \) to \( \Omega^+ \). Throughout the chapter, we use the standard notation \( (\cdot, \cdot)_\omega \) to denote the \( L^2 \) inner product on \( \omega \subset \Omega \). We omit subscript \( \omega \) if \( \omega = \Omega \). Note that the Stokes equation (3.1) can be simplified when the viscosity coefficient \( \mu(x) \) is a (piecewise) constant. In this case, the incompressibility condition (3.2) yields

\[
\frac{\partial \varepsilon_{11}(\mathbf{u})}{\partial x} + \frac{\partial \varepsilon_{12}(\mathbf{u})}{\partial y} = \frac{1}{2} \left( \frac{\partial^2 u_1}{\partial x^2} + \frac{\partial}{\partial x} \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) + \frac{\partial^2 u_1}{\partial y^2} \right) = \frac{1}{2} \Delta u_1,
\]

(3.8)

and similarly,

\[
\frac{\partial \varepsilon_{21}(\mathbf{u})}{\partial x} + \frac{\partial \varepsilon_{22}(\mathbf{u})}{\partial y} = \frac{1}{2} \left( \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial}{\partial y} \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) + \frac{\partial^2 u_2}{\partial y^2} \right) = \frac{1}{2} \Delta u_2.
\]

(3.9)
Therefore, the momentum equation (3.1) can be written as

$$-\mu \Delta \mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega^+ \cup \Omega^-.$$  

(3.10)

In this framework, the stress interface jump condition (3.7) is modified to

$$[(\mu \nabla \mathbf{u} - p \mathbf{I}) \mathbf{n}]_\Gamma = 0.$$  

(3.11)

For more details about derivation of the Stokes equations, we refer the reader to [27].

3.2 Numerical Methods for Solving the Stokes Problem

Stokes equations are used to model multiphase flow with jumps in velocity and pressure, as well as other physical parameters. The Stokes problem has been studied for many years due to its ability to model natural phenomena. The Stokes problem is an important equation in studying multiphase flows with a moving interface. In application, the Stokes problem is studied for use in modeling blood flow in the heart [82], modeling the complexities of the Cochlea (inner ear) [48], and modeling energy production [57], to name a few.

The Taylor-Hood finite elements have been the most commonly used finite element spaces for solving the classical Stokes problem [26, 80, 96]. The family of Taylor-Hood finite elements uses conforming $P_k$-$P_{k-1}$ pairs to approximate the velocity and the pressure, requiring the polynomial degree $k \geq 2$ in [89]. Crouzeix and Raviart, in 1973, introduced a nonconforming $P_1$-$P_0$ finite element space, formally known as $CR$-$P_0$, to solve the classical Stokes problem on a triangular mesh [22]. On quadrilateral meshes, Rannacher and Turek developed a nonconforming rotated-$Q_1$ element in [85]. Nonconforming finite elements make use of low-order polynomials and are elementwise divergence-free [7, 60]. In [55], a mixed conforming-nonconforming finite element
space is introduced to solve the elasticity and the Stokes problems.

Traditional numerical methods use interface-fitted meshes for solving interface problems. For fluid flow interface problems, the arbitrary Lagrangian-Eulerian-based finite element is a popular numerical method [23, 58, 94]. There have been a few numerical methods based on unfitted meshes for Stokes equations including CutFEM [40], Nitsche’s Extended FEM [92], XFEM [28], fictitious domain FEM [79, 88], immersed interface method [65].

In recent years, the Stokes problem with an interface has been studied by many researchers. In 2015, Adjerid, Chaabane, and Lin introduced an immersed discontinuous Galerkin (IDG) $Q_1-Q_0$ finite element space to solve the Stokes interface problem [2, 18]. This space satisfies the interface jump conditions while maintaining the approximation capabilities of the finite element method. Another approach studies a nonconforming $P_1-P_0$ Nitsche’s extended FEM for Stokes interface problem with interface-unfitted meshes [92]. Nitsche’s method is used for the weak enforcement of the essential boundary conditions and can also be modified to enforce jump conditions at the interface [52, 39, 92]. Recently, a $P_2-P_1$ Taylor-Hood IFE space was introduced in [19]. The partially penalized IFE scheme is used with ghost penalty for enhancing the stability of numerical scheme especially for the pressure approximation.

We seek to extend the applications of the immersed finite element method to the Stokes interface problem. In this chapter, we develop two IFE approximations for the steady-state Stokes equations. Our methods are based on the nonconforming FEM framework. It is well-known that the nonconforming $P_1$ finite element, widely known as Crouzeix-Raviart (CR) element [22] defined on triangular meshes and the nonconforming $Q_1$ element, known as Rannacher-Turek element [85] or the rotated-$Q_1$ ($RQ_1$) element defined on rectangular meshes are both stable finite
element pairs for Stokes equations [7, 60]. Comparing with the Taylor-Hood finite elements [89], these nonconforming finite elements can use low-order polynomials and they are element-wise divergence-free [20].

The proposed IFE methods locally modify the $CR-P_0$, and the $RQ_1-Q_0$ FE basis functions on interface elements [50]. Trying to keep the original FE structure as much as possible, we use standard FE basis functions on non-interface elements. On interface elements, we construct new basis functions to incorporate the interface jump conditions. Note that unlike the Poisson equation [74], the IFE basis functions on interface elements are vector-valued since the stress interface condition (3.7) couples together the velocity and the pressure variables. Vector-valued IFE basis functions have been developed for the elasticity system in [75, 73] and for the Stokes equation in [2]. Comparing to other unfitted-mesh FEMs [40, 92, 28], the proposed IFE spaces are isomorphic to the standard $CR-P_0$ or $RQ_1-Q_0$ FE spaces on the same mesh. In other words, the number and the location of the degrees of freedom of the IFE space are identical to the corresponding FE space as if there was no interface. This structure-preserving feature is desirable in solving a moving interface problem [3] and can also adopt existing fast solvers from standard FEM.

Comparing with the $Q_1-Q_0$ IDG method [2], the proposed IFE methods has significantly less computational cost. To be more specific, on a Cartesian mesh with $N \times N$ rectangles, the $Q_1-Q_0$ IDG method has $9N^2$ degrees of freedom, but the new $RQ_1-Q_0$ IFEM has only $5N^2$ degrees of freedom. Cutting each rectangle into two triangles, our new $CR-P_0$ IFEM has $8N^2$ degrees of freedom, still less than the IDG method. In addition, since the IDG method does not enforce continuity across the elements, the computational algorithm must contain additional consistency and stability terms. However, our nonconforming IFE spaces impose weak continuity such that the
average integral value across the edges is continuous [35, 74]. There is no need to include these additional terms in our numerical scheme. Thus, the new IFE algorithm has a much simpler form comparing to the IDG scheme in [2]. The proposed nonconforming IFE method is probably one of the simplest unfitted schemes for Stokes interface problems.

3.3 CR-P₀ Immersed Finite Element Space

In this section, we introduce the CR-P₀ immersed finite element space for the Stokes interface problem. To this point, we assume that Ω is a polygonal domain. Let \( \mathcal{T}_h = \{T_k\}_{k=1}^N \) be an unfitted shape-regular triangulation of Ω where \( N = |\mathcal{T}_h| \) denotes the number of triangles. If an element \( T \in \mathcal{T}_h \) is cut through by the interface \( \Gamma \), we call it an interface element; otherwise, we call it a non-interface element. Denote the collections of interface elements and non-interface elements by \( \mathcal{T}_h^i \) and \( \mathcal{T}_h^n \), respectively. Let \( \mathcal{E}_h \) be the set of all edges in \( \mathcal{T}_h \). The collections of interface edges and non-interface edges are denoted by \( \mathcal{E}_h^i \) and \( \mathcal{E}_h^n \), respectively. The collections of internal edges and boundary edges are denoted by \( \mathcal{E}_h^0 \) and \( \mathcal{E}_h^b \), respectively. Moreover, on a given triangular mesh \( \mathcal{T}_h \), we assume that it satisfies the following hypotheses:

- (H1) The interface \( \Gamma \) cannot intersect an edge of any element at more than two points unless the edge is part of \( \Gamma \).
- (H2) If \( \Gamma \) intersects the boundary of an element at two points, these intersection points must be on different edges of this element.
- (H3) The interface \( \Gamma \) is a piecewise \( C^2 \)-continuous function, and the mesh \( \mathcal{T}_h \) is formed such that the subset of \( \Gamma \) in every interface element is \( C^2 \)-continuous.

3.3.1 CR-P₀ Finite Element Shape Functions

On non-interface elements, the standard CR-P₀ finite element functions are used for approximating the velocity and the pressure. Let \( T \in \mathcal{T}_h^n \) be a non-interface element with vertices \( A_1, A_2, \ldots, A_N \). The standard CR-P₀ finite element functions are given by:

\[
\phi_i(x) = \begin{cases} 
1 & \text{if } x = A_i \\
\sum_{j \neq i} \frac{1}{|A_j - A_i|^2} & \text{otherwise}
\end{cases}
\]
3 oriented counterclockwise. We label the edges of $T$ by $e_1 = \overline{A_1A_2}$, $e_2 = \overline{A_2A_3}$, and $e_3 = \overline{A_3A_1}$.

The degrees of freedom of the CR finite element is determined by the average value over edges. More precisely, the CR local shape functions $\psi_{j,T} \in P_1$, $j = 1, 2, 3$ satisfy

$$\frac{1}{|e_i|} \int_{e_i} \psi_{j,T}(x, y) ds = \delta_{ij}, \quad i, j = 1, 2, 3$$

(3.12)

where $\delta_{ij}$ is the Kronecker function. We approximate the pressure by the piecewise constant function space denoted by $P_0$. For the two-dimensional Stokes problem, the components of the velocity and the pressure constitute a vector-valued finite element space, denoted by $S_h^u(T) = P_1 \times P_1 \times P_0$. There are seven local shape functions as follows:

$$\psi_{j,T} = \begin{bmatrix} \psi_{j,T} \\ 0 \\ 0 \end{bmatrix} \quad \text{for} \quad j = 1, 2, 3, \quad \psi_{j,T} = \begin{bmatrix} 0 \\ \psi_{j-3,T} \\ 0 \end{bmatrix} \quad \text{for} \quad j = 4, 5, 6, \quad \psi_{7,T} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$  

(3.13)

The local CR-$P_0$ finite element space can be written as $S_h^n(T) = \text{span}\{\psi_{j,T} : 1 \leq j \leq 7\}$.

### 3.3.2 CR-$P_0$ Immersed Finite Element Shape Functions

On an interface element $T \in T^I_h$, simply using polynomial approximations will not be accurate, since the exact solution is not smooth across the interface. We need to modify the functions in $S_h^n(T)$ to accommodate the interface jump conditions. Without loss of generality, we consider the following reference triangle $T$ whose vertices are given by

$$\hat{A}_1 = (0, 0), \quad \hat{A}_2 = (1, 0), \quad \hat{A}_3 = (0, 1).$$

(3.14)
Note that an arbitrary triangle with vertices $A_i = (x_i, y_i), i = 1, 2, 3$ can be mapped to this reference triangle by the following mapping

$$
\begin{bmatrix}
\hat{x} \\
\hat{y}
\end{bmatrix} = \begin{bmatrix}
x_2 - x_1 & x_3 - x_1 \\
y_2 - y_1 & y_3 - y_1
\end{bmatrix}^{-1} \begin{bmatrix}
x - x_1 \\
y - y_1
\end{bmatrix}.
$$

(3.15)

For simplicity, we still use symbols without hat on reference interface elements. Based on the hypotheses (H1) and (H2), there are three geometrical configurations of the interface element. See Figure 3.2 for an illustration. Type I refers to the case where the interface $\Gamma$ separates $A_1$ from $A_2, A_3$; Type II refers to the case where the interface $\Gamma$ separates $A_2$ from $A_1, A_3$; and Type III refer to the case that the interface $\Gamma$ separates $A_3$ from $A_1, A_2$. We also let $D = (x_d, y_d)$ and $E = (x_e, y_e)$ be the two intersection points of $\Gamma$ with $\partial T$. We use the line segment $\Gamma_T = DE$ to approximate the actual interface $\Gamma_T = \Gamma \cap T$ inside the element. The element $T$ is subdivided by $\Gamma_T$ into two sub-elements $T^*$ and $T^-$. The intersection points $D$ and $E$ can be written as a convex combination of vertices. For instance, for Type I interface elements, $D = (1 - d)A_1 + dA_2$ and $E = (1 - e)A_1 + eA_3$ where $0 < d, e < 1$.

Now we are ready to construct the local IFE shape functions. Note that for systems of PDEs, the unknown functions are often coupled together through the interface jump conditions. For this Stokes system, the velocity $u$ and the pressure $p$ are coupled together through the stress interface condition (3.7). Thus, vector-valued IFE basis functions must be constructed. Define the following
Types of interface elements. The red curve $\Gamma$ is the actual interface, and $\Gamma_T = DE$ is the line approximation of the interface.

Each vector-valued IFE shape function $\phi_{j,T}$ has 14 undetermined coefficients $a_{1j}^\pm, a_{2j}^\pm, b_{1j}^\pm, b_{2j}^\pm, c_{1j}^\pm, c_{2j}^\pm, d_j^\pm$. These coefficients can be determined by seven local degrees of freedom (average edge values and mean pressure condition), and additional seven interface jump conditions stated below:
• Six edge-value conditions

\[
\frac{1}{|e_k|} \int_{e_k} \phi_{j,T} ds = \begin{bmatrix}
\delta_{jk} \\
0 \\
0
\end{bmatrix}, \quad k = 1, 2, 3, \quad (3.17)
\]

\[
\frac{1}{|e_{k-3}|} \int_{e_{k-3}} \phi_{j,T} ds = \begin{bmatrix}
0 \\
\delta_{j,k} \\
0
\end{bmatrix}, \quad k = 4, 5, 6. \quad (3.18)
\]

• One mean pressure condition

\[
\frac{1}{|T|} \int_T \phi_{j,T} dxdy = \begin{bmatrix}
0 \\
0 \\
\delta_{jk}
\end{bmatrix}, \quad k = 7. \quad (3.19)
\]

• Four continuity conditions of the velocity

\[\phi_{1,j}(D)] = [\phi_{2,j}(D)] = [\phi_{1,j}(E)] = [\phi_{2,j}(E)] = 0. \quad (3.20)\]

• Two stress continuity conditions

\[
[\mu \left(2\partial_x \phi_{1,j} n_1 + (\partial_y \phi_{1,j} + \partial_x \phi_{2,j}) n_2\right) - \phi_{p,j} n_1] = 0, \quad (3.21)
\]

\[
[\mu \left((\partial_x \phi_{2,j} + \partial_y \phi_{1,j}) n_1 + 2\partial_y \phi_{2,j} n_2\right) - \phi_{p,j} n_2] = 0. \quad (3.22)
\]

• One continuity of the divergence condition

\[\left[\partial_x \phi_{1,j} + \partial_y \phi_{2,j}\right]_{TE} = 0. \quad (3.23)\]

Combining (3.17) - (3.23) we obtain a 14 × 14 linear system for the Type I interface element:

\[M_1 \mathbf{c}_j = \mathbf{e}_j. \quad (3.24)\]
where the coefficient matrix $M_1$ is given by

$$M_1 = \begin{pmatrix}
\frac{1}{2}d^2 & 0 & d & \frac{1-d}{2} & 0 & 1-d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{e^2}{2} & e & 0 & \frac{1-e^2}{2} & 1-e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & d^2/2 & 0 & d & \frac{1-d^2}{2} & 0 & 1-d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 1/2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^2/2 & e & 0 & \frac{1-e^2}{2} & 1-e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & de & 1-de \\
-d & 0 & -1 & d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -e & -1 & 0 & e & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -d & 0 & -1 & d & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -e & -1 & 0 & e & 1 & 0 & 0 \\
2e\mu^- & d\mu^- & 0 & -2e\mu^+ & -d\mu^+ & 0 & d\mu^- & 0 & 0 & -d\mu^+ & 0 & 0 & -e & e \\
0 & e\mu^- & 0 & 0 & -e\mu^+ & 0 & e\mu^- & 2d\mu^- & 0 & -e\mu^+ & -2d\mu^+ & 0 & -d & d \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
$$

(3.25)

The unknown vector $\mathbf{c}_j$ and the right-hand side vector $\mathbf{e}_j$ are

$$\mathbf{c}_j = \begin{bmatrix}
a_{1j}^+, b_{1j}^+, c_{1j}^+, a_{1j}^-, b_{1j}^-, c_{1j}^-, a_{2j}^+, b_{2j}^+, c_{2j}^+, a_{2j}^-, b_{2j}^-, c_{2j}^-, d_j^+, d_j^- \end{bmatrix}', \quad (3.26)$$

$$\mathbf{e}_j = \begin{bmatrix}
\delta_{j1}, \delta_{j2}, \delta_{j3}, \delta_{j4}, \delta_{j5}, \delta_{j6}, \delta_{j7}, 0, 0, 0, 0, 0, 0 \end{bmatrix}'. \quad (3.27)$$

With each vector $\mathbf{e}_j, j = 1, 2, \cdots, 7$, we solve the linear system (3.24) to obtain $\mathbf{c}_j$. Substituting the vector $\mathbf{c}_j$ in (3.16), we obtain the $CR-P_0$ vector-valued IFE shape function $\phi_{j,T}$. The derivation of
Type II and Type III interface elements are similar. The coefficient matrix $M_2$ for Type II interface element is

$$M_2 = \begin{pmatrix}
d^2/2 & 0 & d & \frac{1-d^2}{2} & 0 & 1-d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1-e^2}{2} & \frac{1-e^2}{2} & 1-e & m_{2,4} & e^2/2 & e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & d^2/2 & 0 & d & \frac{1-d^2}{2} & 0 & 1-d & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{5,7} & m_{5,8} & 1-e & m_{5,10} & e^2/2 & e & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & m_{7,13} & m_{7,14} \\
-d & 0 & -1 & d & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1+e & -e & -1 & 1-e & e & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -d & 0 & -1 & d & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1+e & -e & -1 & 1-e & e & 1 & 0 & 0 \\
2\mu^- m_{12,2} & 0 & -2\mu^+ m_{12,5} & 0 & m_{12,7} & 0 & 0 & m_{12,10} & 0 & 0 & -e & e & 0 & 0 & 0 \\
e\mu^- & 0 & 0 & -\mu^+ & 0 & e\mu^- & m_{13,8} & 0 & -\mu^+ & m_{13,11} & 0 & m_{13,13} & m_{13,14} & 0 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}$$

(3.28)

where,

$m_{2,4} = \frac{1}{2}(-2 + e)e$, $m_{5,7} = m_{2,1}$, $m_{5,8} = m_{2,2}$, $m_{5,10} = m_{2,4}$, $m_{7,13} = 1 + (-1 + d)e$, $m_{7,14} = e - de$, $m_{12,2} = (-1 + d + e)\mu^-$, $m_{13,2} = (-d + e)\mu^-$, $m_{13,7} = m_{13,2}$, $m_{12,5} = -(1 + d + e)\mu^+$, $m_{12,7} = (-1 + d + e)\mu^-$, $m_{12,10} = -(1 + d + e)\mu^+$, $m_{13,8} = 2(1 + d + e)\mu^-$, $m_{13,11} = -2(1 + d + e)\mu^+$, $m_{13,13} = (1 - d - e)\mu^+$, $m_{13,14} = -1 + d + e\mu^+$. 

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The coefficient matrix $M_3$ for Type III interface element is

$$
M_3 = \begin{pmatrix}
1/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{d^2}{2} & d & \frac{(1-d)^2}{2} & \frac{1-d^2}{2} & 1-d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & e^2/2 & e & 0 & \frac{1-e^2}{2} & 1-e & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & m_{5,7} & d^2/2 & d & \frac{(1-d)^2}{2} & \frac{1-d^2}{2} & 1-d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & e^2/2 & e & 0 & 1-e & 1-e & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1+d & -d & -1 & 1-d & d & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -e & -1 & 0 & e & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1+d & -d & -1 & 1-d & d & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -e & -1 & 0 & e & 1 & 0 & 0 & 0 \\
m_{12,1} & m_{12,2} & 0 & m_{12,4} & m_{12,5} & 0 & m_{12,7} & 0 & 0 & m_{12,10} & 0 & 0 & d-e & -d+e \\
0 & m_{13,2} & 0 & 0 & m_{13,5} & 0 & m_{13,7} & m_{13,8} & 0 & m_{13,10} & m_{13,11} & 0 & -1+d & 1-d \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}
$$

where,

$$
m_{2,1} = -\frac{1}{2}(-2+d)d, \quad m_{5,7} = m_{2,1}, \quad m_{7,13} = d + e - de, \quad m_{7,14} = (-1+d)(-1+e), \quad m_{12,1} = 2(-d+e)\mu^-, \quad m_{12,2} = \mu^- - d\mu^-, \quad m_{12,4} = 2(d-e)\mu^+, \quad m_{12,5} = (-1+d)\mu^+, \quad m_{12,7} = \mu^- - d\mu^-, \quad m_{12,10} = (-1+d)\mu^+, \quad m_{13,2} = (-d+e)\mu^-, \quad m_{13,5} = (d-e)\mu^+, \quad m_{13,7} = (-d+e)\mu^-, \quad m_{13,8} = -2(-1+d)\mu^-, \quad m_{13,10} = (d-e)\mu^+, \quad m_{13,11} = 2(-1+d)\mu^+.
$$

In Figure 3.3, we plot the local $CR-P_0$ IFE vector-valued shape function $\phi_{3,T}$ and the standard FE vector-valued shape function $\psi_{3,T}$ as a comparison. There is a kink on all three components of $\phi_{3,T}$ across the interface, which is designed to satisfy the stress conditions across the interface.
Moreover, unlike the FE shape function $\psi_{3,T}$, the second and the third components of $\phi_{3,T}$ are not entirely zero. We define the local IFE space to be $S_i^h(T) = \text{span}\{\phi_j : 1 \leq j \leq 7\}$. To unify the notation, we denote the local FE/IFE space on each triangle $T \in \mathcal{T}_h$ by

$$S_h(T) = \begin{cases} S_i^h(T), & \text{if } T \in \mathcal{T}_h^i, \\ S_n^h(T), & \text{if } T \in \mathcal{T}_h^n. \end{cases}$$  \hspace{1cm} (3.30)$$

The global $CR-P_0$ IFE space is defined to be

$$S_h(\mathcal{T}_h) = \{v = [v_1, v_2, v_p]^t \in [L^2(\Omega)]^3 : v|_T \in S_h(T), \forall T \in \mathcal{T}_h, \text{ and}$$

$$\int_e [v_i] ds = 0, \forall e \in \mathcal{E}_h^0, i = 1, 2\}.$$  \hspace{1cm} (3.31)
The subspace with vanishing velocity boundary value is defined as

\[
S^0_h(T_h) = \{ \mathbf{v} = [v_1, v_2, v_p]^t \in S_h(T_h) : \int_{e_i} v_i ds = 0, \forall e \in E^b_h, i = 1, 2 \}. \tag{3.32}
\]

### 3.4 \( RQ_1-Q_0 \) Immersed Finite Element Space

In this section, we introduce the \( RQ_1-Q_0 \) immersed finite element space for the Stokes interface problem. Let \( \Omega \subset \mathbb{R}^2 \) be a rectangular domain or a union of rectangular domains. Assume that \( \Omega \) is partitioned by an interface-unfitted rectangular mesh denoted \( R_h = \{ R_k \}_{k=1}^N \). As before, we let \( R^i_h \) and \( R^n_h \) be the collection of interface elements and non-interface elements, respectively. Define \( E_h \) to be the set of all edges in \( R_h \). Let \( E^i_h \) and \( E^n_h \) denote the collection of interface edges and non-interface edges, respectively. The collection of internal edges and boundary edges are denoted by \( E^0_h \) and \( E^b_h \), respectively. In addition, we assume the rectangular mesh \( R_h \) satisfies the same hypotheses (H1), (H2), and (H3) as the triangular mesh \( T_h \).

#### 3.4.1 \( RQ_1-Q_0 \) Finite Element Shape Functions

We recall the nonconforming rotated-\( Q_1 \) finite elements which are used to approximate the velocity. Suppose \( R \in R_h \) is a non-interface element with vertices \( A_1, A_2, A_3, A_4 \) which are oriented counterclockwise. The edges of \( R \) are labeled by \( e_1 = \overline{A_1A_2}, e_2 = \overline{A_2A_3}, e_3 = \overline{A_3A_4}, \) and \( e_4 = \overline{A_4A_1} \). See the left plot of Figure 3.4. The local \( RQ_1 \) space is the \( Q_1 \) space rotated by \( 45^\circ \), i.e.,

\[
RQ_1 = span\{1, x, y, x^2 - y^2 \}, \tag{3.33}
\]

where the local basis functions, denoted by \( \psi_{j,R} \), satisfy

\[
\frac{1}{|e_i|} \int_{e_i} \psi_{j,R}(x, y) ds = \delta_{ij}, \quad i, j = 1, 2, 3, 4,
\]

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where $\delta_{ij}$ is the Kronecker function. The pressure is approximated by the piecewise constant function space denoted by $Q_0$. The coupled velocity-pressure components create a vector-valued finite element space on each element $R \in \mathcal{R}_h$, denoted by $\mathbf{S}_h^n(R) = RQ_1 \times RQ_1 \times Q_0$. This vector-valued finite element space has nine local shape functions as follows:

$$\psi_{j,R} = \begin{bmatrix} \psi_{j,R} \\ 0 \\ 0 \end{bmatrix} \quad \text{for } j = 1, 2, 3, 4, \quad \psi_{j,R} = \begin{bmatrix} 0 \\ \psi_{j-4,R} \\ 0 \end{bmatrix} \quad \text{for } j = 5, 6, 7, 8, \quad \psi_{9,R} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

We can also write the local $RQ_1-Q_0$ finite element space as $\mathbf{S}_h^n(R) = \text{span}\{\psi_{j,R} : 1 \leq j \leq 9\}$.

### 3.4.2 $RQ_1-Q_0$ Immersed Finite Element Shape Functions

Let $R \in \mathcal{R}_h^i$ be an interface element such that the interface curve $\Gamma$ intersects $R$ at the two points, denoted by $D = (x_d, y_d)$ and $E = (x_e, y_e)$. We use the line segment $DE$ to approximate the interface curve within $R$. This line segment separates the element $R$ into two subelements, $R^+ \in \Omega^+$ and $R^- \in \Omega^-$. There are generally two geometrical configurations associated with rectangular interface elements. If the interface intersects an element at two adjacent edges, the element is called a Type I interface element. If the intersection points are on two opposite edges, the element is called a Type II interface element. See Figure 3.4.

For simplicity, we present the construction and the analysis of the $RQ_1-Q_0$ IFE shape functions on a reference element $\hat{R} = \hat{A}_1\hat{A}_2\hat{A}_3\hat{A}_4$, where

$$\hat{A}_1 = (0, 0), \quad \hat{A}_2 = (1, 0), \quad \hat{A}_3 = (1, 1), \quad \hat{A}_4 = (0, 1).$$

Through straightforward scaling, the reference element $\hat{R}$ can be mapped to an arbitrary rectangular element $R$. We drop the hat for simplicity of the analysis. For the Type I interface element, the
A noninterface element (left), Type I interface element (middle), and Type II interface element (right).

Intersection points $D$ and $E$ can be written as a convex combinations of vertices, i.e., $D = (d, 0)$ and $E = (0, e)$; and for Type II interface element $D = (1, d)$ and $E = (0, e)$ for $0 < d, e < 1$.

Similar to (3.16), we construct $RQ_1Q_0$ vector-valued function $\phi_{j,R}$, where $\phi_{j,R} = \phi^s_{j,R}$ on $R^s$, $s = +, -$ and $\phi^s_{j,R} = (\phi^s_{1,j}, \phi^s_{2,j}, \phi^s_{p,j}) \in RQ_1 \times RQ_1 \times Q_0$ such that

$$\phi_{j,R}(x, y) = \begin{cases} 
\phi^+_j(x, y) = \begin{bmatrix} \phi^+_{1,j}(x, y) \\
\phi^+_{2,j}(x, y) \\
\phi^+_{p,j}(x, y) 
\end{bmatrix} = \begin{bmatrix} a^+_{1,j} + b^+_{1,j}x + c^+_{1,j}y + d_{1,j}(x^2 - y^2) \\
a^+_{2,j} + b^+_{2,j}x + c^+_{2,j}y + d_{2,j}(x^2 - y^2) \\
d^+_p 
\end{bmatrix}, \\
\phi^-_j(x, y) = \begin{bmatrix} \phi^-_{1,j}(x, y) \\
\phi^-_{2,j}(x, y) \\
\phi^-_{p,j}(x, y) 
\end{bmatrix} = \begin{bmatrix} a^-_{1,j} + b^-_{1,j}x + c^-_{1,j}y + d_{1,j}(x^2 - y^2) \\
a^-_{2,j} + b^-_{2,j}x + c^-_{2,j}y + d_{2,j}(x^2 - y^2) \\
d^-_p 
\end{bmatrix}, 
\end{cases}$$

\begin{align*}
&j = 1, 2, \cdots, 9.
\end{align*}
This linear system has 16 unknowns $a^x_{1j}, a^x_{2j}, b^x_{1j}, b^x_{2j}, c^x_{1j}, c^x_{2j}, d_{1j}, d_{2j},$ and $d^x_{p,j}$, which will be determined by the following 16 conditions:

- Eight edge-value conditions

\[
\frac{1}{|e_k|} \int_{e_k} \phi_{j,R} ds = \begin{bmatrix} \delta_{jk} \\ 0 \\ 0 \end{bmatrix}, \quad k = 1, 2, 3, 4, \quad (3.36)
\]

\[
\frac{1}{|e_{k-4}|} \int_{e_{k-4}} \phi_{j,R} ds = \begin{bmatrix} 0 \\ \delta_{j,k} \\ 0 \end{bmatrix}, \quad k = 5, 6, 7, 8. \quad (3.37)
\]

- One mean value condition

\[
\frac{1}{|R|} \int_{R} \phi_{j,R} dxdy = \begin{bmatrix} 0 \\ 0 \\ \delta_{jk} \end{bmatrix}, \quad k = 9. \quad (3.38)
\]

- Four velocity conditions across the interface

\[
[\phi_{i,j}(D)] = [\phi_{i,j}(E)] = 0, \quad i = 1, 2. \quad (3.39)
\]

- Two weakly imposed stress jump conditions

\[
\int_{DE} [\mu \left( 2\partial_x \phi_{1,j} n_1 + (\partial_y \phi_{1,j} + \partial_x \phi_{2,j}) n_2 \right) - \phi_{p,j} n_1]_{DE} ds = 0, \quad (3.40)
\]

\[
\int_{DE} [\mu \left( (\partial_x \phi_{2,j} + \partial_y \phi_{1,j}) n_1 + 2\partial_y \phi_{2,j} n_2 \right) - \phi_{p,j} n_2]_{DE} ds = 0. \quad (3.41)
\]

- One weakly imposed continuity of the divergence condition

\[
\int_{DE} [\partial_x \phi_{1,j} + \partial_y \phi_{2,j}]_{DE} ds = 0. \quad (3.42)
\]

Combining (3.36) - (3.42), we obtain a $16 \times 16$ linear system:

\[
M^R_i e_j = e_j, \quad j = 1, 2, \cdots, 9, \quad (3.43)
\]

where $M^R_i, i = 1, 2$ denote the matrix on Type I and Type II interface element, respectively. As before, we choose $e_j \in \mathbb{R}^{16}$ to be canonical vectors, and solve for $e_j$, we can obtain the IFE local
Comparison of the IFE shape functions (Type I and II) $\phi_{4,R}$ and the FE shape function $\psi_{4,R}$. FE Basis functions element. Top figures left to right: $u_1$, $u_2$, $p$. Middle figures: IFE Basis functions for Type I interface element $(d, e) = (0.7, 0.8)$, and $(\mu^-, \mu^+) = (1, 5)$. From left: $u_1$, $u_2$, $p$. Bottom figures: IFE Basis functions for Type II interface element $(d, e) = (0.7, 0.2)$, and $(\mu^-, \mu^+) = (1, 5)$. From left: $u_1$, $u_2$, $p$. 

Figure 3.5
shape functions \( \phi_{j,R} \). In Figure 3.5, the standard FE vector-valued shape function \( \psi_{3,R} \) and the local IFE vector-valued shape function \( \phi_{3,R} \) are plotted for comparison. The local \( RQ_1-Q_0 \) IFE space on the interface rectangle \( R \in \mathcal{R}_h \) is \( P_h(R) = \text{span}\{\phi_{j,R} : 1 \leq j \leq 9\} \). We also unify the notation by

\[
P_h(R) = \begin{cases} 
P^i_h(R), & \text{if } R \in \mathcal{R}^i_h, \\ 
P^n_h(R), & \text{if } R \in \mathcal{R}^n_h. 
\end{cases} \tag{3.44}
\]

The global \( RQ_1-Q_0 \) IFE space and the zero-boundary subspace are defined to be

\[
P_h(\mathcal{R}_h) = \{v = [v_1, v_2, v_p]^t \in [L^2(\Omega)]^3 : v|_R \in P_h(R), \forall R \in \mathcal{R}_h, \}
\]

\[\text{and } \int_e [v_i]ds = 0, \forall e \in \mathcal{E}^0_h, i = 1, 2\}. \tag{3.45}\]

\[
P^0_h(\mathcal{R}_h) = \{v = [v_1, v_2, v_p]^t \in P_h(\mathcal{R}_h) : \int_e v_i ds = 0, \forall e \in \mathcal{E}^b_h, i = 1, 2\} \tag{3.46}\]

### 3.5 Properties of Nonconforming Immersed Finite Element Spaces

In this section, we present some basic properties of the new nonconforming vector-valued IFE spaces.

**Theorem 1**

(Unisolvency) The nonconforming IFE shape functions can be uniquely determined by the prescribed edge values of the velocity and the mean pressure value, regardless the interface locations and the jumps of viscosity coefficients \( \mu^\pm > 0 \). More precisely, we have

- The \( CR-P_0 \) IFE shape functions \( \phi_{j,T} \), \( 1 \leq j \leq 7 \) can be uniquely determined by conditions (3.17) - (3.19).

- The \( RQ_1-Q_0 \) IFE shape functions \( \phi_{j,R} \), \( 1 \leq j \leq 9 \) can be uniquely determined by conditions (3.36) - (3.38).
Proof: We prove this unisolvency by investigating the invertibility of the coefficient matrices. For the \( CR-P_0 \) IFE space, through direct computation, we have

\[
det(M_1) = \frac{1}{16} \left( (1 - de) \mu^- + de \mu^+ \right) \left( d^2 + e^2 \right)^2 > 0, \tag{3.47}
\]

\[
det(M_2) = \frac{1}{16} \left( (1 - d - e)^2 + e^2 \right)^2 \left( \mu^+ (1 - e + de) + \mu^- e (1 - d) \right) > 0, \tag{3.48}
\]

\[
det(M_3) = \frac{1}{16} \left( (1 - d)^2 + (d - e)^2 \right)^2 \left( \mu^- (1 - d) (1 - e) + \mu^+ (d + e (1 - d)) \right) > 0. \tag{3.49}
\]

For \( RQ_1-Q_0 \) IFE space of Type I interface element, through direct computation we have

\[
det(M_1^R) = D_5 + D_6, \tag{3.50}
\]

where

\[
D_5 = \frac{\mu^+}{36} (d^2 + e^2) \left[ de \left( 3(d - e)^2 + 2(d^2 + e^2) \right) \right] > 0, \]

\[
D_6 = \frac{\mu^-}{36} (d^2 + e^2) \left[ 4d^2 - 5d^3 e + 6d^2 e^2 + 4e^2 - 5d e^3 \right] > \frac{\mu^-}{36} (d^2 + e^2) \left[ d^2 (4 - 5e + 3e^2) + e^2 (4 - 5d + 3d^2) \right] > 0.
\]

For Type II interface element, we have

\[
det(M_2^R) = \frac{\mu^+}{36} (1 + (d - e))^2 D_7 + \frac{\mu^-}{36} (1 + (e - d))^2 D_8, \tag{3.51}
\]

where

\[
D_7 = \left[ 3ds^2 + 3et + d(2 - e)t + d(2 - e)t + e(2 - d)s + 2(d^3 + e^3) \right] > 0,
\]

\[
D_8 = \left[ 3sd^2 + 3te + ste + tsd + 2(s^3 + t^3) \right] > 0.
\]

Let \( s = 1 - d \) and \( t = 1 - e \). Thus, each of the \( CR-P_0 \) and the \( RQ_1-Q_0 \) IFE functions of all types are uniquely solvable.
We can use the gradient stress interface condition (3.11) to replace the original stress interface condition (3.7). That is to replace the conditions (3.21) - (3.22) by the following conditions

\[
\mu \left( \partial_x \phi_1,j n_1 + \partial_y \phi_1,j n_2 \right) - \phi_{p,j} n_1 = 0, \quad (3.52)
\]

\[
\mu \left( \partial_x \phi_2,j n_1 + \partial_y \phi_2,j n_2 \right) - \phi_{p,j} n_2 = 0, \quad (3.53)
\]

in constructing the \( CR-P_0 \) IFE shape functions. In this case, the new coefficient matrices, denoted by \( \tilde{M}_i, i = 1, 2, 3 \) are formed by updating the 12th and 13th rows of the matrices \( M_i, i = 1, 2, 3 \) in (3.25) - (3.29). It is an interesting observation that \( \det(\tilde{M}_i) = \det(M_i) \) for all \( 1 \leq i \leq 3 \), although these matrices are not entirely the same. The IFE basis functions using these two stress conditions are very close. Since the determinants are identical, the unisolvency result (Theorem 5.1) also hold true for this configuration. The same results are observed for \( RQ_1-Q_0 \) IFE functions.

**Theorem 2**

*(Continuity of Velocity)* The velocity components of the vector-valued IFE shape functions are continuous within each interface element. To be more accurate,

- Let \( T \in T^i_h \) be an interface triangle and let \( \phi_{j,T} = (\phi_{1,j}, \phi_{2,j}, \phi_{p,j}), 1 \leq j \leq 7 \) be the \( CR-P_0 \) IFE shape functions. Then \( \phi_{i,j} \in C^0(T), i = 1, 2 \).

- Let \( R \in R^i_h \) be an interface rectangle and let \( \phi_{j,R} = (\phi_{1,j}, \phi_{2,j}, \phi_{p,j}), 1 \leq j \leq 9 \) be the \( RQ_1-Q_0 \) IFE shape functions. Then \( \phi_{i,j} \in C^0(R), i = 1, 2 \).

Proof: The construction of \( CR-P_0 \) IFE functions uses the velocity jump condition (3.20). Note that two linear functions \( \phi^+_{i,j} \) and \( \phi^-_{i,j} \) coincide along the line segment \( DE \) if they match at distinct endpoints \( D \) and \( E \). This means the velocity components \( \phi_{1,j} \) and \( \phi_{2,j} \) are both continuous across the line segment \( DE \), thus continuous within the whole element \( T \). For \( RQ_1-Q_0 \) IFE function (3.35), since the coefficients of the high-order terms \( x^2 - y^2 \) in the velocity components are equal
on $R^+$ and $R^-$, their difference is also a linear polynomial. This ensures the continuity of the velocity components over the entire interface element $R$.

The next two theorems show the consistency of IFE functions with standard FE functions.

**Theorem 3**

(Consistency I) The IFE shape functions become the standard FE shape functions if $\mu^+ = \mu^-$. More precisely,

- Let $T \in T_h^i$ be an interface triangle and let $\phi_{j,T}$, $1 \leq j \leq 7$ be the $CR-P_0$ IFE shape functions. Then $\phi_{j,T}$ becomes $\psi_{j,T}$, if $\mu^+ = \mu^-$. 

- Let $R \in R_h^i$ be an interface rectangle and let $\phi_{j,R}$, $1 \leq j \leq 9$ be the $RQ_1-Q_0$ IFE shape functions. Then $\phi_{j,R}$ becomes $\psi_{j,R}$, if $\mu^+ = \mu^-$. 

Proof: It can be verified by direct calculation that when $\mu^+ = \mu^-$, the $CR-P_0$ IFE shape functions $\phi_{j,T}$ become

$$
\begin{align*}
\phi^\pm_{1,T} &= \begin{bmatrix} 1 - 2y \\ 0 \\ 0 \end{bmatrix}, & \phi^\pm_{2,T} &= \begin{bmatrix} 2x + 2y - 1 \\ 0 \\ 0 \end{bmatrix}, & \phi^\pm_{3,T} &= \begin{bmatrix} 1 - 2x \\ 0 \\ 0 \end{bmatrix} , \\
\phi^\pm_{4,T} &= \begin{bmatrix} 0 \\ 1 - 2y \\ 0 \end{bmatrix}, & \phi^\pm_{5,T} &= \begin{bmatrix} 0 \\ 2x + 2y - 1 \\ 0 \end{bmatrix}, & \phi^\pm_{6,T} &= \begin{bmatrix} 0 \\ 1 - 2x \\ 0 \end{bmatrix}, & \phi^\pm_{7,T} &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} ,
\end{align*}
$$

for all three types of the interface configurations. These are the same as standard $CR-P_0$ FE shape functions on the reference triangle. A similar argument can be used for the $RQ_1-Q_0$ IFE shape functions for all interface types.
Theorem 4

(Consistency II) The IFE shape functions become the standard FE shape functions if the interface moves out of the element. More precisely,

- Let $T \in T_h^i$ be an interface triangle and let $\phi_{j,T}$, $1 \leq j \leq 7$ be the CR-$P_0$ IFE shape functions. Then,
  \[ \phi_{j,T} \to \psi_{j,T}, \quad \text{as} \quad \frac{\min\{|T^-|,|T^+|\}}{|T|} \to 0. \]

- Let $R \in R_h^i$ be an interface rectangle and let $\phi_{j,R}$, $1 \leq j \leq 9$ be the $RQ_1$-$Q_0$ IFE shape functions. Then,
  \[ \phi_{j,R} \to \psi_{j,R}, \quad \text{as} \quad \frac{\min\{|R^-|,|R^+|\}}{|R|} \to 0. \]

Proof: We first consider the $CR$-$P_0$ IFE case when $|T^-| \to 0$, then

- for Type I element: $d \to 0$ or $e \to 0$.
- for Type II element: $d \to 0$ and $e \to 1$.
- for Type III element: $d \to 0$ and $e \to 1$.

In all the above cases, we have verified by direct calculation that $\phi_{j,T} \to \phi_{j,T}^+ = \psi_{j,T}$ for all $1 \leq j \leq 7$. Next, if $|T^+| \to 0$, then

- for Type I element: $d \to 1$ and $e \to 1$.
- for Type II element: $d \to 1$ or $e \to 0$.
- for Type III element: $d \to 1$ or $e \to 0$.

In all these cases, we have verified by direct calculation that $\phi_{j,T} \to \phi_{j,T}^- = \psi_{j,T}$ for all $1 \leq j \leq 7$.

Similar argument can be used to verify the consistency for the $RQ_1$-$Q_0$ IFE shape functions. We first consider the case when $|R^-| \to 0$, then

- for Type I element: $d \to 0$ or $e \to 0$.
- for Type II element: $d \to 0$ and $e \to 0$. 


In all the above cases, we have verified by direct calculation that \( \phi_{j,R}(x,y) \rightarrow \phi_{j,R}^+(x,y) = \psi_{j,R}(x,y), \quad 1 \leq j \leq 9 \). Next, if \( |T^+| \rightarrow 0 \)

- for Type I element: \( d \rightarrow 1 \) or \( e \rightarrow 0 \).
- for Type II element: \( d \rightarrow 1 \) and \( e \rightarrow 0 \).

In all cases, we have verified by direct calculation that

\[
\phi_{j,R}(x,y) \rightarrow \phi_{j,R}^-(x,y) = \psi_{j,R}(x,y), \quad 1 \leq j \leq 9.
\]

The next theorem is concerning the partition of unity property of local IFE shape functions. This property is verified through direct calculation.

**Theorem 5**

*(Partition of Unity)* The vector-valued IFE functions satisfy the partition of unity property. More precisely,

- Let \( T \in T_h^i \) be an interface triangle and let \( \phi_{j,T}, \ 1 \leq j \leq 7 \) be the CR-P\(_0\) IFE shape functions. Then
  \[
  \sum_{j=1}^{3} \phi_{j,T}(x,y) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \sum_{j=4}^{6} \phi_{j,T}(x,y) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \phi_{7,T}(x,y) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \forall (x,y) \in T. \tag{3.56}
  \]

- Let \( R \in R_h^i \) be an interface rectangle and let \( \phi_{j,R}, \ 1 \leq j \leq 9 \) be the RQ\(_1\)-Q\(_0\) IFE shape functions. Then,
  \[
  \sum_{j=1}^{4} \phi_{j,R}(x,y) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \sum_{j=5}^{8} \phi_{j,R}(x,y) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \phi_{9,R}(x,y) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \forall (x,y) \in R. \tag{3.57}
  \]

The CR-P\(_0\) IFE basis \( \phi_{7,T} \) is a constant vector \([0,0,1]^t\) for any interface location and any coefficient jump. An explanation of this phenomenon is that since the viscosity coefficient \( \mu \) is only a multiple factor of velocity component in (3.21) and (3.22). When the velocity components \( \phi_{1,7} = \phi_{2,7} = 0 \), then the stress interface conditions (3.21) and (3.22) degenerate to \([\phi_{p,7}] = 0\). Thus, the piecewise
constant function $\phi_{p,7}$ must be continuous within $T$, so it must be a constant. The mean-value condition (3.19) further implies that $\phi_{p,7} = 1$. Finally, the unisolvent property ensures that $\phi_{7,T} = [0, 0, 1]'$ is the only basis to satisfy all edge value conditions. This idea holds for the $RQ_1-Q_0$ IFE basis functions as well.

### 3.6 Nonconforming Immersed Finite Element Method

In this section, we present the nonconforming IFEM for solving the Stokes interface problem (3.1)-(3.7). First, we derive the weak formulation for the Stokes system.

Multiplying the momentum equation (3.1) by $v \in [H^1_0(\Omega)]^2$ and integration by parts over $\Omega^-$ yields,

$$
\int_{\Omega^-} (2\mu \varepsilon(u) - p I) : \nabla v dx - \int_{\partial \Omega^-} (2\mu \varepsilon(u) - p I) n_{\partial \Omega^-} \cdot v ds = \int_{\Omega^-} f \cdot v dx. \quad (3.58)
$$

Here the tensor product operator for $A = [a_{ij}]$ and $B = [b_{ij}]$ is defined to be $A : B = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ij}$. Since $v$ vanishes on the outer boundary $\partial \Omega$, and $n_{\Gamma}$ is from $\Omega^-$ to $\Omega^+$, then we have

$$
\int_{\Omega^-} (2\mu \varepsilon(u) - p I) : \nabla v dx - \int_{\Gamma} (2\mu \varepsilon(u) - p I) n_{\Gamma} \cdot v ds = \int_{\Omega^-} f \cdot v dx. \quad (3.59)
$$

Similarly, on $\Omega^+$, we have

$$
\int_{\Omega^+} (2\mu \varepsilon(u) - p I) : \nabla v dx + \int_{\Gamma} (2\mu \varepsilon(u) - p I) n_{\Gamma} \cdot v ds = \int_{\Omega^+} f \cdot v dx. \quad (3.60)
$$

Summing up these two equations, we have

$$
\int_{\Omega} (2\mu \varepsilon(u) - p I) : \nabla v dx - \int_{\Gamma} [2\mu \varepsilon(u) - p I] n_{\Gamma} \cdot v ds = \int_{\Omega} f \cdot v dx. \quad (3.61)
$$

Applying the stress jump condition (3.7), we have

$$
\int_{\Omega} (2\mu \varepsilon(u) - p I) : \nabla v dx = \int_{\Omega} f \cdot v dx. \quad (3.62)
$$
Using the identity \(2\mu \varepsilon(u) - p I : \nabla v = 2\mu \varepsilon(u) : \varepsilon(v) - p(\nabla \cdot v)\), we have

\[
\int_{\Omega} 2\mu \varepsilon(u) : \varepsilon(v) d\mathbf{x} - \int_{\Omega} p(\nabla \cdot v) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} d\mathbf{x}.
\]  

(3.63)

Multiplying a test function \(q \in L^2(\Omega)\) to (3.2), and integration by parts we have

\[
\int_{\Omega} q (\nabla \cdot u) d\mathbf{x} = 0
\]

(3.64)

At the discretization level, we use the IFE space \(S_h\) to approximate \(H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)\).

The nonconforming IFE method is to find \((u_h, p_h) \in S_h\) such that

\[
\int_{\Omega} 2\mu \varepsilon(u_h) : \varepsilon(v_h) d\mathbf{x} - \int_{\Omega} p(\nabla \cdot v_h) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h d\mathbf{x}, \quad \forall (v_h, q_h) \in S_h.
\]

(3.65)

\[
\int_{\Omega} q_h (\nabla \cdot u_h) d\mathbf{x} = 0,
\]

Alternatively, since the viscosity coefficient \(\mu\) is piecewise constant, we can also use the simplified momentum equation (3.10) and the gradient stress condition (3.11). The corresponding IFE method is to find \((u_h, p_h) \in \tilde{S}_h\) such that

\[
\int_{\Omega} \mu \nabla u_h : \nabla v_h d\mathbf{x} - \int_{\Omega} p(\nabla \cdot v_h) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v}_h d\mathbf{x}, \quad \forall (v_h, q_h) \in \tilde{S}_h.
\]

(3.66)

\[
\int_{\Omega} q_h (\nabla \cdot u_h) d\mathbf{x} = 0,
\]

3.7 Numerical Examples

In this section, we test the accuracy and the convergency of each class of IFE methods for the Stokes interface problem through a series of numerical experiments. We will consider the accuracy of both the interpolation and the IFE solution with various configurations of the interface and coefficient jumps. Interpolation errors and IFE solution errors for the velocity and the pressure

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are measured by the $L^2$-norm and the broken $H^1$-norm. Define the $CR-P_0$ IFE interpolation operator $I_h : H^1(\Omega)^2 \times L^2(\Omega) \to S_h(T_h)$ such that

$$I_h(u, p)|_T = I_{h,T}(u, p) = \begin{cases} 
\sum_{j=1}^7 c_j \phi_{j,T}, & \text{if } T \in T_h^i, \\
\sum_{j=1}^7 c_j \psi_{j,T}, & \text{if } T \in T_h^n,
\end{cases}$$

(3.67)

where $\phi_{j,T}$ and $\psi_{j,T}$ are the local $CR-P_0$ IFE shape functions and the standard $CR-P_0$ FE shape functions $T$, respectively. The coefficients $c_j$ take the values

$$c_j = \frac{1}{|e_j|} \int_{e_j} u_1(x, y)ds, \quad j = 1, 2, 3, \quad c_j = \frac{1}{|e_{j-3}|} \int_{e_{j-3}} u_2(x, y)ds, \quad j = 4, 5, 6,$$

(3.68)

$$c_7 = \frac{1}{|T|} \int_T p(x, y)dxdy,$$

(3.69)

where $e_j$, $j = 1, 2, 3$ are the boundary edges of the triangle $T$. Similarly, the $RQ1-Q_0$ IFE interpolation operator $I_h : H^1(\Omega)^2 \times L^2(\Omega) \to P_h(R_h)$ is defined to be

$$I_h(u, p)|_R = I_{h,R}(u, p) = \begin{cases} 
\sum_{j=1}^9 c_j \phi_{j,R}, & \text{if } R \in R_h^i, \\
\sum_{j=1}^9 c_j \psi_{j,R}, & \text{if } R \in R_h^n,
\end{cases}$$

(3.70)

where $\phi_{j,R}$ and $\psi_{j,R}$ are the local $RQ1-Q_0$ IFE shape functions and the standard $RQ1-Q_0$ FE shape functions $R$, respectively. The coefficients $c_j$ take the values

$$c_j = \frac{1}{|e_j|} \int_{e_j} u_1(x, y)ds, \quad j = 1, 2, 3, 4, \quad c_j = \frac{1}{|e_{j-4}|} \int_{e_{j-4}} u_2(x, y)ds, \quad j = 5, 6, 7, 8,$$

(3.71)

$$c_9 = \frac{1}{|R|} \int_R p(x, y)dxdy.$$

(3.72)

with $e_j$, $j = 1, 2, 3, 4$ the boundary edges of the rectangle $R$.

Since $I_h(u, p)$ is a vector-valued interpolation, we have the following relations $(I_h(u, p))_1 \approx u_1$, $(I_h(u, p))_2 \approx u_2$, and $(I_h(u, p))_3 \approx p$. The error of the IFE interpolation for each component is denoted by

$$e_{1,T} = u_1 - I_h(u, p)_1, \quad e_{2,T} = u_2 - I_h(u, p)_2, \quad e_{p,T} = p - I_h(u, p)_3.$$
Similarly, the error of the IFE solution for approximating \( u_1, u_2 \) and \( p \) are denoted by

\[
e_{1,h} = u_1 - u_{1h}, \quad e_{2,h} = u_2 - u_{2h}, \quad e_{p,h} = p - p_h.
\]

The rate of convergence on two consecutive triangular meshes \( T_h \) and \( T_{h/2} \) (or rectangular meshes \( R_h \) and \( R_{h/2} \)) is calculated by

\[
r = \frac{\log(e_h/e_{h/2})}{\log(2)}.
\]

Numerical examples for the \( RQ_1-Q_0 \) IFE method are performed on unfitted Cartesian meshes with \( N \times N \) rectangles. For the \( CR-P_0 \) IFE method, we further divide each rectangle into two triangles by its diagonal with the positive slope. The IFE spaces reported are based on stress jump conditions (3.52)-(3.53). We also test all numerical examples using the stress conditions (3.21)-(3.22), and the results are very close. We note that these nonconforming IFE discretizations (3.65) and (3.66) of the Stokes system lead to a saddle-point problem. We use a GMRES solver with preconditioners designed by an iterative projection method.

### 3.7.1 Example 3.1: Straight-Line Interface

In this example, we consider a Stokes interface problem with a straight line interface. Let \( \Omega = [-1, 1]^2 \), and the interface \( \Gamma = \{(x, y) : y = \frac{\pi}{6}\} \). The interface divides the domain \( \Omega \) into two subdomains \( \Omega^- = \{(x, y) : y < \frac{\pi}{6}\} \) and \( \Omega^+ = \{(x, y) : y > \frac{\pi}{6}\} \). Let \( \mu^- = 1 \) and \( \mu^+ = 10 \). The exact
solutions $u$ and $p$ are chosen as follows:

$$
\mathbf{u}(x, y) = \begin{cases} 
\frac{1}{\mu x} (y - \frac{\pi}{6}) x^2, & \text{if } (x, y) \in \Omega^+, \\
\frac{1}{\mu y} (y - \frac{\pi}{6}) y^2, & \text{if } (x, y) \in \Omega^-,
\end{cases}
$$

$$
u_1(x, y) = \begin{cases} 
-\frac{1}{\mu} x y (y - \frac{\pi}{6}), & \text{if } (x, y) \in \Omega^+, \\
-\frac{1}{\mu} x (y - \frac{\pi}{6})^2, & \text{if } (x, y) \in \Omega^-,
\end{cases}
$$

$$v_2(x, y) = \begin{cases} 
\frac{1}{\mu} y (y - \frac{\pi}{6}), & \text{if } (x, y) \in \Omega^+, \\
\frac{1}{\mu} x^2 - \frac{1}{\mu} y^2, & \text{if } (x, y) \in \Omega^-.
\end{cases}
$$

$$p(x, y) = e^x - e^y. \quad (3.76)$$

In Tables 3.1 and 3.2, we present the errors and the convergence rates of the IFE interpolations. It can be seen that the convergence rates for velocity components $u_1$ and $u_2$ are $O(h^2)$ in the $L^2$-norm and $O(h)$ in the broken $H^1$-norm. Convergence rates for the pressure $p$ is $O(h)$ in the $L^2$-norm. This result is consistent with our expectation based on the degrees of polynomials we used for the approximation. In Tables 3.3 and 3.4, we report the errors and the convergence rates for the IFE solution. The convergence rates for all the norms mentioned above are optimal.

### 3.7.2 Example 3.2: Curved Interface

In this example, we consider a circular interface problem which has been used in Example 1 in [2]. Let $\Omega = [-1, 1]^2$ and the interface $\Gamma = \{(x, y) : x^2 + y^2 = 0.3\}$. The circular interface separates the domain $\Omega$ into two subdomains $\Omega^- = \{(x, y) : x^2 + y^2 < 0.3\}$ and $\Omega^+ = \{(x, y) : x^2 + y^2 > 0.3\}$. 

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Table 3.1

Errors of $CR-P_0$ IFE interpolation for Example 3.1 with $\mu^- = 1$ and $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_{1,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{P,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{1,l}|_{H^1(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{H^1(\Omega)}$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7.16e-3 n/a</td>
<td>8.22e-3 n/a</td>
<td>1.71e-1 n/a</td>
<td>1.97e-1 n/a</td>
<td>2.25e-1 n/a</td>
</tr>
<tr>
<td>20</td>
<td>1.80e-3 1.99</td>
<td>2.06e-3 2.00</td>
<td>7.08e-2 1.27</td>
<td>9.87e-2 1.00</td>
<td>1.13e-1 1.00</td>
</tr>
<tr>
<td>40</td>
<td>4.50e-4 2.00</td>
<td>5.15e-4 2.00</td>
<td>3.64e-2 0.96</td>
<td>4.94e-2 1.00</td>
<td>5.64e-2 1.00</td>
</tr>
<tr>
<td>80</td>
<td>1.13e-4 2.00</td>
<td>1.29e-4 2.00</td>
<td>1.80e-2 1.01</td>
<td>2.47e-2 1.00</td>
<td>2.82e-2 1.00</td>
</tr>
<tr>
<td>160</td>
<td>2.82e-5 2.00</td>
<td>3.22e-5 2.00</td>
<td>8.97e-3 1.01</td>
<td>1.24e-2 1.00</td>
<td>1.41e-2 1.00</td>
</tr>
<tr>
<td>320</td>
<td>7.05e-6 2.00</td>
<td>8.05e-6 2.00</td>
<td>4.46e-3 1.01</td>
<td>6.18e-3 1.00</td>
<td>7.06e-3 1.00</td>
</tr>
</tbody>
</table>

Table 3.2

Errors of $RQ_1-Q_0$ IFE interpolation for Example 3.1 with $\mu^- = 1$ and $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_{1,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{P,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{1,l}|_{H^1(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{H^1(\Omega)}$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8.39e-3 n/a</td>
<td>1.08e-2 n/a</td>
<td>2.22e-1 n/a</td>
<td>2.06e-1 n/a</td>
<td>2.64e-1 n/a</td>
</tr>
<tr>
<td>20</td>
<td>2.10e-3 2.00</td>
<td>2.70e-3 2.00</td>
<td>1.10e-1 1.02</td>
<td>1.03e-1 1.00</td>
<td>1.32e-1 1.00</td>
</tr>
<tr>
<td>40</td>
<td>5.27e-4 2.00</td>
<td>6.73e-4 2.00</td>
<td>5.50e-2 1.00</td>
<td>5.17e-2 1.00</td>
<td>6.60e-2 1.00</td>
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<tr>
<td>80</td>
<td>1.32e-4 2.00</td>
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<td>2.59e-2 1.00</td>
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</tr>
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</tr>
<tr>
<td>320</td>
<td>8.26e-6 2.00</td>
<td>1.05e-5 2.00</td>
<td>6.87e-3 1.00</td>
<td>6.47e-3 1.00</td>
<td>8.25e-3 1.00</td>
</tr>
</tbody>
</table>
### Table 3.3

Errors of $CR-P_0$ IFE solutions for Example 3.1 with $\mu^- = 1$ and $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_{1h}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2h}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,h}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{1h}|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2h}|_{H^1(\Omega)}$</th>
<th>Rate</th>
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</thead>
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<tr>
<td>10</td>
<td>1.70e-2</td>
<td>n/a</td>
<td>1.29e-2</td>
<td>n/a</td>
<td>2.20e-1</td>
<td>n/a</td>
<td>2.77e-1</td>
<td>n/a</td>
<td>2.49e-1</td>
<td>n/a</td>
</tr>
<tr>
<td>20</td>
<td>4.24e-3</td>
<td>2.00</td>
<td>3.37e-3</td>
<td>1.94</td>
<td>9.90e-2</td>
<td>1.15</td>
<td>1.40e-1</td>
<td>0.98</td>
<td>1.26e-1</td>
<td>0.98</td>
</tr>
<tr>
<td>40</td>
<td>1.21e-3</td>
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</tr>
<tr>
<td>80</td>
<td>2.67e-4</td>
<td>2.18</td>
<td>2.15e-4</td>
<td>2.06</td>
<td>2.41e-2</td>
<td>1.23</td>
<td>3.53e-2</td>
<td>1.01</td>
<td>3.17e-2</td>
<td>1.00</td>
</tr>
<tr>
<td>160</td>
<td>6.70e-5</td>
<td>1.99</td>
<td>5.38e-5</td>
<td>2.00</td>
<td>1.20e-2</td>
<td>1.00</td>
<td>1.76e-2</td>
<td>1.00</td>
<td>1.59e-2</td>
<td>1.00</td>
</tr>
<tr>
<td>320</td>
<td>1.70e-5</td>
<td>1.98</td>
<td>1.34e-5</td>
<td>2.00</td>
<td>6.04e-3</td>
<td>1.00</td>
<td>8.85e-3</td>
<td>1.00</td>
<td>7.93e-3</td>
<td>1.00</td>
</tr>
</tbody>
</table>

### Table 3.4

Errors of $RQ_1-Q_0$ IFE solutions for Example 3.1 with $\mu^- = 1$ and $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_{1h}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2h}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,h}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{1h}|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2h}|_{H^1(\Omega)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>8.07e-3</td>
<td>n/a</td>
<td>1.09e-2</td>
<td>n/a</td>
<td>2.30e-1</td>
<td>n/a</td>
<td>2.27e-1</td>
<td>n/a</td>
<td>2.71e-1</td>
<td>n/a</td>
</tr>
<tr>
<td>20</td>
<td>1.83e-3</td>
<td>2.14</td>
<td>2.70e-3</td>
<td>2.02</td>
<td>1.11e-1</td>
<td>1.05</td>
<td>1.14e-1</td>
<td>1.00</td>
<td>1.36e-1</td>
<td>1.00</td>
</tr>
<tr>
<td>40</td>
<td>6.32e-4</td>
<td>1.53</td>
<td>7.15e-4</td>
<td>1.92</td>
<td>6.21e-2</td>
<td>0.84</td>
<td>5.80e-2</td>
<td>0.97</td>
<td>6.84e-2</td>
<td>0.99</td>
</tr>
<tr>
<td>80</td>
<td>1.14e-4</td>
<td>2.47</td>
<td>1.70e-4</td>
<td>2.08</td>
<td>2.75e-2</td>
<td>1.17</td>
<td>2.85e-2</td>
<td>1.02</td>
<td>3.40e-2</td>
<td>1.01</td>
</tr>
<tr>
<td>160</td>
<td>2.86e-5</td>
<td>1.99</td>
<td>4.23e-5</td>
<td>2.00</td>
<td>1.38e-2</td>
<td>1.00</td>
<td>1.43e-2</td>
<td>1.00</td>
<td>1.70e-2</td>
<td>1.00</td>
</tr>
<tr>
<td>320</td>
<td>7.39e-6</td>
<td>1.95</td>
<td>1.06e-5</td>
<td>1.99</td>
<td>6.90e-3</td>
<td>1.00</td>
<td>7.14e-3</td>
<td>1.00</td>
<td>8.52e-3</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Let \( \mu^- = 1 \) and \( \mu^+ = 10 \). The exact solutions \( u = [u_1, u_2]' \) and \( p \) are chosen as follows:

\[
\begin{align*}
u_1 &= \begin{cases} 
\frac{y(x^2+y^2-0.3)}{\mu^+}, & \text{if } (x,y) \in \Omega^+, \\
\frac{y(x^2+y^2-0.3)}{\mu^-}, & \text{if } (x,y) \in \Omega^-, 
\end{cases} \\
u_2 &= \begin{cases} 
\frac{-x(x^2+y^2-0.3)}{\mu^+}, & \text{if } (x,y) \in \Omega^+, \\
\frac{-x(x^2+y^2-0.3)}{\mu^-}, & \text{if } (x,y) \in \Omega^-, 
\end{cases}
\end{align*}
\]

(3.77)

and

\[
p = \frac{1}{10} (x^3 - y^3).
\]

(3.78)

Errors of the IFE interpolation and the IFE solution for this problem are reported in Tables 3.5 - 3.8, respectively. Again, the convergence rates for both the interpolation and the IFE solution are optimal in all norms. The \( CR-P_0 \) IFE solution on the \( 160 \times 160 \) mesh is plotted in Figure 3.6. One can observe that the numerical solution around the interface is resolved accurately. The velocity vector field is plotted in the left plot of Figure 3.7.

Table 3.5

Errors of \( CR-P_0 \) IFE interpolation for Example 3.2 with \( \mu^- = 1, \mu^+ = 10 \).

| N  | \( ||e_{1,I}||_{L^2(\Omega)} \) Rate | \( ||e_{2,I}||_{L^2(\Omega)} \) Rate | \( ||e_{P,I}||_{L^2(\Omega)} \) Rate | \( ||e_{1,I}||_{H^1(\Omega)} \) Rate | \( ||e_{2,I}||_{H^1(\Omega)} \) Rate |
|----|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| 10 | 3.36e-3  n/a                      | 3.36e-3  n/a                      | 4.00e-2  n/a                      | 9.02e-2  n/a                      | 9.06e-2  n/a                      |
| 20 | 9.01e-4  1.90                     | 9.01e-4  1.90                     | 1.29e-2  1.63                     | 4.59e-2  0.97                     | 4.61e-2  0.97                     |
| 40 | 2.34e-4  1.94                     | 2.34e-4  1.94                     | 6.85e-3  0.92                     | 2.36e-2  0.96                     | 2.36e-2  0.97                     |
| 80 | 5.94e-5  1.98                     | 5.94e-5  1.98                     | 2.77e-3  1.31                     | 1.21e-2  0.96                     | 1.20e-2  0.98                     |
| 160| 1.49e-5  1.99                     | 1.49e-5  1.99                     | 1.19e-3  1.22                     | 6.01e-3  1.01                     | 5.94e-3  1.02                     |
| 320| 3.74e-6  1.99                     | 3.74e-6  1.99                     | 5.32e-4  1.16                     | 2.98e-3  1.01                     | 2.95e-3  1.01                     |
### Table 3.6

Errors of $RQ_1-Q_0$ IFE interpolation for Example 3.2 with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_{1,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{p,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{1,l}|_{H^1(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{H^1(\Omega)}$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.11e-3 n/a</td>
<td>4.11e-3 n/a</td>
<td>2.17e-2 n/a</td>
<td>1.07e-1 n/a</td>
<td>1.07e-1 n/a</td>
</tr>
<tr>
<td>20</td>
<td>1.04e-3 1.98</td>
<td>1.04e-3 1.04</td>
<td>1.10e-2 0.98</td>
<td>5.17e-2 1.06</td>
<td>5.16e-2 1.06</td>
</tr>
<tr>
<td>40</td>
<td>2.69e-4 1.95</td>
<td>2.69e-4 1.95</td>
<td>5.48e-3 1.00</td>
<td>2.68e-2 0.95</td>
<td>2.68e-2 0.95</td>
</tr>
<tr>
<td>80</td>
<td>6.74e-5 2.00</td>
<td>6.74e-5 2.00</td>
<td>2.74e-3 1.00</td>
<td>1.36e-2 0.97</td>
<td>1.37e-2 0.97</td>
</tr>
<tr>
<td>160</td>
<td>1.68e-5 2.00</td>
<td>1.68e-5 2.00</td>
<td>1.37e-3 1.00</td>
<td>6.72e-3 1.02</td>
<td>6.73e-3 1.02</td>
</tr>
<tr>
<td>320</td>
<td>4.20e-6 2.00</td>
<td>4.20e-6 2.00</td>
<td>6.85e-4 1.00</td>
<td>3.33e-3 1.01</td>
<td>3.33e-3 1.01</td>
</tr>
</tbody>
</table>

### Table 3.7

Errors of $CR-P_0$ IFE solution for Example 3.2 with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_{1,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{p,l}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{1,l}|_{H^1(\Omega)}$ Rate</th>
<th>$|e_{2,l}|_{H^1(\Omega)}$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.20e-3 n/a</td>
<td>4.20e-3 n/a</td>
<td>7.60e-2 n/a</td>
<td>1.02e-1 n/a</td>
<td>1.02e-1 n/a</td>
</tr>
<tr>
<td>20</td>
<td>1.14e-3 1.88</td>
<td>1.14e-3 1.88</td>
<td>3.24e-2 1.23</td>
<td>5.31e-2 0.93</td>
<td>5.33e-2 0.94</td>
</tr>
<tr>
<td>40</td>
<td>2.96e-4 1.94</td>
<td>2.96e-4 1.94</td>
<td>1.60e-2 1.03</td>
<td>2.75e-2 0.95</td>
<td>2.75e-2 0.96</td>
</tr>
<tr>
<td>80</td>
<td>7.62e-5 1.96</td>
<td>7.62e-5 1.96</td>
<td>7.53e-3 1.09</td>
<td>1.41e-2 0.96</td>
<td>1.40e-2 0.97</td>
</tr>
<tr>
<td>160</td>
<td>1.91e-5 1.99</td>
<td>1.91e-5 1.99</td>
<td>3.66e-3 1.04</td>
<td>7.03e-3 1.00</td>
<td>6.98e-3 1.01</td>
</tr>
<tr>
<td>320</td>
<td>4.81e-6 1.99</td>
<td>4.81e-6 1.99</td>
<td>1.80e-3 1.03</td>
<td>3.50e-3 1.01</td>
<td>3.48e-3 1.00</td>
</tr>
</tbody>
</table>
IFE Solutions $u_{1h}, u_{2h},$ and $p_h$ on the $160 \times 160$ mesh of Example 2 with $\mu^− = 1, \mu^+ = 10.$
Table 3.8

Errors of $RQ_1$-$Q_0$ IFE solution for Example 3.2 with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_1,1|_{L^2(\Omega)}$ Rate</th>
<th>$|e_2,1|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{p,1}|_{L^2(\Omega)}$ Rate</th>
<th>$|e_{1,1}|_{H^1(\Omega)}$ Rate</th>
<th>$|e_{2,1}|_{H^1(\Omega)}$ Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.11e-3 n/a</td>
<td>4.11e-3 n/a</td>
<td>2.17e-2 n/a</td>
<td>1.07e-1 n/a</td>
<td>1.07e-1 n/a</td>
</tr>
<tr>
<td>20</td>
<td>1.04e-3 1.98</td>
<td>1.04e-3 1.04</td>
<td>1.10e-2 0.98</td>
<td>5.17e-2 1.06</td>
<td>5.16e-2 1.06</td>
</tr>
<tr>
<td>40</td>
<td>2.69e-4 1.95</td>
<td>2.69e-4 1.95</td>
<td>5.48e-3 1.00</td>
<td>2.68e-2 0.95</td>
<td>2.68e-2 0.95</td>
</tr>
<tr>
<td>80</td>
<td>6.74e-5 2.00</td>
<td>6.74e-5 2.00</td>
<td>2.74e-3 1.00</td>
<td>1.36e-2 0.97</td>
<td>1.37e-2 0.97</td>
</tr>
<tr>
<td>160</td>
<td>1.68e-5 2.00</td>
<td>1.68e-5 2.00</td>
<td>1.37e-3 1.00</td>
<td>6.72e-3 1.02</td>
<td>6.73e-3 1.02</td>
</tr>
<tr>
<td>320</td>
<td>4.20e-6 2.00</td>
<td>4.20e-6 2.00</td>
<td>6.85e-4 1.00</td>
<td>3.33e-3 1.01</td>
<td>3.33e-3 1.01</td>
</tr>
</tbody>
</table>

Figure 3.7

Velocity vector field of Example 3.2 and Example 3.3 with various coefficients. From left:

$(\mu^-, \mu^+) = (1, 10), (10, 1), \text{and} (1, 1000)$. 

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3.7.3 Example 3.3: Flipped Coefficients and Large Coefficient Contrast

In this example, we consider the curved interface problem in Example 3.2 again with different jump ratios. This time we only report the IFE solution errors in consideration of the page limit. The convergence rates of the IFE interpolation are optimal as usual. In Tables 3.9 - 3.12 we report the error of IFE solutions for the case when the coefficient is flipped \((\mu^-, \mu^+) = (10, 1)\), and when the coefficient has a large jump \((\mu^-, \mu^+) = (1, 1000)\), respectively. As before, we can see that the convergence rates are optimal for both cases. In Figure 3.7, we plot the velocity vector field of these cases.

Table 3.9

| N  | \[|e_{1,l}|\]_{L^2(\Omega)} \ Rate | \[|e_{2,l}|\]_{L^2(\Omega)} \ Rate | \[|e_{p,l}|\]_{L^2(\Omega)} \ Rate | \[|e_{1,l}|\]_{H^1(\Omega)} \ Rate | \[|e_{2,l}|\]_{H^1(\Omega)} \ Rate |
|----|----------------|----------------|----------------|----------------|----------------|
| 10 | 1.88e-2 n/a | 1.88e-2 n/a | 6.84e-2 n/a | 4.27e-1 n/a | 4.27e-1 n/a |
| 20 | 4.85e-3 1.96 | 4.85e-3 1.96 | 3.71e-2 0.88 | 2.15e-1 0.99 | 2.16e-1 0.99 |
| 40 | 1.22e-3 1.99 | 1.22e-3 1.99 | 1.62e-2 1.20 | 1.08e-1 0.99 | 1.08e-1 0.99 |
| 80 | 3.07e-4 1.99 | 3.07e-4 1.99 | 7.38e-3 1.13 | 5.43e-2 1.00 | 5.42e-2 1.00 |
| 160| 7.69e-5 2.00 | 7.69e-5 2.00 | 3.63e-3 1.02 | 2.71e-2 1.00 | 2.71e-2 1.00 |
| 320| 1.92e-5 2.00 | 1.92e-5 2.00 | 1.79e-3 1.02 | 1.36e-2 1.00 | 1.36e-2 1.00 |
Table 3.10

Errors of $RQ_1-Q_0$ IFE solution for Example 3.3 with $\mu^- = 10$, $\mu^+ = 1$.

| N  | $||e_{1,I}||_{L^2(\Omega)}$ Rate | $||e_{2,I}||_{L^2(\Omega)}$ Rate | $||e_{p,I}||_{L^2(\Omega)}$ Rate | $||e_{1,I}||_{H^1(\Omega)}$ Rate | $||e_{2,I}||_{H^1(\Omega)}$ Rate |
|----|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| 10 | 1.21e-2                          | n/a                              | 1.21e-2                          | n/a                              | 2.56e-2                          | n/a                              |
|    |                                  |                                  | 4.10e-1                          | n/a                              | 4.10e-1                          | n/a                              |
| 20 | 3.06e-3                          | 1.98                             | 3.06e-3                          | 1.98                             | 1.51e-2                          | 0.76                             |
|    |                                  |                                  | 2.05e-1                          | 1.00                             | 2.05e-1                          | 1.00                             |
| 40 | 7.74e-4                          | 1.98                             | 7.74e-4                          | 1.98                             | 6.00e-3                          | 1.33                             |
|    |                                  |                                  | 1.03e-1                          | 1.00                             | 1.03e-1                          | 1.00                             |
| 80 | 1.94e-4                          | 1.99                             | 1.94e-4                          | 1.99                             | 2.93e-3                          | 1.04                             |
|    |                                  |                                  | 5.14e-2                          | 1.00                             | 5.15e-2                          | 1.00                             |
| 160| 4.84e-5                          | 2.01                             | 4.84e-5                          | 2.01                             | 1.42e-3                          | 1.05                             |
|    |                                  |                                  | 2.57e-2                          | 1.00                             | 2.57e-2                          | 1.00                             |
| 320| 1.21e-5                          | 2.00                             | 1.21e-5                          | 2.00                             | 6.97e-4                          | 1.02                             |
|    |                                  |                                  | 1.28e-2                          | 1.00                             | 1.28e-2                          | 1.00                             |

Table 3.11

Errors of $CR-P_0$ IFE solution for Example 3.3 with $\mu^- = 1$, $\mu^+ = 1000$.

| N  | $||e_{1,I}||_{L^2(\Omega)}$ Rate | $||e_{2,I}||_{L^2(\Omega)}$ Rate | $||e_{p,I}||_{L^2(\Omega)}$ Rate | $||e_{1,I}||_{H^1(\Omega)}$ Rate | $||e_{2,I}||_{H^1(\Omega)}$ Rate |
|----|----------------------------------|----------------------------------|----------------------------------|----------------------------------|----------------------------------|
| 10 | 1.22e-2                          | n/a                              | 1.22e-2                          | n/a                              | 8.27e-1                          | n/a                              |
|    |                                  |                                  | 1.35e-1                          | n/a                              | 1.36e-1                          | n/a                              |
| 20 | 2.37e-3                          | 2.36                             | 2.37e-3                          | 2.36                             | 6.39e-1                          | 0.37                             |
|    |                                  |                                  | 5.56e-2                          | 1.28                             | 5.58e-2                          | 1.28                             |
| 40 | 5.58e-4                          | 2.09                             | 5.58e-4                          | 2.09                             | 3.43e-1                          | 0.90                             |
|    |                                  |                                  | 2.64e-2                          | 1.08                             | 2.64e-2                          | 1.08                             |
| 80 | 1.10e-4                          | 2.34                             | 1.10e-4                          | 2.34                             | 1.37e-1                          | 1.32                             |
|    |                                  |                                  | 1.32e-2                          | 0.99                             | 1.31e-2                          | 1.01                             |
| 160| 2.25e-5                          | 2.29                             | 2.25e-5                          | 2.29                             | 4.96e-2                          | 1.47                             |
|    |                                  |                                  | 6.56e-3                          | 1.01                             | 6.49e-3                          | 1.02                             |
| 320| 4.87e-6                          | 2.21                             | 4.87e-6                          | 2.21                             | 1.73e-2                          | 1.52                             |
|    |                                  |                                  | 3.24e-3                          | 1.02                             | 3.21e-2                          | 1.01                             |
| N  | $||e_{1,I}||_{L^2(\Omega)}$ | Rate | $||e_{2,I}||_{L^2(\Omega)}$ | Rate | $||e_{p,I}||_{L^2(\Omega)}$ | Rate | $||e_{1,I}||_{H^1(\Omega)}$ | Rate | $||e_{2,I}||_{H^1(\Omega)}$ | Rate |
|----|-----------------------------|------|-----------------------------|------|-----------------------------|------|-----------------------------|------|-----------------------------|------|
| 10 | 1.90e-2                     | n/a  | 1.90e-2                     | n/a  | 9.75e-1                     | n/a  | 1.70e-1                     | n/a  | 1.70e-1                     | n/a  |
| 20 | 3.19e-3                     | 2.57 | 3.19e-3                     | 2.57 | 6.02e-1                     | 0.69 | 5.54e-2                     | 1.62 | 5.53e-2                     | 1.62 |
| 40 | 1.35e-3                     | 1.24 | 1.35e-3                     | 1.24 | 2.57e-1                     | 1.23 | 3.07e-2                     | 0.85 | 3.07e-2                     | 0.85 |
| 80 | 2.20e-4                     | 2.62 | 2.20e-4                     | 2.62 | 1.01e-1                     | 1.35 | 1.33e-2                     | 1.20 | 1.34e-2                     | 1.20 |
| 160| 3.47e-5                     | 2.66 | 3.47e-5                     | 2.66 | 3.58e-2                     | 1.49 | 6.35e-3                     | 1.07 | 6.37e-3                     | 1.07 |
| 320| 6.45e-6                     | 2.43 | 6.45e-6                     | 2.43 | 1.30e-2                     | 1.46 | 3.10e-3                     | 1.06 | 3.11e-3                     | 1.04 |

Table 3.12

Errors of $RQ_1$-$Q_0$ IFE solution for Example 3.3 with $\mu^- = 1$, $\mu^+ = 1000$. 

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CHAPTER IV

A CONFORMING-NONCONFORMING IMMERSED FINITE ELEMENT METHOD FOR
STOKES INTERFACE PROBLEMS

In this chapter, we develop lowest-order conforming-nonconforming mixed immersed finite element spaces for the Stokes interface problem based on [55]. The proposed IFE spaces use conforming linear elements for one velocity component and non-conforming linear elements for the other component. The pressure is approximated by piecewise constant. Unisolvency, among other fundamental properties of the new vector-valued IFE functions, is analyzed [51].

Comparing with the IDG method in [2], our new IFE method has no additional terms for consistency and stability, so the numerical formulation is much simpler and the degrees of freedom are much less. Comparing with the $CR-P_0$ IFE space, there is a good amount of saving in the degrees of freedom due to the conformity of one velocity component. In fact, on the same triangular mesh, there are only two-thirds of degrees of freedom for velocity in this new mixed IFEM. Besides, the mixed conforming-nonconforming finite element is robust for handling both Dirichlet and Neumann boundary conditions, while the CR finite element space is only stable for Dirichlet boundary conditions [55]. Numerical experiments are carried out to demonstrate the performance of this new IFE method.
4.1 Statement of the Problem

Consider the governing incompressible Stokes equations:

\[-\nabla \cdot S(u, p) = f \quad \text{in } \Omega^+ \cup \Omega^- \tag{4.1}\]
\[\nabla \cdot u = 0 \quad \text{in } \Omega \tag{4.2}\]
\[u = 0 \quad \text{in } \partial \Omega \tag{4.3}\]

where \(u\) and \(p\) denote the velocity and the pressure, respectively. \(S(u, p)\) is the stress tensor defined as

\[S(u, p) = 2\mu \epsilon(u) - pI \tag{4.4}\]

where \(\epsilon(u) = (\nabla u + (\nabla u)^T)/2\) is the strain tensor, and \(I\) is the identity tensor. The viscosity function \(\mu(x)\) is assumed to have a finite jump across the interface \(\Gamma\). For simplicity, we assume that \(\mu(x)\) is a piecewise constant function

\[\mu(x) = \begin{cases} 
\mu^- & \text{in } \Omega^-, \\
\mu^+ & \text{in } \Omega^+,
\end{cases} \tag{4.5}\]

where \(\mu^\pm\) are positive constants and \(x = (x, y)\). Across the fluid interface \(\Gamma\), the solution is assumed to satisfy the following velocity and stress jump conditions:

\[[u]_{\Gamma} = 0, \tag{4.6}\]
\[[S(u, p)n]_{\Gamma} = 0, \tag{4.7}\]

where the jump \([v(x)]_{\Gamma} := v^+(x)|_{\Gamma} - v^-(x)|_{\Gamma}\), and \(n\) denotes the unit normal vector to the interface \(\Gamma\) pointing from \(\Omega^-\) to \(\Omega^+\). Throughout the paper, we use the standard notation \((\cdot, \cdot)_\omega\) to denote the \(L^2\) inner product on \(\omega \subset \Omega\). We omit subscript \(\omega\) if \(\omega = \Omega\).
4.2 Mixed Conforming-Nonconforming Immersed Finite Element Spaces

In this section, we introduce the mixed conforming-nonconforming IFE spaces for the Stokes interface problem. Let $\mathcal{T}_h = \{T_k\}_{k=1}^N$ be an unfitted shape-regular triangulation of $\Omega$ where $N = |\mathcal{T}_h|$ denotes the number of triangles. Let $\mathcal{N}_h$ and $\mathcal{E}_h$ denote the collections of nodes and edges of the mesh $\mathcal{T}_h$, respectively. Elements in $\mathcal{T}_h$ are divided into two categories: an interface element if $T$ is cut through by the interface $\Gamma$, and a non-interface element otherwise. The collections of interface elements and non-interface elements are denoted by $\mathcal{T}^I_h$ and $\mathcal{T}^n_h$, respectively. Similarly, for each edge $e \in \mathcal{E}_h$, if $e$ intersects the interface, it is called an interface edge; otherwise, it is a non-interface edge. The collections of interface edges and non-interface edges are denoted by $\mathcal{E}^I_h$ and $\mathcal{E}^n_h$, respectively. Additionally, we let $\mathcal{E}_h^i$ and $\mathcal{E}_h^b$ be the collections of internal edges and boundary edges, respectively. Let $\mathcal{N}_h$ and $\mathcal{N}_h^b$ be the collections of internal nodes and boundary nodes, respectively. We also assume that the triangulation $\mathcal{T}_h$ satisfies the following hypotheses [74]:

- **(H1)** The interface $\Gamma$ cannot intersect an edge of any element at more than two points unless the edge is part of $\Gamma$.

- **(H2)** If $\Gamma$ intersects the boundary of an element at two points, these intersection points must be on different edges of this element.

- **(H3)** The interface $\Gamma$ is a piecewise $C^2$-continuous function, and the mesh $\mathcal{T}_h$ is formed such that the subset of $\Gamma$ in every interface element is $C^2$-continuous.
4.2.1 Conforming-Nonconforming CR-$P_1$-$P_0$ FE Spaces

Let $T \in \mathcal{T}_h^n$ be a non-interface element with vertices $A_1, A_2, A_3$ oriented counterclockwise. We label the edges of $T$ by $e_1 = A_1A_2$, $e_2 = A_2A_3$, and $e_3 = A_3A_1$. Let $\lambda_{j,T} \in \mathcal{P}_1$ be the Lagrange linear nodal basis functions such that

$$
\lambda_{j,T}(A_i) = \delta_{ij}, \quad i, j = 1, 2, 3,
$$

(4.8)

where $\delta_{ij}$ is the Kronecker function. Define $\psi_{j,T} = 1 - 2\lambda_{k_{ij},T}$ with $k_1 = 3$, $k_2 = 1$, and $k_3 = 2$. It can be verified that $\psi_{j,T}$ satisfies the mean-value conditions, namely,

$$
\frac{1}{|e_i|} \int_{e_i} \psi_{j,T}(x,y) ds = \delta_{ij}, \quad i, j = 1, 2, 3.
$$

(4.9)

Thus $\psi_{j,T}$, $j = 1, 2, 3$ are nonconforming-$P_1$ (CR) basis functions on $T$. The pressure is approximated by the piecewise constant function space denoted by $\mathcal{P}_0$. On each non-interface triangle $T \in \mathcal{T}_h^n$, the vector-valued CR-$P_1$-$P_0$ finite element space can be written as $S^n_h(T) = \mathcal{P}_1 \times \mathcal{P}_1 \times \mathcal{P}_0$, or equivalently, $S^n_h(T) = span\{\psi_{j,T} : 1 \leq j \leq 7\}$ where the vector-valued basis functions are as follows:

$$
\psi_{j,T} = \begin{bmatrix} \psi_{j,T} \\ 0 \\ 0 \end{bmatrix}, \quad j = 1, 2, 3, \quad \psi_{j,T} = \begin{bmatrix} 0 \\ \lambda_{j-3,T} \\ 0 \end{bmatrix}, \quad j = 4, 5, 6, \quad \psi_{7,T} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
$$

(4.10)
Similarly, we can also form the $P_1-CR-P_0$ finite element space using conforming-$P_1$ bases for the first component, and the nonconforming-$P_1$ bases in the second component, then the basis functions are:

$$\tilde{\psi}_{j,T} = \begin{bmatrix} \lambda_{j,T} \\ 0 \\ 0 \end{bmatrix}, \quad j = 1, 2, 3, \quad \tilde{\psi}_{j,T} = \begin{bmatrix} 0 \\ \psi_{j-3, T} \\ 0 \end{bmatrix}, \quad j = 4, 5, 6, \quad \tilde{\psi}_{7,T} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \quad (4.11)$$

The $P_1-CR-P_0$ finite element space is $\tilde{S}_h^n(T) = span\{\tilde{\psi}_{j,T} : 1 \leq j \leq 7\}$. Here, we note that these two spaces $S_h^n(T)$ and $\tilde{S}_h^n(T)$ are identical. In fact, they both are $P_1 \times P_1 \times P_0$, but the degrees of freedom of $S_h^n(T)$ and $\tilde{S}_h^n(T)$ are different, as indicated in (4.8) and (4.9). For more details of the conforming-nonconforming finite elements, we refer readers to [55].

### 4.2.2 Conforming-Nonconforming CR-$P_1-P_0$ IFE Spaces

We extend these conforming-nonconforming finite elements to IFE spaces defined on each interface triangle $T \in T_h^i$. Let $A_i = (x_i, y_i), i = 1, 2, 3$ be the vertices of the triangle. Without loss of generality, we consider the following reference triangle $T$ whose vertices are given by

$$\hat{A}_1 = (0, 0), \quad \hat{A}_2 = (1, 0), \quad \hat{A}_3 = (0, 1). \quad (4.12)$$

Note that an arbitrary triangle with vertices $A_i = (x_i, y_i), i = 1, 2, 3$ can be mapped to this reference triangle by affine mapping.

To simplify the notation, we still use $(x, y)$ rather than $(\hat{x}, \hat{y})$ on the reference triangle. According to the hypotheses (H1)-(H3), there are two distinct intersection points on each interface triangle, denoted by $D = (x_d, y_d)$ and $E = (x_e, y_e)$, on two different edges. There are generally three types of interface triangles as depicted in Figure 4.1. The line segment $\overline{DE}$ is used to approximate the
Types of interface elements. From left: Type I, Type II, Type III. The red curve $\Gamma$ is the actual interface, and $\Gamma_T = DE$ is the line approximation of the interface.

actual interface curve $\Gamma \cap T$, and it divides the element $T$ into two subelements, denoted by $T^+$ and $T^-$. For example, on a Type I interface element, $D = A_1 + d(A_2 - A_1)$ and $E = A_1 + e(A_3 - A_1)$ where $0 < d, e < 1$. We construct the vector-valued IFE shape functions in terms of the FE functions $\psi_{j,T}$ in (4.10). To be more precise, we have

$$\phi_{j,T}(x,y) = \begin{cases} \sum_{i=1}^{7} c_{ij}^+ \psi_{j,T}, & \text{if } (x,y) \in T^+, \\ \sum_{i=1}^{7} c_{ij}^- \psi_{j,T}, & \text{if } (x,y) \in T^- \end{cases}$$

(4.13)

It can be observed that each vector-valued IFE shape function $\phi_{j,T}$ has 14 unknown coefficients $c_{ij}^\pm$, $1 \leq j \leq 7$. These coefficients are determined by seven local degrees of freedom (prescribed nodal values, edge values, and mean pressure value), and an additional seven interface jump conditions stated below:

- Three edge-value conditions:

$$\frac{1}{|e_k|} \int_{e_k} \phi_{j,T} ds = \begin{pmatrix} \delta_{jk} \\ 0 \\ 0 \end{pmatrix}, \quad k = 1, 2, 3.$$  

(4.14)
• Three nodal-value conditions:

\[ \phi_{j,T}(A_{k-3}) = \begin{pmatrix} 0 \\ \delta_{j,k} \\ 0 \end{pmatrix}, \quad k = 4, 5, 6. \]  
(4.15)

• One mean-pressure-value condition:

\[ \frac{1}{|T|} \int_T \phi_{j,T} \, dx \, dy = \begin{pmatrix} 0 \\ 0 \\ \delta_{j,k} \end{pmatrix}, \quad k = 7. \]  
(4.16)

• Four continuity conditions of the velocity to incorporate (4.6):

\[ [\phi_{1,j}(D)] = [\phi_{2,j}(D)] = [\phi_{1,j}(E)] = [\phi_{2,j}(E)] = 0. \]  
(4.17)

• Two stress continuity conditions to incorporate (4.7):

\[ [\mu (\partial_x \phi_{1,j} n_1 + \partial_y \phi_{1,j} n_2) - \phi_{p,j} n_1]_{D_E} = 0, \]  
(4.18)

\[ [\mu (\partial_x \phi_{2,j} n_1 + \partial_y \phi_{2,j} n_2) - \phi_{p,j} n_2]_{D_E} = 0. \]  
(4.19)

• One continuity of the divergence condition to incorporate (4.2):

\[ [\partial_x \phi_{1,j} + \partial_y \phi_{2,j}]_{D_E} = 0. \]  
(4.20)

Here, in (4.17)-(4.20), the scalar function \( \phi_{i,j} \) denotes the \( i \)-th component of \( \phi_{j,T} \). More precisely, we have \( \phi_{j,T} = (\phi_{1,j}, \phi_{2,j}, \phi_{p,j}) \) such that \( \phi_{j,T}|_{T^s} = \phi_{j,T}^s = (\phi_{1,j}^s, \phi_{2,j}^s, \phi_{p,j}^s) \in P_1 \times P_1 \times P_0 \), with \( s = +, - \). Combining the conditions (4.14)-(4.20) yields a linear system of fourteen unknowns. On Type I interface element, we have

\[ M_j c_j = e_j, \]  
(4.21)
where the coefficient matrix $M_I$ is

$$M_I = \begin{pmatrix}
    d & d^2 - d & d - d^2 & 0 & 0 & 0 & 0 & 1 - d & d - d^2 & d^2 - d & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
    e - e^2 & e^2 - e & e & 0 & 0 & 0 & 0 & e^2 - e & e - e^2 & 1 - e & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 0 & 0 & de & 0 & 0 & 0 & 0 & 0 & 1 - de \\
    -1 & 1 - 2d & 2d - 1 & 0 & 0 & 0 & 0 & 1 & 2d - 1 & 1 - 2d & 0 & 0 & 0 \\
    0 & 0 & 0 & d - 1 & -d & 0 & 0 & 0 & 0 & 1 - d & d & 0 & 0 \\
    2e - 1 & 1 - 2e & -1 & 0 & 0 & 0 & 1 - 2e & 2e - 1 & 1 & 0 & 0 & 0 & 0 \\
    0 & 0 & 0 & e - 1 & 0 & -e & 0 & 0 & 0 & 1 - e & 0 & e & 0 \\
    -2d & 2d + 4e & -4e & -d & d & 0 & -e & 2dp & -2(d + 2e)\rho & 4e\rho & dp & -dp & 0 & e \\
    -2e & 2e & 0 & 2d - e & e & 2d & -d & 2ep & -2ep & 0 & (2d + e)\rho & -e\rho & -2dp & d \\
    0 & -2 & 2 & 1 & 0 & -1 & 0 & 0 & 2 & -2 & -1 & 0 & 1 & 0
\end{pmatrix}$$

(4.22)

with $\rho = \mu^+ / \mu^-$ denoting the jump ratio. The unknown vector $c_j$ and the right-hand-side vector $e_j$ take the form

$$c_j = \begin{bmatrix} c_{ij}^+, c_{ij}^2, c_{ij}^3, c_{ij}^4, c_{ij}^5, c_{ij}^6, c_{ij}^7, c_{ij}^- \end{bmatrix}^T,$$

$$e_j = \begin{bmatrix} \delta_{j1}, \delta_{j2}, \delta_{j3}, \delta_{j4}, \delta_{j5}, \delta_{j6}, \delta_{j7}, 0, 0, 0, 0, 0, 0 \end{bmatrix}^T.$$

We can obtain the vector-valued IFE shape functions $\phi_{j,T}$ by solving for $c_j$ given each vector $e_j$, $j = 1, 2, \cdots, 7$. Note that the matrices for Type II and Type III interface elements, denoted by $M_{II}$ and $M_{III}$, can be derived in similar fashion.

As an illustration, we plot the three components of the $CR-P_1-P_0$ IFE shape function $\phi_{4,T}$ in Figure (4.2). As a comparison, we plot the standard $CR-P_1-P_0$ FE shape function $\psi_{4,T}$. We note
that both FE and IFE shape functions are configured such that their second velocity components have the value one at the node $A_1$. However, due to the coupled stress jump condition (4.7), the first velocity component and the pressure component of the IFE shape function $\phi_{4,T}$ are not completely zero, as the FE shape function. This is a similar phenomenon that also occurs in other vector-valued IFE functions [2, 75, 73]. The local $CR-P_1-P_0$ IFE space is formed by

\[ S_{\eta}^i(T) = \text{span}\{\phi_{j,T} : 1 \leq j \leq 7\}, \]

and the global $CR-P_1-P_0$ IFE space is defined to be

\[ S_{\eta}(T_h) = \left\{ \mathbf{v} = [v_1, v_2, v_p]^T \in [L^2(\Omega)]^3 : \mathbf{v}|_T \in S_{\eta}^i(T), \text{ if } T \in T_h \right\}; \]

Figure 4.2

A comparison of the vector-valued IFE shape function $\phi_{4,T}$ with $\mu^- = 1, \mu^+ = 5$ (top), and the corresponding FE shape function $\psi_{4,T}$ (bottom) on the reference triangle.
\[ \mathbf{v}|_T \in \mathbf{S}_h^i(T) \text{ if } T \in \mathcal{T}_h^i, \text{ and } \int_e [v_1] ds = 0, \forall e \in \hat{\mathcal{E}}_h, \]

and \( v_2 \) is continuous at every point \((x, y) \in \mathcal{N}_h\). \hfill (4.23)

We can construct the \( P_1-CR-P_0 \) IFE space in a similar manner. In this case the edge-value conditions (4.14) are imposed on the second velocity component, and the nodal-value conditions (4.15) will apply to the first velocity component. The remaining conditions (4.16)-(4.20) are the same. Let \( \tilde{S}_h^i(T) \) be the local \( P_1-CR-P_0 \) IFE space, then the global mixed IFE space \( \tilde{S}_h(T_h) \) is given by

\[
\tilde{S}_h(T_h) = \left\{ \mathbf{v} = [v_1, v_2, v_p]^T \in [L^2(\Omega)]^3 : \mathbf{v}|_T \in \tilde{S}_h^i(T), \text{ if } T \in \mathcal{T}_h^n; \mathbf{v}|_T \in \tilde{S}_h^i(T) \text{ if } T \in \mathcal{T}_h^i, \right. \\
\left. v_1 \text{ is continuous at every point } (x, y) \in \mathcal{N}_h, \text{ and } \int_e [v_2] ds = 0, \forall e \in \hat{\mathcal{E}}_h \right\}. \hfill (4.24)

4.3 Properties of the Mixed Conforming-Nonconforming IFE Spaces

In this section, we present some basic properties of the mixed conforming-nonconforming IFE spaces. Note that the properties are explicitly derived for the \( CR-P_1-P_0 \) IFE shape functions but also hold true for the \( P_1-CR-P_0 \) IFE shape functions.

**Theorem 6 (Unisolvency)**

The \( CR-P_1-P_0 \) IFE shape functions \( \phi_{j,T}, 1 \leq j \leq 7 \) can be uniquely determined by the prescribed edge values, the nodal values, and the mean pressure value, regardless of the interface locations and the jumps of viscosity coefficients \( \mu^\pm > 0 \).
Proof: We show the unisolvency property by considering the invertibility of the coefficient matrices \( M_I, M_{II}, \) and \( M_{III}. \) For the Type I interface triangle, by direct calculation we have

\[
\det(M_I) = -4\left(d^4(1-de) + d^2e^2(2-d-e) + e^4(1-d) + \rho de(d^4 + de^2 + d^2e^2 + e^3)\right) < 0.
\]

For the Type II interface element, we have

\[
\det(M_{II}) = D_1 + \rho D_2
\]

where

\[
D_1 = -4(1-d)e\left((-1+d)^4 + 4(-1+d)^3e + 7(-1+d)^2e^2 + (-5+6d)e^3 + 2e^4\right)
\]

\[
= -4(1-d)e\left((1-d)^3 - 4(1-d)^3e + 7(1-d)^2e^2 - 6(1-d)e^3 + e^3 + 2e^4\right)
\]

\[
\leq -4(1-d)e\left((1-d)^3 - 4(1-d)^3e + 7(1-d)^2e^2 - 6(1-d)e^3 + 3e^4\right)
\]

\[
= -4(1-d)e\left((1-d)^2(1-d-2e)^2 + 3e^2(1-d-e)^2\right) < 0,
\]

and with \( s = 1-d, \) we have

\[
D_2 = -4\left(4e^4 - 8e^3s - e^4s - 2e^5s + 8e^2s^2 + 6e^4s^2 - 4es^3 - 7e^3s^3 + s^4 + 4e^2s^3 - es^5\right)
\]

\[
= -4\left(4e^2(e-s)^2 + s^2(2e-s)^2\right) - 4es\left(2e^4 + e^3(1-6s) + 7e^2s^2 - 4es^3 + s^4\right)
\]

\[
\leq -4\left(4e^2(e-s)^2 + s^2(2e-s)^2\right) - 4es\left(3e^4 - 6e^3s + 7e^2s^2 - 4es^3 + s^4\right)
\]

\[
= -4\left(4e^2(e-s)^2 + s^2(2e-s)^2\right) - 4es\left(3e^2(e-s)^2 + s^2(2e-s)^2\right) < 0.
\]

For the Type III interface element, we have

\[
\det(M_{III}) = D_3 + \rho D_4
\]
where

\[
D_3 = -4(-1 + d)^2(1 - 2(-1 + d)^2d + d(-4 + d(-1 + 2d))e \\
\quad -(-2 + d)(1 + 2d)e^2 + (-2 + d)e^3 \\
= -4s^2\left(s(1 - t)^2t + s^2(1 - t) + t(2s^3 + t^2 - 2s^2t)\right) \\
\leq -4s^2\left(s(1 - t)^2t + s^2(1 - t) + ts^3 + t(s^2 - t)^2\right) < 0
\]

and with \( s = 1 - d \) and \( t = 1 - e \),

\[
D_4 = -4\left(3s^4 - 9s^3t + s^4t - 2s^5t + 8s^2t^2 + 2s^3t^2 + 2s^4t^2 - 4st^3 - s^3t^3 + t^4\right) \\
< 0.
\]

The determinants of coefficient matrices are uniformly nonzero for all \( 0 \leq d \leq 1, 0 \leq e \leq 1, \rho > 0 \) and for all three types of interface elements. This ensures the unisolvency of the IFE functions.

The following theorems provide basic properties of the new IFE functions. The proof of these results are similar to the proofs of nonconforming \( CR-P_0 \) and \( RQ_1-Q_0 \) in Chapter 3, hence we omit the proofs in this chapter. For more details, we refer the readers to the previous chapters for a more explicit discussion.

**Theorem 7 (Consistency)**

*Let \( T \in \mathcal{T}_h^i \) be an interface triangle.*

- *If \( \mu^+ = \mu^- \), the IFE shape functions \( \phi_{j,T} \) become the FE shape functions \( \psi_{j,T} \), \( j = 1, 2, \cdots, 7 \).*

- *If the interface moves out of a triangle \( T \), i.e.,

\[
\frac{\min\{|T^-|, |T^+|\}}{|T|} \to 0,
\]

the IFE shape functions \( \phi_{j,T} \) become the FE shape functions \( \psi_{j,T} \), \( j = 1, 2, \cdots, 7 \).*
The second consistency result enables us to use IFE functions for solving Stokes moving interface problem efficiently. In fact, as the interface moves out of an element, the IFE functions smoothly convert to the FE functions. No extra condition is needed to enforce this transition.

**Theorem 8 (Continuity of Velocity)**

Let \( T \in T_h \) be an interface element and \( \phi_{j,T} \) be the vector-valued shape functions. Then the velocity components \( \phi_{i,j} \in C(T) \), for \( i = 1, 2 \), and \( j = 1, 2, \cdots, 7 \).

**Theorem 9 (Partition of Unity)**

Let \( T \in T_h \) be an interface element. The vector-valued IFE shape functions \( \phi_{j,T}, j = 1, 2, \cdots, 7 \), satisfy the partition of unity property, namely:

\[
\sum_{j=1}^{3} \phi_{j,T}(x, y) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \sum_{j=4}^{6} \phi_{j,T}(x, y) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \phi_{7,T}(x, y) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \forall (x, y) \in T. \tag{4.26}
\]

**4.4 Mixed Conforming-Nonconforming Immersed Finite Element Method**

In this section, we present the mixed conforming-nonconforming \( CR-P_1-P_0 \) IFEM for solving the Stokes interface problem (4.1) - (4.7). For this weak formulation, we omit the detail. We refer the reader to Chapter 3 for a more descriptive derivation of the weak formulation for the Stokes interface problem.

Multiplying the momentum equation (4.1) by \( v \in [H_0^1(\Omega)]^2 \), integrating by parts over \( \Omega^+ \), and applying the stress jump condition (4.7), we have

\[
\int_{\Omega} (2\mu\varepsilon(u) - pI) : \nabla v dx = \int_{\Omega} f \cdot v dx. \quad (4.27)
\]
Using the identity $(2\mu\varepsilon(u) - pI) : \nabla v = 2\mu\varepsilon(u) : \varepsilon(v) - p(\nabla \cdot v)$, we have

$$\int_{\Omega} 2\mu\varepsilon(u) : \varepsilon(v) d\mathbf{x} - \int_{\Omega} p(\nabla \cdot v) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot v d\mathbf{x}. \tag{4.28}$$

Multiplying a test function $q \in L^2(\Omega)$ to (4.2), and integration by parts we have

$$\int_{\Omega} q(\nabla \cdot u) d\mathbf{x} = 0. \tag{4.29}$$

At the discretization level, we use the IFE space $S_h$ to approximate $H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega)$. The mixed conforming-nonconforming IFE method is to find $(u_h, p_h) \in S_h$ such that

$$\int_{\Omega} 2\mu\varepsilon(u_h) : \varepsilon(v_h) d\mathbf{x} - \int_{\Omega} p(\nabla \cdot v_h) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot v_h d\mathbf{x},$$

$$\int_{\Omega} q_h(\nabla \cdot u_h) d\mathbf{x} = 0,$$

$$\forall (v_h, q_h) \in S_h. \tag{4.30}$$

### 4.5 Numerical Examples

In this section, we report some numerical experiments for the mixed conforming-nonconforming IFE methods for the Stokes interface problems. We test both the interpolation and the IFE solution with various configurations of the interface and coefficient jumps. All of our numerical experiments are performed on a family of Cartesian triangular meshes which are obtained by first partitioning the domain into $N_s \times N_s$ congruent rectangles, and then further dividing each rectangle into two triangles by its diagonal with the positive slope.

We investigate the approximation property of IFE space by the interpolation. Define the $CR-P_1-P_0$ IFE interpolation operator is defined to be $I_h : H^1(\Omega) \times C(\Omega) \times L^2(\Omega) \rightarrow S_h(\mathcal{T}_h)$ such that

$$I_h(u, p)|_T = I_{h,T}(u, p) = \begin{cases} \sum_{j=1}^{7} c_j \phi_{j,T}, & \text{if } T \in \mathcal{T}_h^i, \\
\sum_{j=1}^{7} c_j \psi_{j,T}, & \text{if } T \in \mathcal{T}_h^n, \end{cases} \tag{4.31}$$
where \( \phi_{j,T} \) and \( \psi_{j,T} \) are the local IFE/FE shape functions given in (4.10) and (4.13), respectively.

The coefficients \( c_j \) take the values

\[
c_j = \frac{1}{|e_j|} \int_{e_j} u_1(x, y) \, ds, \quad j = 1, 2, 3, \quad c_j = u_2(A_j), \quad j = 4, 5, 6,
\]

(4.32)

\[
c_7 = \frac{1}{|T|} \int_T p(x, y) \, dx \, dy,
\]

(4.33)

where \( A_j \) and \( e_j, j = 1, 2, 3 \) are the vertices and edges of the triangle \( T \), respectively. The \( P_1-CR-P_0 \) interpolation can be defined similarly. The errors of the IFE interpolations are measured in \( L^2 \) and semi-\( H^1 \) norms as follows

\[
e^0(u_{1,l}) = \|u_1 - u_{1,l}\|_{L^2(\Omega)}, \quad e^0(u_{2,l}) = \|u_2 - u_{2,l}\|_{L^2(\Omega)}, \quad e^0(p_I) = \|p - p_I\|_{L^2(\Omega)},
\]

(4.34)

\[
e^1(u_{1,l}) = |u_1 - u_{1,l}|_{H^1(\Omega)}, \quad e^1(u_{2,l}) = |u_2 - u_{2,l}|_{H^1(\Omega)},
\]

(4.35)

where \( u_{1,l}, u_{2,l}, p_I \) are components of the vector-valued function \( I_h(u, p) \).

### 4.5.1 Example 4.1: Straight-Line Interface

In this example, we consider a Stokes interface problem with a straight-line interface. Let \( \Omega = [-1, 1]^2 \), and the interface \( \Gamma = \{(x, y) : y = \frac{\pi}{6}\} \). The interface divides the domain \( \Omega \) into two subdomains \( \Omega^- = \{(x, y) : y < \frac{\pi}{6}\} \) and \( \Omega^+ = \{(x, y) : y > \frac{\pi}{6}\} \). Let \( \mu^- = 1 \) and \( \mu^+ = 10 \). The exact...
solutions $u$ and $p$ are chosen as follows:

$$u(x, y) = \begin{cases} \frac{1}{\mu^+} (y - \frac{\pi}{6}) x^2, & \text{if } (x, y) \in \Omega^+, \\ \frac{1}{\mu^-} (y - \frac{\pi}{6}) x^2, & \text{if } (x, y) \in \Omega^-, \end{cases}$$

$$u_2(x, y) = \begin{cases} -\frac{1}{\mu^+} x(y - \frac{\pi}{6})^2, & \text{if } (x, y) \in \Omega^+, \\ -\frac{1}{\mu^-} x(y - \frac{\pi}{6})^2, & \text{if } (x, y) \in \Omega^-, \end{cases}$$

(4.36)

$$p(x, y) = e^x - e^y.$$  

(4.37)

Table 4.1

| N   | $||e_1||_{L^2(\Omega)}$ Rate | $||e_2||_{L^2(\Omega)}$ Rate | $||e_{p,1}||_{L^2(\Omega)}$ Rate | $||e_{p,2}||_{L^2(\Omega)}$ Rate | $||e_{p,1}||_{H^1(\Omega)}$ Rate | $||e_{p,2}||_{H^1(\Omega)}$ Rate |
|-----|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| 10  | 7.16e-3 n/a                   | 1.49e-2 n/a                   | 1.70e-1 n/a                   | 1.97e-1 n/a                   | 3.73e-1 n/a                   | 3.73e-1 n/a                   |
| 20  | 1.80e-3 1.99                   | 3.72e-3 2.00                   | 7.06e-2 1.27                   | 9.87e-2 1.00                   | 1.87e-1 1.00                   | 1.87e-1 1.00                   |
| 40  | 4.50e-4 2.00                   | 9.31e-4 2.00                   | 3.58e-2 0.98                   | 4.94e-2 1.00                   | 9.33e-2 1.00                   | 9.33e-2 1.00                   |
| 80  | 1.13e-4 2.00                   | 2.33e-4 2.00                   | 1.85e-2 0.95                   | 2.47e-2 1.00                   | 4.67e-2 1.00                   | 4.67e-2 1.00                   |
| 160 | 2.82e-5 2.00                   | 5.82e-5 2.00                   | 9.14e-3 1.02                   | 1.24e-2 1.00                   | 2.33e-2 1.00                   | 2.33e-2 1.00                   |
| 320 | 7.05e-6 2.00                   | 1.46e-5 2.00                   | 4.48e-3 1.01                   | 6.18e-3 1.00                   | 1.17e-2 1.00                   | 1.17e-2 1.00                   |
Table 4.2

Errors of $P_1$-CR-$P_0$ IFE interpolation for Example 4.1 with $\mu^- = 1$ and $\mu^+ = 10$.

| N  | $||e_{1,l}||_{L^2(\Omega)}$ | Rate | $||e_{2,l}||_{L^2(\Omega)}$ | Rate | $||e_{P,l}||_{L^2(\Omega)}$ | Rate | $||e_{1,l}||_{H^1(\Omega)}$ | Rate | $||e_{2,l}||_{H^1(\Omega)}$ | Rate |
|----|----------------|-------|----------------|-------|----------------|-------|----------------|-------|----------------|-------|
| 10 | 1.40e-2        | n/a   | 8.22e-3        | n/a   | 2.44e-1        | n/a   | 2.91e-1        | n/a   | 2.25e-1        | n/a   |
| 20 | 3.52e-3        | 1.99  | 2.06e-3        | 2.00  | 7.16e-2        | 1.77  | 1.46e-1        | 0.99  | 1.13e-1        | 1.00  |
| 40 | 8.80e-4        | 2.00  | 5.15e-4        | 2.00  | 3.98e-2        | 0.85  | 7.32e-2        | 1.00  | 5.64e-2        | 1.00  |
| 80 | 2.20e-4        | 2.00  | 1.29e-4        | 2.00  | 1.92e-2        | 1.05  | 3.66e-2        | 1.00  | 2.82e-2        | 1.00  |
| 160| 5.50e-5        | 2.00  | 3.22e-5        | 2.00  | 9.40e-3        | 1.03  | 1.83e-2        | 1.00  | 1.41e-2        | 1.00  |
| 320| 1.37e-5        | 2.00  | 8.05e-6        | 2.00  | 4.58e-3        | 1.04  | 9.15e-3        | 1.00  | 7.06e-3        | 1.00  |

Table 4.3

Errors of CR-$P_1$-$P_0$ IFE solutions for Example 4.1 with $\mu^- = 1$ and $\mu^+ = 10$.

| N  | $||e_{1h}||_{L^2(\Omega)}$ | Rate | $||e_{2h}||_{L^2(\Omega)}$ | Rate | $||e_{P h}||_{L^2(\Omega)}$ | Rate | $||e_{1h}||_{H^1(\Omega)}$ | Rate | $||e_{2h}||_{H^1(\Omega)}$ | Rate |
|----|----------------|-------|----------------|-------|----------------|-------|----------------|-------|----------------|-------|
| 10 | 1.70e-2        | n/a   | 1.29e-2        | n/a   | 2.20e-1        | n/a   | 2.77e-1        | n/a   | 2.49e-1        | n/a   |
| 20 | 4.24e-3        | 2.00  | 3.37e-3        | 1.94  | 9.90e-2        | 1.15  | 1.40e-1        | 0.98  | 1.26e-1        | 0.98  |
| 40 | 1.21e-3        | 1.81  | 8.95e-4        | 1.91  | 5.64e-2        | 0.81  | 7.12e-2        | 0.98  | 6.35e-2        | 0.99  |
| 80 | 2.67e-4        | 2.18  | 2.15e-4        | 2.06  | 2.41e-2        | 1.23  | 3.53e-2        | 1.01  | 3.17e-2        | 1.00  |
| 160| 6.70e-5        | 1.99  | 5.38e-5        | 2.00  | 1.20e-2        | 1.00  | 1.76e-2        | 1.00  | 1.59e-2        | 1.00  |
| 320| 1.70e-5        | 1.98  | 1.34e-5        | 2.00  | 6.04e-3        | 1.00  | 8.85e-3        | 1.00  | 7.93e-3        | 1.00  |
Table 4.4

Errors of $P_1-CR-P_0$ IFE solutions for Example 4.1 with $\mu^-=1$ and $\mu^+=10$.

| N  | $||e_{1h}||_{L^2(\Omega)}$ Rate | $||e_{2h}||_{L^2(\Omega)}$ Rate | $||e_{\rho h}||_{L^2(\Omega)}$ Rate | $||e_{1h}||_{H^1(\Omega)}$ Rate | $||e_{2h}||_{H^1(\Omega)}$ Rate |
|----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 10 | 8.07e-3 n/a                     | 1.09e-2 n/a                     | 2.30e-1 n/a                     | 2.27e-1 n/a                     | 2.71e-1 n/a                     |
| 20 | 1.83e-3 2.14                    | 2.70e-3 2.02                    | 1.11e-1 1.05                    | 1.14e-1 1.00                    | 1.36e-1 1.00                    |
| 40 | 6.32e-4 1.53                    | 7.15e-4 1.92                    | 6.21e-2 0.84                    | 5.80e-2 0.97                    | 6.84e-2 0.99                    |
| 80 | 1.14e-4 2.47                    | 1.70e-4 2.08                    | 2.75e-2 1.17                    | 2.85e-2 1.02                    | 3.40e-2 1.01                    |
| 160| 2.86e-5 1.99                    | 4.23e-5 2.00                    | 1.38e-2 1.00                    | 1.43e-2 1.00                    | 1.70e-2 1.00                    |
| 320| 7.39e-6 1.95                    | 1.06e-5 1.99                    | 6.90e-3 1.00                    | 7.14e-3 1.00                    | 8.52e-3 1.00                    |

4.5.2 Example 4.2: Curved Interface

In this example, we consider a circular interface problem which has been used in Example 1 in [2]. Let $\Omega = [-1,1]^2$ and the interface $\Gamma = \{ (x,y) : x^2+y^2 = 0.3 \}$. The circular interface separates the domain $\Omega$ into two subdomains $\Omega^- = \{ (x,y) : x^2+y^2 < 0.3 \}$ and $\Omega^+ = \{ (x,y) : x^2+y^2 > 0.3 \}$.

Let $\mu^-=1$ and $\mu^+=10$. The exact solutions $u = [u_1,u_2]^T$ and $p$ are chosen as follows:

\[
\begin{align*}
    u_1 &= \begin{cases} 
    \frac{y(x^2+y^2-0.3)}{\mu^+}, & \text{if } (x,y) \in \Omega^+, \\
    \frac{y(x^2+y^2-0.3)}{\mu^-}, & \text{if } (x,y) \in \Omega^-,
    \end{cases} \\
    u_2 &= \begin{cases} 
    -\frac{x(x^2+y^2-0.3)}{\mu^+}, & \text{if } (x,y) \in \Omega^+, \\
    -\frac{x(x^2+y^2-0.3)}{\mu^-}, & \text{if } (x,y) \in \Omega^-.
    \end{cases}
\end{align*}
\]

and

\[
p = \frac{1}{10}(x^3-y^3).
\]
Table 4.5

Errors of $CR-P_1-P_0$ IFE interpolation for Example 4.2 with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_1|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_2|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_1|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_2|_{H^1(\Omega)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.49e-3</td>
<td>n/a</td>
<td>7.27e-3</td>
<td>n/a</td>
<td>1.56e-1</td>
<td>n/a</td>
<td>9.22e-2</td>
<td>n/a</td>
<td>1.16e-1</td>
<td>n/a</td>
</tr>
<tr>
<td>20</td>
<td>9.20e-4</td>
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<td>1.99e-3</td>
<td>1.87</td>
<td>4.86e-2</td>
<td>1.68</td>
<td>4.65e-2</td>
<td>0.99</td>
<td>5.98e-2</td>
<td>0.95</td>
</tr>
<tr>
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<td>2.36e-4</td>
<td>1.96</td>
<td>5.15e-4</td>
<td>1.95</td>
<td>2.14e-2</td>
<td>1.18</td>
<td>2.37e-2</td>
<td>0.97</td>
<td>3.08e-2</td>
<td>0.96</td>
</tr>
<tr>
<td>80</td>
<td>5.96e-5</td>
<td>1.99</td>
<td>1.31e-4</td>
<td>1.98</td>
<td>8.21e-3</td>
<td>1.39</td>
<td>1.22e-2</td>
<td>0.96</td>
<td>1.57e-2</td>
<td>0.98</td>
</tr>
<tr>
<td>160</td>
<td>1.50e-5</td>
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<td>3.30e-5</td>
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<td>8.29e-6</td>
<td>1.99</td>
<td>9.99e-4</td>
<td>1.49</td>
<td>2.98e-3</td>
<td>1.01</td>
<td>3.90e-3</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 4.6

Errors of $P_1-CR-P_0$ IFE interpolation for Example 4.2 with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_1|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_2|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_1|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_2|_{H^1(\Omega)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7.27e-3</td>
<td>n/a</td>
<td>3.49e-3</td>
<td>n/a</td>
<td>1.56e-1</td>
<td>n/a</td>
<td>1.15e-1</td>
<td>n/a</td>
<td>9.24e-2</td>
<td>n/a</td>
</tr>
<tr>
<td>20</td>
<td>1.99e-3</td>
<td>1.87</td>
<td>9.20e-4</td>
<td>1.92</td>
<td>4.86e-2</td>
<td>1.68</td>
<td>5.97e-2</td>
<td>0.95</td>
<td>4.67e-2</td>
<td>0.98</td>
</tr>
<tr>
<td>40</td>
<td>5.15e-4</td>
<td>1.95</td>
<td>2.36e-4</td>
<td>1.96</td>
<td>2.14e-2</td>
<td>1.18</td>
<td>3.08e-2</td>
<td>0.96</td>
<td>2.37e-2</td>
<td>0.98</td>
</tr>
<tr>
<td>80</td>
<td>1.31e-4</td>
<td>1.98</td>
<td>5.96e-5</td>
<td>1.99</td>
<td>8.21e-3</td>
<td>1.39</td>
<td>1.57e-2</td>
<td>0.97</td>
<td>1.20e-2</td>
<td>0.98</td>
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<td>1.99</td>
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<td>3.75e-6</td>
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<td>9.99e-4</td>
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<td>3.91e-3</td>
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<td>2.96e-3</td>
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</table>
Table 4.7

Errors of $CR-P_1-P_0$ IFE solution for Example 4.2 with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_{1,I}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,I}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,I}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{1,I}|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,I}|_{H^1(\Omega)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
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<td>n/a</td>
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</table>

Table 4.8

Errors of $P_1-CR-P_0$ IFE solution for Example 4.2 with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>N</th>
<th>$|e_{1,I}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,I}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,I}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{1,I}|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,I}|_{H^1(\Omega)}$</th>
<th>Rate</th>
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<td>9.65e-3</td>
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</table>
4.5.3 Example 4.3: Flipped Coefficients and Large Coefficient Contrast

In this example, we consider the curved interface problem in Example 4.2 again with different jump ratios. This time we only report the IFE solution errors in consideration of the page limit. The convergence rates of the IFE interpolation are optimal as usual. In Tables 4.9 - 4.12 we report the error of IFE solutions for the case when the coefficient is flipped \((\mu^-, \mu^+) = (10, 1)\), and when the coefficient has a large jump \((\mu^-, \mu^+) = (1, 200)\), respectively.

Table 4.9

Errors of \(CR-P_1-P_0\) IFE solution for Example 4.3 with \(\mu^- = 10, \mu^+ = 1\).

<table>
<thead>
<tr>
<th>N</th>
<th>(| e_1 |_{L^2(\Omega)})</th>
<th>Rate</th>
<th>(| e_2 |_{L^2(\Omega)})</th>
<th>Rate</th>
<th>(| e_{p,i} |_{L^2(\Omega)})</th>
<th>Rate</th>
<th>(| e_{1,i} |_{H^1(\Omega)})</th>
<th>Rate</th>
<th>(| e_{2,i} |_{H^1(\Omega)})</th>
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<td>1.09e-3</td>
<td>2.05</td>
<td>9.06e-2</td>
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<td>8.94e-2</td>
<td>0.97</td>
<td>9.35e-2</td>
<td>1.00</td>
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<tr>
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<td>2.42e-2</td>
<td>1.90</td>
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<td>4.67e-2</td>
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<tr>
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<td>2.02</td>
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<td>2.20e-2</td>
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<td>1.75e-5</td>
<td>2.03</td>
<td>5.65e-3</td>
<td>1.04</td>
<td>1.10e-2</td>
<td>1.00</td>
<td>1.17e-2</td>
<td>1.00</td>
</tr>
</tbody>
</table>
Table 4.10

Errors of $P_1$-$CR$-$P_0$ IFE solution for Example 4.3 with $\mu^-=10$, $\mu^+=1$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_{1,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{1,l}|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,l}|_{H^1(\Omega)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.69e-2</td>
<td>n/a</td>
<td>1.50e-2</td>
<td>n/a</td>
<td>4.41e-1</td>
<td>n/a</td>
<td>2.94e-1</td>
<td>n/a</td>
<td>2.99e-1</td>
<td>n/a</td>
</tr>
<tr>
<td>20</td>
<td>4.25e-3</td>
<td>1.99</td>
<td>3.95e-3</td>
<td>1.92</td>
<td>1.21e-1</td>
<td>1.86</td>
<td>1.47e-1</td>
<td>1.00</td>
<td>1.51e-1</td>
<td>0.99</td>
</tr>
<tr>
<td>40</td>
<td>1.09e-3</td>
<td>1.97</td>
<td>1.00e-3</td>
<td>1.98</td>
<td>6.40e-2</td>
<td>0.92</td>
<td>7.36e-2</td>
<td>1.00</td>
<td>7.56e-2</td>
<td>1.00</td>
</tr>
<tr>
<td>80</td>
<td>2.68e-4</td>
<td>2.02</td>
<td>2.50e-4</td>
<td>2.00</td>
<td>1.98e-2</td>
<td>1.69</td>
<td>3.66e-2</td>
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<td>3.79e-2</td>
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</tr>
<tr>
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<td>6.56e-5</td>
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<td>0.98</td>
<td>1.83e-2</td>
<td>1.00</td>
<td>1.90e-2</td>
<td>0.99</td>
</tr>
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<td>1.82</td>
<td>5.16e-3</td>
<td>0.96</td>
<td>9.21e-3</td>
<td>0.99</td>
<td>9.65e-3</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Table 4.11

Errors of $CR$-$P_1$-$P_0$ IFE solution for Example 4.3 with $\mu^-=1$, $\mu^+=200$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$|e_{1,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{p,l}|_{L^2(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{1,l}|_{H^1(\Omega)}$</th>
<th>Rate</th>
<th>$|e_{2,l}|_{H^1(\Omega)}$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>1.20e-2</td>
<td>n/a</td>
<td>1.35e-2</td>
<td>n/a</td>
<td>9.18e-1</td>
<td>n/a</td>
<td>1.31e-1</td>
<td>n/a</td>
<td>1.22e-1</td>
<td>n/a</td>
</tr>
<tr>
<td>20</td>
<td>2.68e-3</td>
<td>2.36</td>
<td>3.12e-3</td>
<td>2.36</td>
<td>4.07e-1</td>
<td>0.37</td>
<td>6.03e-2</td>
<td>1.28</td>
<td>5.65e-2</td>
<td>1.28</td>
</tr>
<tr>
<td>40</td>
<td>6.17e-4</td>
<td>2.12</td>
<td>7.04e-4</td>
<td>2.15</td>
<td>2.14e-1</td>
<td>0.93</td>
<td>2.78e-2</td>
<td>1.11</td>
<td>2.85e-2</td>
<td>0.99</td>
</tr>
<tr>
<td>80</td>
<td>1.32e-4</td>
<td>2.22</td>
<td>1.53e-4</td>
<td>2.20</td>
<td>8.99e-2</td>
<td>1.25</td>
<td>1.39e-2</td>
<td>1.00</td>
<td>1.45e-2</td>
<td>0.97</td>
</tr>
<tr>
<td>160</td>
<td>2.92e-5</td>
<td>2.18</td>
<td>3.31e-5</td>
<td>2.21</td>
<td>3.94e-2</td>
<td>1.19</td>
<td>6.77e-3</td>
<td>1.04</td>
<td>7.27e-3</td>
<td>1.00</td>
</tr>
<tr>
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<td>6.90e-6</td>
<td>2.08</td>
<td>6.92e-6</td>
<td>2.26</td>
<td>1.90e-2</td>
<td>1.05</td>
<td>3.34e-3</td>
<td>1.02</td>
<td>3.68e-3</td>
<td>0.98</td>
</tr>
</tbody>
</table>
Table 4.12

Errors of $P_1$-$CR$-$P_0$ IFE solution for Example 4.3 with $\mu^- = 1$, $\mu^+ = 200$.

| N  | $||e_{1,I}||_{L^2(\Omega)}$ Rate | $||e_{2,I}||_{L^2(\Omega)}$ Rate | $||e_{P,I}||_{L^2(\Omega)}$ Rate | $||e_{1,I}||_{H^1(\Omega)}$ Rate | $||e_{2,I}||_{H^1(\Omega)}$ Rate |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 10  | 1.55e-2 n/a                     | 1.39e-2 n/a                     | 1.37e+0 n/a                     | 1.32e-1 n/a                     | 1.43e-1 n/a                     |
| 20  | 3.97e-3 1.97                     | 3.36e-3 2.05                     | 6.73e-1 1.03                     | 5.92e-2 1.16                     | 6.66e-2 1.10                     |
| 40  | 9.18e-4 2.11                     | 7.67e-4 2.13                     | 3.92e-1 0.78                     | 2.90e-2 1.03                     | 2.90e-2 1.20                     |
| 80  | 1.86e-4 2.30                     | 1.55e-4 2.30                     | 1.56e-1 1.33                     | 1.47e-2 0.98                     | 1.42e-2 1.03                     |
| 160 | 4.00e-5 2.22                     | 3.31e-5 2.23                     | 6.69e-2 1.22                     | 7.37e-3 1.00                     | 6.83e-3 1.05                     |
| 320 | 8.12e-6 2.30                     | 7.43e-6 2.16                     | 3.07e-2 1.12                     | 3.72e-3 0.99                     | 3.36e-3 1.02                     |
CHAPTER V
IMMERSED FINITE ELEMENT METHODS FOR UNSTEADY STOKES PROBLEMS WITH MOVING INTERFACES

In this chapter, we construct a class of immersed finite element methods to solve the Stokes problem with a moving interface. We compare each class of IFE methods presented in the past two chapters with various interface configurations. Based on the new IFE spaces, semi-discrete and full-discrete schemes are developed for solving the unsteady Stokes equations with a stationary or a moving interface. Since the immersed finite element method uses a mesh independent of the interface location, a structured Cartesian mesh is used throughout the simulation. Numerical experiments are carried out to demonstrate the performance of these new IFE methods.

5.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain separated by a time-dependent smooth interface $\Gamma(t)$. The evolving interface $\Gamma(t)$ divides the domain $\Omega$ into two open subdomains $\Omega^+(t)$ and $\Omega^-(t)$
such that $\Omega = \Omega^+(t) \cup \Omega^-(t) \cup \Gamma(t)$, see Figure 5.1. Consider the following initial-boundary-value problems of the Stokes equations

\[
\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (\mu \nabla \mathbf{u} - p\mathbf{I}) = \mathbf{f} \quad \text{in } \Omega \times [0, T],
\]

(5.1)

\[
\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times [0, T],
\]

(5.2)

\[
\mathbf{u} = 0 \quad \text{on } \partial \Omega \times [0, T],
\]

(5.3)

\[
\mathbf{u}(x, 0) = \mathbf{u}_0, \quad p(x, 0) = p_0 \quad \text{on } \Omega,
\]

(5.4)

where $\mathbf{u}$ and $p$ denote the flow velocity and the pressure, respectively. The function $\mathbf{f}$ is given on $[0, T] \times \Omega$. $\mathbf{u}_0$ and $p_0$ are given initial velocity and pressure. $\mathbf{I}$ denotes the identity tensor. The movement of the interface is assumed to be guided by a given velocity field $v(x, t)$ as follows

\[
\frac{dx}{dt} = v(x, t), \quad \text{on } \Gamma(t) \times [0, T].
\]

(5.5)

The viscosity function $\mu(x)$ is assumed to have a finite jump across the interface $\Gamma(t)$. For simplicity, we assume that $\mu(x)$ is a piecewise constant function

\[
\mu(x) = \begin{cases} 
\mu^- \text{ in } \Omega^-, \\
\mu^+ \text{ in } \Omega^+,
\end{cases}
\]

(5.6)

where $\mu^\pm > 0$ and $x = (x, y)$. At any time $t$, the velocity and the stress tensors satisfy the following homogeneous interface jump condition

\[
[u]_\Gamma = 0,
\]

(5.7)

\[
[(\mu \nabla \mathbf{u} - p\mathbf{I})\mathbf{n}]_\Gamma = 0,
\]

(5.8)

where the jump $[v(x)]_\Gamma := v^+(x)|_\Gamma - v^-(x)|_\Gamma$, and $\mathbf{n}$ denotes the unit normal vector to the interface $\Gamma$ pointing from $\Omega^-(t)$ to $\Omega^+(t)$. 

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The geometrical setup of a moving interface problem.

5.2 Semi-discrete and Full-discrete Schemes

In this section, we first derive the weak formulation of the unsteady Stokes interface problem (5.1)-(5.8), and develop the semi-discrete and full-discrete IFE schemes.

5.2.1 Weak Formulation

Taking the inner product with \( v \in [H^1_0(\Omega)]^2 \) on the equation (5.1) and integrating by parts over \( \Omega^- \) yields,

\[
(u_t, v)_{\Omega^-} + (\mu \nabla u - p I, \nabla v)_{\Omega^-} - ((\mu \nabla u - p I) \mathbf{n}_{\partial \Omega^-}, v)_{\partial \Omega^-} = (f, v)_{\Omega^-}. \tag{5.9}
\]

Here the second term is the inner product of two tensors \( A = [A_{ij}] \) and \( B = [B_{ij}] \), which is defined by \( (A, B) = \sum_{i,j} (A_{ij}, B_{ij}) \). Note that \( \mathbf{n}_\Gamma \) is pointing from \( \Omega^- \) to \( \Omega^+ \) and \( v \) vanishes on the outer boundary \( \partial \Omega \). We have

\[
(u_t, v)_{\Omega^-} + (\mu \nabla u - p I, \nabla v)_{\Omega^-} - ((\mu \nabla u - p I) \mathbf{n}_{\Gamma}, v)_{\Gamma} = (f, v)_{\Omega^-}. \tag{5.10}
\]
Similar argument applying to the subdomain $\Omega^+$ yields

$$(u, v)_{\Omega^+} + (\mu \nabla u - p I, \nabla v)_{\Omega^+} + ((\mu \nabla u - p I) n_{\Gamma}, v)_{\Gamma} = (f, v)_{\Omega^+}. \quad (5.11)$$

Adding the two equations above, and applying the interface jump condition (5.8), we have

$$(u, v) + (\mu \nabla u, \nabla v) - (p, \nabla \cdot v) = (f, v). \quad (5.12)$$

Multiplying $q \in L^2(\Omega)$ to (5.2), and integrating by parts we have

$$(q, \nabla \cdot u) = 0. \quad (5.13)$$

Define the bilinear form and the linear form

$$a(w, v) = (\mu \nabla w, \nabla v), \quad \forall w, v \in [H^1_0(\Omega)]^2, \quad (5.14)$$
$$b(v, q) = -(q, \nabla \cdot v), \quad \forall v \in [H^1_0(\Omega)]^2, \quad \forall q \in L^2_0(\Omega). \quad (5.15)$$

Here, $L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q d\mathbf{x} = 0\}$. The weak form of the unsteady Stokes interface problem (5.1)-(5.8) is given as follows.

**Weak Form:** Find $u \in H^1(0, T; [H^1_0(\Omega)]^2)$ and $p \in L^2(0, T; L^2_0(\Omega))$ such that for each $t \in [0, T]$

$$(u_t, v) + a(u, v) + b(v, p) = (f, v), \quad \forall v \in [H^1_0(\Omega)]^2, \quad (5.16)$$
$$b(u, q) = 0, \quad \forall q \in L^2_0(\Omega), \quad (5.17)$$

and subject to the initial conditions $u(x, 0) = u_0(x), p(x, 0) = p_0(x)$.

### 5.2.2 Semi-Discrete Scheme

For semi-discretization in space, we use the finite element space $S_h(\mathcal{T}_h)$ to approximate to approximate $[H^1_0(\Omega)]^2 \times L^2(\Omega)$. We write the vector-valued IFE space $S_h(\mathcal{T}_h) = U_{1h} \times U_{2h} \times \mathcal{W}_h$. 100
Then we propose the semi-discrete scheme as follows.

**Semi-discrete IFE Scheme**: 

Find \( (u_h, p_h) = (u_{1h}, u_{2h}, p_h) \) \( \in H^1(0, T; U_{1h}) \times H^1(0, T; U_{2h}) \times L^2(0, T; W_h) \) such that

\[
\begin{align*}
\left( \partial_t u_h, v_h \right) + a(u_h, v_h) + b(v_h, p_h) &= \left( f_h, v_h \right), \quad \forall v_h \in U_{1h} \times U_{2h} \quad (5.18) \\
b(u_h, q_h) &= 0, \quad \forall q_h \in W_h \quad (5.19)
\end{align*}
\]

and subject to the initial conditions

\[
\begin{align*}
u_h(x, 0) &= u_{0,h}(x), \quad p(x, 0) = p_{0,h}(x). \quad (5.20)
\end{align*}
\]

\( u_{0,h} \) and \( p_{0,h} \) are some approximations (e.g. the interpolation) of \( u_0 \) and \( p_0 \) in \( U_{1h} \times U_{2h} \) and \( W_h \). We rewrite the semi-discrete scheme in the following matrix form.

**Matrix Form**: Find the vector function \( \mathbf{U}(t) \) such that

\[
\begin{align*}
M(t)\mathbf{U}'(t) + A(t)\mathbf{U}(t) &= \mathbf{F}(t), \quad (5.21) \\
\mathbf{U}(0) &= \mathbf{U}^0, \quad (5.22)
\end{align*}
\]

where \( M(t) \) and \( A(t) \) denote the IFE mass and stiffness matrices, and \( \mathbf{F}(t) \) is the vector corresponding to the right-hand side of (5.18)-(5.19). The initial vector \( \mathbf{U}^0 \) takes the values of the coefficients of the interpolation \( \mathbf{I}_h(u_0, p_0) \). More details will be given in the next sub-section.

Since the interface \( \Gamma(t) \) is a function of time \( t \), the IFE spaces \( S_h(\mathcal{T}_h) = U_{1h} \times U_{2h} \times W_h \) depend on the interface location, hence are time-dependent. Although the background mesh \( \mathcal{T}_h \) is time-independent, the collections of interface elements \( \mathcal{T}_h^{f(t)} \) and non-interface elements \( \mathcal{T}_h^{n(t)} \) vary by time. That is why the mass matrix \( M(t) \) and stiffness matrix \( A(t) \) are both time-dependent.
5.2.3 Full-Discrete Scheme

Let \( 0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T \) be a partition of the time interval \([0, T]\) with the uniform step size \( \tau \), i.e., \( \tau = T / N \), and \( t_n = n \tau \). Evaluating (5.21) at \( t = t_{n+\theta} = t_n + \theta \Delta t \), and using the following finite-difference approximations

\[
M(t_{n+\theta}) U'(t_{n+\theta}) \approx M(t_{n+\theta}) \frac{U(t_{n+1}) - U(t_n)}{\tau} \approx \frac{1}{\tau} \left( M(t_{n+1}) U(t_{n+1}) - M(t_n) U(t_n) \right).
\]

\[ (5.23) \]

\[
A(t_{n+\theta}) U(t_{n+\theta}) \approx (1 - \theta) A(t_n) U(t_n) + \theta A(t_{n+1}) U(t_{n+1}).
\]

\[ (5.24) \]

\[
F(t_{n+\theta}) \approx (1 - \theta) F(t_n) + \theta F(t_{n+1}).
\]

\[ (5.25) \]

We can obtain the following full-discrete IFE scheme.

**Full-discrete IFE Scheme:** Given the initial vector \( U^0 \), find \( U^{n+1} \) for each \( n = 0, 1, \cdots, N - 1 \) in

\[
\left( \frac{1}{\tau} M^{n+1} + \theta A^{n+1} \right) U^{n+1} = \left( \frac{1}{\tau} M^n - (1 - \theta) A^n \right) U^n + (1 - \theta) F^n + \theta F^{n+1}.
\]

\[ (5.26) \]

Note that when \( \theta = 1 \), the method becomes the **Backward-Euler method**:

\[
\left( \frac{1}{\tau} M^{n+1} + A^{n+1} \right) U^{n+1} = \frac{1}{\tau} M^n U^n + F^{n+1}.
\]

\[ (5.27) \]

When \( \theta = \frac{1}{2} \), the method is the **Crank-Nicolson method**:

\[
\left( \frac{1}{\tau} M^{n+1} + \frac{1}{2} A^{n+1} \right) U^{n+1} = \left( \frac{1}{\tau} M^n - \frac{1}{2} A^n \right) U^n + \frac{1}{2} (F^n + F^{n+1}).
\]

\[ (5.28) \]

For the time-dependent Stokes interface problem with a stationary interface, i.e., \( \Gamma \) is time-independent, the matrices \( M \) and \( A \) in the full-discrete scheme (5.26) will remain unchanged as time evolves. At each time level, only the vector \( F^n \) needs to be updated.
For the time-dependent Stokes interface problem with a moving interface, although the matrices $M^n$ and $A^n$ depend on the interface location, which further depends on time, these matrices are efficiently generated by locally modifying the matrices from the previous time step. A unique feature of IFEM is that the computational mesh and the number of and the location of the degrees of freedom remain unchanged. In two consecutive time steps, only the yellow elements change their interface configurations, as shown in Figure 5.2. Consequently, we only need to modify local stiffness and mass matrices on those elements. Majority of the global matrices remain unchanged. This feature is important in the analysis of moving interface problems, see [29].

Figure 5.2

Interface moves in two consecutive steps. Elements in dark yellow indicate interface configuration changes, and elements in dark blue remain unchanged.
5.3 Numerical Examples

In this section, we test the accuracy and the convergence of each class of IFE methods for the time dependent Stokes interface problem through a series of numerical experiments. We will consider the accuracy of the IFE solution with various configurations of the interface and coefficient jumps. The time interval is set to be $[0, 1]$, and it is partition uniformly to $N_t$ subintervals. We use both Backward-Euler and Crank-Nicolson schemes with the step size $\tau = 2h$. The errors are measured at the final time $t = 1$:

$$e^0(u_{1h}) = \|u_1(\cdot, 1) - u_{1h}(\cdot, 1)\|_{L^2(\Omega)}, \quad e^0(u_{2h}) = \|u_2(\cdot, 1) - u_{2h}(\cdot, 1)\|_{L^2(\Omega)},$$ (5.29)

$$e^0(p_h) = \|p(\cdot, 1) - p_h(\cdot, 1)\|_{L^2(\Omega)},$$ (5.30)

$$e^1(u_{1h}) = |u_1(\cdot, 1) - u_{1h}(\cdot, 1)|_{H^1(\Omega)}, \quad e^1(u_{2h}) = |u_2(\cdot, 1) - u_{2h}(\cdot, 1)|_{H^1(\Omega)}.$$ (5.31)

In the tables below, we report the convergence rate based on two consecutive meshes $\mathcal{T}_h$ and $\mathcal{T}_{h/2}$, as well as the overall convergence rate among all meshes using the linear regression.

5.3.1 Example 5.1: Unsteady Stokes Equation with Fixed Interface

In this example, we consider a time-dependent Stokes equation with a fixed interface. Consider the domain $\Omega = [-1, 1]^2$ which is cut by an interface $\Gamma = \{(x, y) : x^2 + y^2 = 0.3\}$. The circular interface separates the domain $\Omega$ into two regions $\Omega^- = \{(x, y) : x^2 + y^2 < 0.3\}$ and $\Omega^+ = \{(x, y) : x^2 + y^2 > 0.3\}$. 

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The initial data $u_0$, $p_0$, the boundary condition, and the source term $f$ are chosen so that the exact solutions of this problem are as follows:

$$u(x, y, t) = \begin{cases} u_1 = \frac{y(x^2+y^2-0.3)}{\mu t} e^{mt}, & \text{if } (x, y) \in \Omega^+, \\ \frac{y(x^2+y^2-0.3)}{\mu t} e^{mt}, & \text{if } (x, y) \in \Omega^-, \\ u_2 = \frac{-x(x^2+y^2-0.3)}{\mu t} e^{mt}, & \text{if } (x, y) \in \Omega^+, \\ \frac{-x(x^2+y^2-0.3)}{\mu t} e^{mt}, & \text{if } (x, y) \in \Omega^- \end{cases}$$

(5.32)

$$p(x, y) = \frac{1}{10}(x^3 - y^3).$$

(5.33)

Tables 5.1 - 5.6 report the Backward-Euler and the Crank-Nicolson IFE solutions at the final time $t = 1$, respectively. The numerical results indicate that the errors of Crank-Nicolson are a little smaller than those of Backward-Euler. They obey the expected convergence rates

$$e^0(u_{ih}) \approx O(h^2 + \tau^k), \quad e^1(u_{ih}) \approx O(h + \tau^k), \quad e^0(p_h) \approx O(h + \tau^k),$$

(5.34)

where $i = 1, 2$, and $k = 1$ for Backward-Euler, and $k = 2$ for Crank-Nicolson. We only report the $P_1-CR-P_0$ IFE solutions, and the results for the $CR-P_1-P_0$ IFE solution are similar. In addition, we report $CR-P_0$ and $RQ_1-Q_0$ IFE solutions.

5.3.2 Example 5.2: Unsteady Stokes Equation with Moving Interface

In this example, we test a Stokes moving interface problem. The interface curve is a circle centered at the origin with a varying radius. The function for the interface curve is given as

$$\Gamma(x, y, t) = x^2 + y^2 - 0.3 \left( \frac{1}{2} \sin(2\pi t) + 1 \right).$$
Table 5.1

$P_1$-$CR$-$P_0$ Backward-Euler IFE solutions for Example 5.1 at $t = 1$ with $\mu^- = 1$, $\mu^+ = 10$, and $m = 1$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$e^0(u_{1h})$</th>
<th>Rate</th>
<th>$e^0(u_{2h})$</th>
<th>Rate</th>
<th>$e^0(p_h)$</th>
<th>Rate</th>
<th>$e^1(u_{1h})$</th>
<th>Rate</th>
<th>$e^1(u_{2h})$</th>
<th>Rate</th>
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<tbody>
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<td>n/a</td>
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<td>3.98e-01</td>
<td>n/a</td>
<td>3.82e-01</td>
<td>n/a</td>
</tr>
<tr>
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<td>1.01e-02</td>
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<td>1.79</td>
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<td>2.04e-01</td>
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<td>1.87e-01</td>
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<td>1.92e-03</td>
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Table 5.2

$P_1$-CR-$P_0$ Crank-Nicolson IFE solutions for Example 5.1 at $t = 1$ with $\mu^- = 1$, $\mu^+ = 10$, and $m = 1$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$e^0(u_{1h})$</th>
<th>Rate</th>
<th>$e^0(u_{2h})$</th>
<th>Rate</th>
<th>$e^0(p_h)$</th>
<th>Rate</th>
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<th>Rate</th>
<th>$e^1(u_{2h})$</th>
<th>Rate</th>
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Table 5.3

$CR-P_0$ Backward-Euler IFE solutions for Example 5.1 at $t = 1$ with $\mu^- = 1$, $\mu^+ = 10$, and $m = 1$.

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<tr>
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<th>$e^0(p_h)$</th>
<th>Rate</th>
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<th>Rate</th>
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<th>Rate</th>
<th>$e^1(u_{2h})$</th>
<th>Rate</th>
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</table>
### Table 5.4

**CR-P₀ Crank-Nicolson IFE solutions for Example 5.1 at \( t = 1 \) with \( \mu^- = 1, \mu^+ = 10, \) and \( m = 1. \)**

<table>
<thead>
<tr>
<th>( N_s )</th>
<th>( e^0(u_{1h}) ) Rate</th>
<th>( e^0(u_{2h}) ) Rate</th>
<th>( e^0(p_h) ) Rate</th>
<th>( e^1(u_{1h}) ) Rate</th>
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<td>8.37e-02 0.99</td>
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<td>1.08e-02 1.00</td>
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</table>

### Table 5.5

**RQ₁-Q₀ Backward-Euler IFE solutions for Example 5.1 at \( t = 1 \) with \( \mu^- = 1, \mu^+ = 10, \) and \( m = 1. \)**

<table>
<thead>
<tr>
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<th>( e^0(u_{1h}) ) Rate</th>
<th>( e^0(u_{2h}) ) Rate</th>
<th>( e^0(p_h) ) Rate</th>
<th>( e^1(u_{1h}) ) Rate</th>
<th>( e^1(u_{2h}) ) Rate</th>
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<td>3.82e-01 n/a</td>
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<td>1.01e-02 1.78</td>
<td>7.08e-03 1.79</td>
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<td>1.87e-01 1.03</td>
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<tr>
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<td>1.92e-03 1.88</td>
<td>3.28e-01 0.95</td>
<td>1.03e-01 0.98</td>
<td>9.63e-02 0.96</td>
</tr>
<tr>
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<td>5.80e-04 2.19</td>
<td>4.37e-04 2.13</td>
<td>1.55e-01 1.08</td>
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Table 5.6

$RQ_1-Q_0$ Crank-Nicolson IFE solutions for Example 5.1 at $t = 1$ with $\mu^- = 1$, $\mu^+ = 10$, and $m = 1$.

<table>
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<th>$e^0(u_{1h})$ Rate</th>
<th>$e^0(u_{2h})$ Rate</th>
<th>$e^0(p_h)$ Rate</th>
<th>$e^1(u_{1h})$ Rate</th>
<th>$e^1(u_{2h})$ Rate</th>
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<td>6.27e-03 1.69</td>
<td>4.44e-01 1.01</td>
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Table 5.7

$P_1-CR-P_0$ Backward-Euler IFE solutions for Example 5.1 at $t = 1$ with $\mu^- = 1$, $\mu^+ = 10$, and $m = 3$.

<table>
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<th>$e^0(p_h)$ Rate</th>
<th>$e^1(u_{1h})$ Rate</th>
<th>$e^1(u_{2h})$ Rate</th>
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Table 5.8

$P_1$-$CR$-$P_0$ Crank-Nicolson IFE solutions for Example 5.1 at $t = 1$ with $\mu^- = 1$, $\mu^+ = 10$, and $m = 3$.

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<th>$e^0(u_{2h})$</th>
<th>Rate</th>
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<th>Rate</th>
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<td>7.94e-04</td>
<td>2.04</td>
<td>5.97e-01</td>
<td>0.89</td>
<td>2.03e-01</td>
<td>0.95</td>
<td>1.87e-01</td>
<td>0.95</td>
</tr>
<tr>
<td>256</td>
<td>2.39e-04</td>
<td>2.06</td>
<td>2.33e-04</td>
<td>1.76</td>
<td>3.20e-01</td>
<td>0.90</td>
<td>1.06e-01</td>
<td>0.93</td>
<td>1.02e-01</td>
<td>0.91</td>
</tr>
<tr>
<td>overall</td>
<td>2.03</td>
<td>1.93</td>
<td>0.97</td>
<td></td>
<td>0.96</td>
<td></td>
<td>0.96</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
It can be seen that at time $t = 0$, the interface is the same as Example 1 with a radius of $r = 0.54772$. As the time $t$ increases, the radius will first increase, then decrease, and finally return to the original one. The maximum and minimum radius $r_{\text{max}} = 0.67082$ and $r_{\text{min}} = 0.3873$ occur at $t = 0.25$ and $t = 0.75$, as shown in Figure 5.3. The exact solution $u$ is written in terms of the level-set interface function $\Gamma$:

$$
\begin{align*}
    u(x, y, t) &= \begin{cases} 
    \frac{1}{\mu^+} y \Gamma(x, y, t), & \text{if } (x, y) \in \Omega^+(t), \\
    \frac{1}{\mu^+} y \Gamma(x, y, t), & \text{if } (x, y) \in \Omega^-(t), \\
    \frac{1}{\mu^-} x \Gamma(x, y, t), & \text{if } (x, y) \in \Omega^+(t), \\
    \frac{1}{\mu^-} x \Gamma(x, y, t), & \text{if } (x, y) \in \Omega^-(t),
\end{cases}
\end{align*}
$$

(5.35)

$$
p(x, y) = \frac{1}{10} (x^3 - y^3).
$$

(5.36)

In this experiment, we set the time step size $\tau = h$. We first test the moderate jump case for this moving interface problem. Tables 5.9 - 5.11 reports the errors at the final time level of the Backward-Euler IFE solutions. The error decay is observed to converge in an optimal order, as stated in (5.34). Figure 5.3 shows the IFE solution $u_1$ and $u_2$ at time $t = 0.25$, $t = 0.75$, and $t = 1$, respectively, on the $64 \times 64$ mesh. For a larger jump case, the errors are reported in Tables 5.12 - 5.14.

The condition numbers of the IFE systems are reported in Tables 5.15 and 5.16. We monitor the condition numbers at $t = 0.25$, $t = 0.75$, and $t = 1$ which correspond to the interface circle listed in Figure 5.3. We test different contrast ratios by fixing the coefficient $\mu^- = 1$ and varying the other coefficient $\mu^+ = 0.01$, 0.1, 1, 10, and 100. Note that when $\mu^+ = 1$, there is no jump in
Table 5.9

CR-P₁-P₀ Backward-Euler IFE solution for Example 5.2 at $t = 1$ with $\mu^- = 1$, $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$e^0(u_{1h})$</th>
<th>Rate</th>
<th>$e^0(u_{2h})$</th>
<th>Rate</th>
<th>$e^0(p_h)$</th>
<th>Rate</th>
<th>$e^1(u_{1h})$</th>
<th>Rate</th>
<th>$e^1(u_{2h})$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>7.85e-03</td>
<td>n/a</td>
<td>1.14e-02</td>
<td>n/a</td>
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<td>1.36e-01</td>
<td>n/a</td>
<td>1.51e-01</td>
<td>n/a</td>
</tr>
<tr>
<td>16</td>
<td>2.05e-03</td>
<td>1.94</td>
<td>2.95e-03</td>
<td>1.95</td>
<td>2.41e-01</td>
<td>1.00</td>
<td>7.02e-02</td>
<td>0.95</td>
<td>7.45e-02</td>
<td>1.02</td>
</tr>
<tr>
<td>32</td>
<td>5.13e-04</td>
<td>2.00</td>
<td>6.54e-04</td>
<td>2.17</td>
<td>1.24e-01</td>
<td>0.96</td>
<td>3.57e-02</td>
<td>0.98</td>
<td>3.82e-02</td>
<td>0.96</td>
</tr>
<tr>
<td>64</td>
<td>1.68e-04</td>
<td>1.61</td>
<td>1.32e-04</td>
<td>2.30</td>
<td>5.78e-02</td>
<td>1.10</td>
<td>1.84e-02</td>
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<td>1.96e-02</td>
<td>0.96</td>
</tr>
<tr>
<td>128</td>
<td>8.54e-05</td>
<td>0.98</td>
<td>6.68e-05</td>
<td>0.99</td>
<td>3.12e-02</td>
<td>0.89</td>
<td>9.52e-03</td>
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<td>1.01e-02</td>
<td>0.95</td>
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<tr>
<td>overall</td>
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<td>0.96</td>
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<td>0.97</td>
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Table 5.10

CR-P₀ Backward-Euler IFE solution for Example 5.2 at $t = 1$ with $\mu^- = 1$ and $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$e^0(u_{1h})$</th>
<th>Rate</th>
<th>$e^0(u_{2h})$</th>
<th>Rate</th>
<th>$e^0(p_h)$</th>
<th>Rate</th>
<th>$e^1(u_{1h})$</th>
<th>Rate</th>
<th>$e^1(u_{2h})$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
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<td>8</td>
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<td>1.67e-01</td>
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<td>1.68e-01</td>
<td>n/a</td>
</tr>
<tr>
<td>16</td>
<td>2.40e-03</td>
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<td>2.40e-03</td>
<td>1.81</td>
<td>5.03e-02</td>
<td>1.27</td>
<td>9.01e-02</td>
<td>0.89</td>
<td>8.96e-02</td>
<td>0.90</td>
</tr>
<tr>
<td>32</td>
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<td>1.91</td>
<td>2.08e-02</td>
<td>1.27</td>
<td>4.61e-02</td>
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<td>4.61e-02</td>
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</tr>
<tr>
<td>64</td>
<td>1.68e-04</td>
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<td>1.69e-04</td>
<td>1.92</td>
<td>1.02e-02</td>
<td>1.03</td>
<td>2.33e-02</td>
<td>0.99</td>
<td>2.33e-02</td>
<td>0.99</td>
</tr>
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<td>1.85</td>
<td>4.73e-03</td>
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<td>1.18e-02</td>
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<td>1.18e-02</td>
<td>0.99</td>
</tr>
<tr>
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<td>1.88</td>
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<td>1.88</td>
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<td>1.17</td>
<td></td>
<td>0.96</td>
<td></td>
<td>0.96</td>
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</tr>
</tbody>
</table>
### Table 5.11

$RQ_1 - Q_0$ Backward-Euler IFE solution for Example 5.2 at $t = 1$ with $\mu^- = 1$ and $\mu^+ = 10$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$e^0(u_{1h})$</th>
<th>Rate</th>
<th>$e^0(u_{2h})$</th>
<th>Rate</th>
<th>$e^0(p_h)$</th>
<th>Rate</th>
<th>$e^1(u_{1h})$</th>
<th>Rate</th>
<th>$e^1(u_{2h})$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
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<td>1.38e-02</td>
<td>n/a</td>
<td>9.96e-02</td>
<td>n/a</td>
<td>2.96e-01</td>
<td>n/a</td>
<td>2.92e-01</td>
<td>n/a</td>
</tr>
<tr>
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<td>3.97e-03</td>
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<td>3.97e-03</td>
<td>1.80</td>
<td>1.96e-02</td>
<td>2.35</td>
<td>1.92e-01</td>
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<td>1.92e-01</td>
<td>0.61</td>
</tr>
<tr>
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<td>1.10e-03</td>
<td>1.85</td>
<td>1.10e-03</td>
<td>1.85</td>
<td>9.91e-03</td>
<td>0.98</td>
<td>1.22e-01</td>
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<td>1.22e-01</td>
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</tr>
<tr>
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<td>3.51e-04</td>
<td>1.65</td>
<td>4.15e-03</td>
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<td>8.09e-02</td>
<td>0.59</td>
</tr>
<tr>
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<td>1.70</td>
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<td>1.14</td>
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<td>5.55e-02</td>
<td>0.54</td>
</tr>
<tr>
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<td></td>
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<td></td>
<td>175</td>
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<td>1.37</td>
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</tr>
</tbody>
</table>

### Table 5.12

$CR-P_1-P_0$ Backward-Euler IFE solution for Example 5.2 at $t = 1$ with $\mu^- = 1$ and $\mu^+ = 200$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$e^0(u_{1h})$</th>
<th>Rate</th>
<th>$e^0(u_{2h})$</th>
<th>Rate</th>
<th>$e^0(p_h)$</th>
<th>Rate</th>
<th>$e^1(u_{1h})$</th>
<th>Rate</th>
<th>$e^1(u_{2h})$</th>
<th>Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.17e-02</td>
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<td>1.29e-02</td>
<td>n/a</td>
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<td>n/a</td>
<td>1.44e-01</td>
<td>n/a</td>
<td>1.41e-01</td>
<td>n/a</td>
</tr>
<tr>
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<td>3.86e-03</td>
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<td>4.56e-03</td>
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<td>8.16e-01</td>
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<td>7.99e-02</td>
<td>0.85</td>
<td>7.01e-01</td>
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</tr>
<tr>
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<td>1.20e-03</td>
<td>1.69</td>
<td>1.42e-03</td>
<td>1.69</td>
<td>5.10e-01</td>
<td>0.68</td>
<td>3.80e-02</td>
<td>1.07</td>
<td>3.56e-02</td>
<td>0.98</td>
</tr>
<tr>
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<td>2.02e-04</td>
<td>2.57</td>
<td>2.50e-04</td>
<td>2.50</td>
<td>2.00e-01</td>
<td>1.35</td>
<td>1.74e-02</td>
<td>1.12</td>
<td>1.78e-02</td>
<td>1.00</td>
</tr>
<tr>
<td>128</td>
<td>3.48e-05</td>
<td>2.54</td>
<td>4.21e-05</td>
<td>2.57</td>
<td>8.70e-02</td>
<td>1.20</td>
<td>8.43e-03</td>
<td>1.05</td>
<td>9.01e-03</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>overall</td>
<td></td>
<td></td>
<td></td>
<td>2.10</td>
<td></td>
<td>2.07</td>
<td></td>
<td>0.97</td>
<td></td>
</tr>
</tbody>
</table>

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### Table 5.13

*CR-P₀ Backward-Euler IFE solution for Example 5.2 at \( t = 1 \) with \( \mu^- = 1 \) and \( \mu^+ = 200 \).*

<table>
<thead>
<tr>
<th>( N_s )</th>
<th>( e^0(u_{1h}) ) Rate</th>
<th>( e^0(u_{2h}) ) Rate</th>
<th>( e^0(p_h) ) Rate</th>
<th>( e^1(u_{1h}) ) Rate</th>
<th>( e^1(u_{2h}) ) Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>1.35e-02 n/a</td>
<td>1.35e-02 n/a</td>
<td>4.92e-01 n/a</td>
<td>1.66e-01 n/a</td>
<td>1.66e-01 n/a</td>
</tr>
<tr>
<td>16</td>
<td>2.78e-03 2.28</td>
<td>2.78e-03 2.28</td>
<td>2.96e-01 0.73</td>
<td>8.53e-02 0.96</td>
<td>8.46e-02 0.98</td>
</tr>
<tr>
<td>32</td>
<td>6.94e-04 2.00</td>
<td>6.94e-04 2.00</td>
<td>1.14e-01 1.38</td>
<td>4.39e-02 0.96</td>
<td>4.39e-02 0.94</td>
</tr>
<tr>
<td>64</td>
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<td>1.79e-04 1.96</td>
<td>5.77e-02 0.99</td>
<td>2.23e-02 0.98</td>
<td>2.23e-02 0.98</td>
</tr>
<tr>
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<td>4.69e-05 1.93</td>
<td>2.16e-02 1.42</td>
<td>1.13e-02 0.98</td>
<td>1.13e-02 0.98</td>
</tr>
<tr>
<td>overall</td>
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<td>2.03</td>
<td>1.14</td>
<td>0.97</td>
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</tr>
</tbody>
</table>

### Table 5.14

*RQ₁-Q₀ Backward-Euler IFE solution for Example 5.2 at \( t = 1 \) with \( \mu^- = 1 \) and \( \mu^+ = 200 \).*

<table>
<thead>
<tr>
<th>( N_s )</th>
<th>( e^0(u_{1h}) ) Rate</th>
<th>( e^0(u_{2h}) ) Rate</th>
<th>( e^0(p_h) ) Rate</th>
<th>( e^1(u_{1h}) ) Rate</th>
<th>( e^1(u_{2h}) ) Rate</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2.23e-02 n/a</td>
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<td>2.94e-01 n/a</td>
<td>2.95e-01 n/a</td>
</tr>
<tr>
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<td>4.84e-03 2.20</td>
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<tr>
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<td>1.32e-03 1.88</td>
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<td>9.92e-02 0.58</td>
</tr>
<tr>
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<td>1.23e-04 1.78</td>
<td>1.44e-02 1.55</td>
<td>6.87e-02 0.53</td>
<td>6.89e-02 0.53</td>
</tr>
<tr>
<td>overall</td>
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<td>1.85</td>
<td>1.40</td>
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<td>0.55</td>
</tr>
</tbody>
</table>

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$CR-P_1-P_0$ IFE Solution of Example 5.2 with $\mu^- = 1$ and $\mu^+ = 10$ on the $64 \times 64$ mesh at times $t = 0.25, 0.75,$ and $1$. Top plots: Interfaces, middle: IFE solutions $u_{1h}$, bottom: IFE solutions $u_{2h}$.

coefficient, hence the IFE scheme becomes the standard FE scheme. Even in this no-jump case, we observe that the condition number is of the order $O(h^{-4})$. We also observe that the condition number increases as the jump ratio enlarges. No significant differences have been noticed for Backward-Euler and Crank-Nicolson in terms of the condition numbers.
Table 5.15

Condition Number for Backward-Euler $CR-P_1-P_0$ Example 5.2 with $\mu^- = 1$.

<table>
<thead>
<tr>
<th>$N_s$</th>
<th>$\mu^+ = 0.01$</th>
<th>$\mu^+ = 0.1$</th>
<th>$\mu^+ = 1$</th>
<th>$\mu^+ = 10$</th>
<th>$\mu^+ = 100$</th>
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</thead>
<tbody>
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<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>3.03e+05</td>
<td>5.94e+04</td>
<td>2.80e+05</td>
<td>1.38e+07</td>
<td>1.31e+09</td>
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<tr>
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<td>1.04e+06</td>
<td>7.82e+05</td>
<td>4.36e+06</td>
<td>1.11e+08</td>
<td>1.40e+10</td>
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<td>$t = 0.25$</td>
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<td>6.06e+06</td>
<td>6.87e+07</td>
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</tr>
<tr>
<td>64</td>
<td>6.51e+10</td>
<td>7.07e+07</td>
<td>1.09e+09</td>
<td>8.46e+09</td>
<td>8.48e+12</td>
</tr>
<tr>
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<td>1.30e+12</td>
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<td>1.74e+10</td>
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</tr>
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<tr>
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<td>7.82e+05</td>
<td>4.36e+06</td>
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<td>2.22e+10</td>
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<tr>
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<td>1.16e+11</td>
<td>2.66e+15</td>
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<td>6.87e+07</td>
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<td>9.94e+10</td>
<td>2.72e+15</td>
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Table 5.16

Condition Number for Crank-Nicolson $CR-P_1-P_0$ Example 5.2 with $\mu^- = 1$.

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<th>$N_s$</th>
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<tr>
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<td>9.37e+06</td>
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<td>1.74e+10</td>
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</tbody>
</table>
In this chapter, we discuss a few research directions that follow from this dissertation. Here we outline some of the possible extensions to this work and provide details on how the ideas may be approached.

6.1 Two-Grid Mesh Method for Stokes Interface Problem

It is well known that the Stokes problem requires a finite element pair that satisfies the inf-sup condition to produce accurate solutions. The Taylor-Hood finite element spaces have been the most commonly used finite element spaces for solving the classical Stokes problem [26, 80, 96]. The family of Taylor-Hood finite elements [89] uses conforming $P_k - P_{k-1}$ pairs to approximate the velocity and the pressure, requiring the polynomial degree $k \geq 2$.

One of our future research topics is to develop a $P_1-P_0$ two grid immersed finite element method to solve the Stokes interface problem. Traditionally, the $P_1-P_0$ finite element method is known to be an unstable pair. However, with a suitable two-grid mesh, optimal convergence is observed for the finite element method. We look to extend this idea to the immersed finite element method for the unsteady Stokes problem for the stationary or the moving interface.
6.2 Non-Homogeneous Stress and Velocity Jump for Stokes Interface Problem

A natural extension of our current work is to incorporate the non-homogeneous stress and velocity jumps across the interface. In 2015, Adjerid, Chaabane, and Lin introduced an immersed discontinuous Galerkin (IDG) $Q_1$-$Q_0$ finite element space to solve the Stokes interface problem [2]. The authors incorporate the nonhomogeneous stress jump into their formulation by constructing a set of particular IFE functions that can capture accurate solutions at the interface jump. We believe this idea can be incorporated into our numerical scheme for the stress jump as well as the velocity jump across the interface.
REFERENCES


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