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The mathematical foundation of the musical scales and overtones

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This thesis addresses the question of mathematical involvement in music, a topic long discussed going all the way back to Plato. It details the mathematical construction of the three main tuning systems (Pythagorean, just intonation, and equal temperament), the methods by which they were built and the mathematics that drives them through the lens of a historical perspective. It also briefly touches on the philosophical aspects of the tuning systems and whether their differences affect listeners. It further details the invention of the Fourier Series and their relation to the sound wave to explain the concept of overtones within the tuning systems.

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TABLE OF CONTENTS

LIST OF KEY TERMINOLOGY

This is not a comprehensive list of terms, but this is a short list of key terms that are fundamental to understanding the concepts that readers will encounter in this thesis. Many of these terms will also be defined within the body of the thesis. Definitions are tailored to the context of this thesis and are influenced by "Mathematical Concepts in Music" by George A. Articolo, and "Math and Music: Harmonious Connections" by Trudi Hammel Garland and Charity Vaughn Kahn:

Frequency – measures the vibrations (or cycles) per unit of time. In relation to music, this would be waves per second, which has a unit of Hertz (Hz)

Interval – measured distance between two notes

Key – a rule imposed on a piece of music that dictates which notes are to be played

Note – the written notation for a tone indicating the duration of the tone; will often be interchanged with "tone" to mean the same thing. This is the case for this work.

Octave – an interval whose frequency ratio is 2, that is, an interval of the same tone with different pitch. It is also called an eighth.

Pitch – perception of a note by the listener, how high or low it sounds; measured in frequency (Hertz)

Scale – finite succession of notes with a defined interval pattern

Tone – a sound that “lasts long enough and is steady enough to have a pitch, quality, and loudness” [6]

Tuning (noun) – “a method for creating intervals that can be expressed in integer ratio” i.e., a defined interval pattern that creates a scale specific to those intervals [6].

Tuning (verb) – the process of adjusting the frequency of the pitch to remove impure waves or beats from tones that are sounded simultaneously. This is done amongst instruments that are playing together.

LIST OF FIGURES

CHAPTER I

THE MATHEMATICAL CONSTRUCTION OF THE MUSICAL SCALES

1.1 Introduction: How did we get here?

For many generations, people have pondered the existence of mathematics in other disciplines. The interplay of mathematics and the arts, literature, religion, and science is undeniable – from the presence of the Golden Ratio in architecture and artwork, the rhythmic structure and form of poetry, the logic and reasoning in religious beliefs, the eternal quality of truth that leads to divine understanding, to the mathematical support of long-speculated scientific credence [13]. Perhaps the most obvious of these connections is the role that mathematics has in music. Math has a hand in what most people would think about when considering music – rhythm, beats, meters, notes, rests, notation, style, etc. On a more complex level, it is involved in the organization of sounds, the way musicians think about chords, and the core fundamentals of how sound is produced. Mathematicians and musicians alike have studied the relationship between the two disciplines, some arguing that they are so intertwined that they are truly one. In this work, however, we will be focusing on the construction of the mainstream musical scales as they are understood by applied mathematics and the tuning process associated with each. There will be an overview of overtones, a brief discussion of the effect that the different tuning systems have on the listener, and then how the Fourier Series are tied into these concepts.

1.2 The Pythagorean Diatonic Scale

It is largely held that Pythagoras of Samos (ca. 585-500 BCE) was the pioneer of research into the mathematics involved with music, although the two disciplines were mentioned together as early as Plato. Pythagoras was a prominent mathematician in his time and was also the leader of a cult-like brotherhood that worshipped numbers. The mantra of the Pythagoreans was "numbers rule the universe" [11]. The group focused on mathematical research and promoting mathematics in the surrounding society, both educationally and philosophically. Many of the discoveries and proofs that are attributed to Pythagoras, such as the Pythagorean Theorem, may have actually been a discovery of an unnamed brother and credited to Pythagoras as the leader; this speculation has led to contention among scholars [11]. There is little other information about the life and person of Pythagoras other than this short description. However, we have insight into the beginning of his musical career. As the tale goes, Pythagoras was walking through town one day and he passed a blacksmith hammering metals beams. He noticed that the sound that the hammering of the beams produced differed based on their respective lengths and became intrigued with this concept [11]. Thus began his professional journey of research in primitive harmonic analysis. He initially used several different sound-emitting objects for his research and eventually built himself a basic instrument called the **monochord**. This instrument consisted of a small, rectangular box with a hole in it, a string that was fixed at either end, and a moveable bridge to adjust the length of the string. It resembled something similar to an elementary guitar.

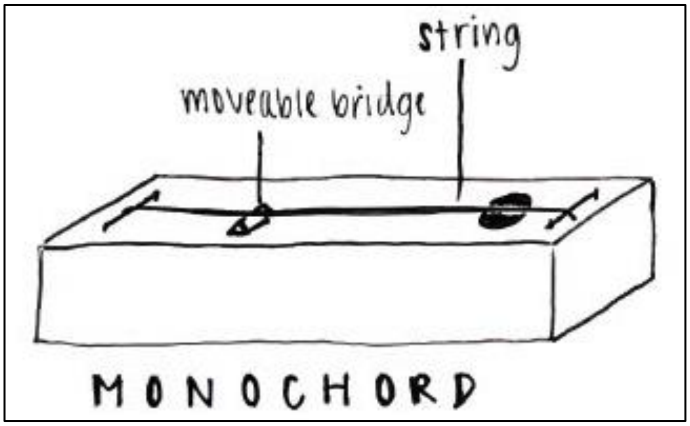

Figure 1.1 The Pythagorean Monochord

The monochord allowed him to test his theories about the relationship between sound and object length on a controlled object without having variants. During his research, he observed that plucking the string at its full length (with the bridge at the very end of the string) yielded a distinct tone while plucking it at half length (with the bridge at the measured halfway mark of the string) yielded the same tone but one octave higher. Similarly, he found that plucking the string when the bridge was stopped at two-thirds of the string length yielded an **interval** — the measured musical distance between any two notes — of a fifth [6]. Doing this again with the bridge stopped at three-fourths of the string length yielded an interval of a fourth. (In this case, the fourth, fifth, and octave are named because of their placement in the fourth, fifth and eighth pitches in the Pythagorean diatonic scale, which will be demonstrated later in this section). Pythagoras called these intervals **perfect consonants** because of the pleasing sound that they produced to the ear [11].

It is important to pause here and define some key terms that may seem intuitive to most people. Most people have a general understanding of what a tone or pitch is by just listening to music, even if there is a lack of familiarity with the musical jargon. But there is a subtle difference between the two concepts that is critical to understanding the mathematics behind the music. **Pitch** is the perception of a note by the listener — essentially how high or low it sounds. It is often defined in terms of **frequency**, which simply put, is the speed of vibrations of an object [6]. It is usually measured in Hertz, number of cycles or waves per second. In the case of the monochord, the faster the string vibrates, the higher the pitch sounds; similarly, the slower the string vibrates, the lower the pitch sounds. (The concept of frequency will be discussed in more detail later in Chapter 2. The focus at the moment is on understanding the music theory). On the other hand, **tone** is a sound that “lasts long enough and is steady enough to have pitch, quality, and loudness” [6]. “Tone” and “note” are used interchangeably throughout this work, though a **note** is technically the notation of a pitch. We must also note that the string length ratios of $\frac{1}{2}$, $\frac{2}{3}$, and $\frac{3}{4}$ that Pythagoras found when conducting his research to produce the heard intervals are inversely proportional to the frequency ratio because string length is inversely proportional to the frequency produced [11]. Thus, the frequency ratios of the aforementioned would be 2, $\frac{3}{2}$, and $\frac{4}{3}$, all within the one octave range. Since it is common practice to refer to intervals in terms of their frequency ratios rather than their string length, that is what we will do unless otherwise noted.

But back to Pythagoras: by establishing his perfect consonants, Pythagoras had the building blocks for his musical scale and a new theory cemented in his mind. He postulated that only the ratios of (relatively) small numbers sounded beautiful and that ratios of larger numbers were dissonant. The religious-like belief of the Pythagorean brotherhood that numbers ruled the universe heavily guided the construction of the Pythagorean diatonic scale. He started by “adding” the intervals together, and he discovered that by taking a fifth of a root and then taking the fourth of said fifth yielded an octave of the root. This can be represented mathematically by multiplication of the frequency ratios: $1 * \frac{3}{2} * \frac{4}{3} = 2$.

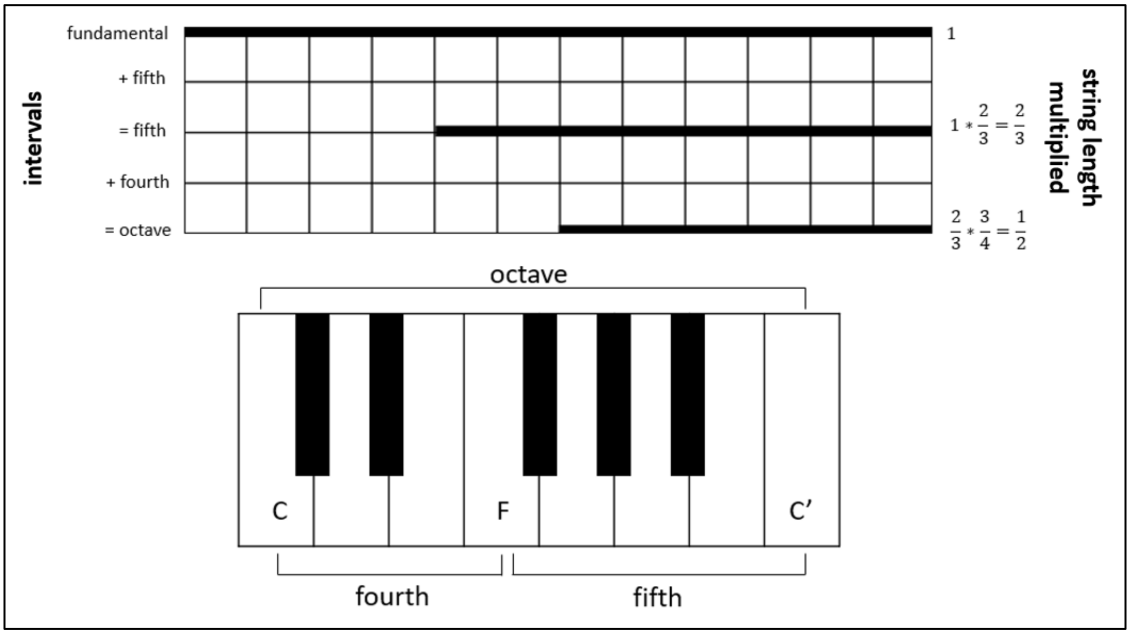

Figure 1.2 Derivation of the octave by string length

Adding a fourth to a fifth yields an octave.

However, this method would not work since it resulted in only octaves of the root pitch. He then experimented by taking positive integer powers of the three intervals he had already solidified:

1. **THE OCTAVE.** The octave has a frequency ratio of 2. By taking increasing positive integer powers of 2, the result is always a number that is divisible by two and therefore can be reduced back down to 2 itself, that is, the original note. In musical terms, this means that increasing powers of 2 yielded continuously higher octaves of the starting pitch, not yielding any new tones. Hence, the frequency ratio of 2 was not a viable option on which to construct a scale.

2. **THE FOURTH.** The fourth was the second interval Pythagoras tested; its frequency ratio is $\frac{4}{3}$. By taking increasing powers of $\frac{4}{3}$, we get reciprocals of the $\frac{3}{2}$, the frequency ratio of the fifth. In musical terms, this is the process of **inversion**, meaning that the bottom note of a chord is placed an octave above its original position to create a new inverted chord. Below are two representations — one mathematical and one musical. By taking powers of $\frac{4}{3}$, we see the inverted fifths displayed in the equation:

$$\left(\frac{4}{3}\right)^1 = \frac{4}{3} = \left(\frac{2}{3}\right)(2)$$

where $\frac{2}{3}$ is the inversion of $\frac{3}{2}$ and 2 is an octave (and therefore negligible).

$$\left(\frac{4}{3}\right)^2 = \frac{16}{9} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(2^2)$$

$$\left(\frac{4}{3}\right)^3 = \frac{64}{27} = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)(2^3)$$

By continuing this process, we get

$$\left(\frac{4}{3}\right)^n = \left(\frac{2}{3}\right)^n (2^n)$$

Thus, we can see that regardless of what *n* is, we will get an inversion of $\frac{3}{2}$. Figure 3 shows the inversion of the frequency ratio $\frac{4}{3}$ into the frequency ratio $\frac{3}{2}$, which is a fourth inverted into a fifth.

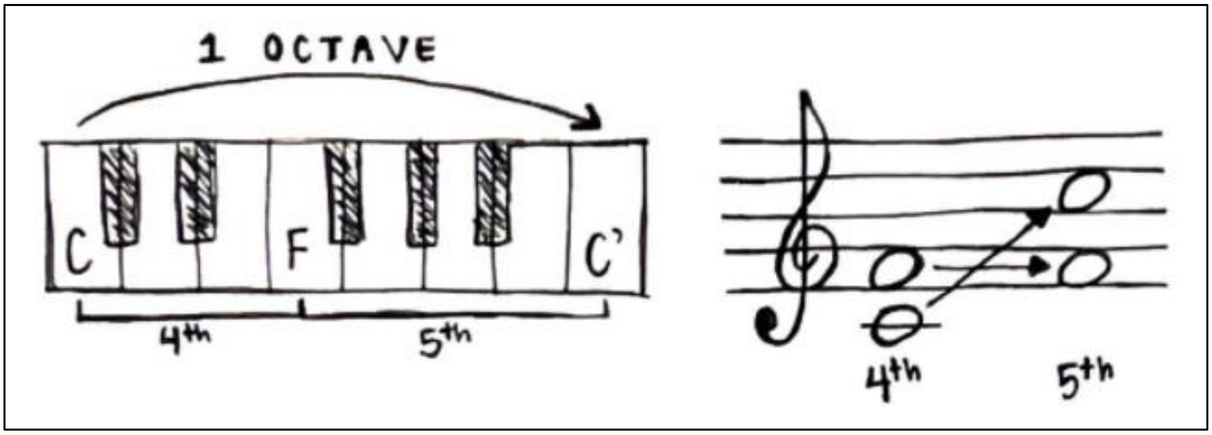

Figure 1.1 Inversion of a fourth shown on the keyboard and the staff

3. **THE FIFTH.** Lastly, Pythagoras tried taking powers of $\frac{3}{2}$ and succeeded. Consider this:

$$\left(\frac{3}{2}\right)^1 = \frac{3}{2}$$

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

$$\left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

$$\left(\frac{3}{2}\right)^4 = \frac{81}{16}$$

$$\left(\frac{3}{2}\right)^5 = \frac{243}{32}$$

Powers of $\frac{3}{2}$ yields different ratios every time, making this the best option to construct the scale. Since we know that multiplying or dividing by 2 simply changes the octave, and one octave can be described in ratios between the frequency ratios 1 and 2, we will apply this to the ratios above to limit them within one octave. The process of reducing ratios down or up to get them within one octave is known as **normalization.** Below are the normalized ratios of the Pythagorean scale:

$$\frac{3}{2} \to \frac{3}{2}$$

$$\frac{9}{4} \to \frac{9}{8}$$

$$\frac{27}{8} \to \frac{27}{16}$$

$$\frac{81}{16} \to \frac{81}{64}$$

$$\frac{243}{32} \to \frac{243}{128}$$

When these ratios are bookended with 1 and 2, we have an almost complete scale, but we are missing one of the fundamental intervals that Pythagoras initially found using the monochord: the fourth. To get this interval mathematically, he started with the power of -1 (rather than the power of 0) and multiplied by 2 to get the ratio within the octave [11]. This is equivalent to using the inverted fifth to get the fourth of the scale:

$$\left(\frac{3}{2}\right)^{-1} = \frac{2}{3} \to \frac{4}{3}$$

By this method, Pythagoras found all the needed ratios to complete his diatonic scale. Arranging them in sequential order provides the numerical representation of the first scale:

$$1 \quad \frac{9}{8} \quad \frac{81}{64} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{27}{16} \quad \frac{243}{128} \quad 2$$

Another way to think about the development of the Pythagorean scale is in terms of string length. The physical version of the math shown above would be if the string of the monochord was continuously divided in two-thirds by moving the bridge appropriately. This is the equivalent of taking the fifth of the fifth (of the fifth of the fifth…) until all the notes of the scale are sounded. (Actually, by this method, all twelve intervals in the octave are sounded. The Pythagorean diatonic scale only uses seven of the twelve intervals). This explains why taking powers worked to build the first scale – the fifth has a frequency ratio of $\frac{3}{2}$, and taking the fifth of the fifth... would be $\left(\frac{3}{2}\right)^n$ for as many positive integers *n* as needed.

This was an amazing breakthrough in his research, but perhaps even more astounding was what this told him about the intervals *between* the notes of the scale. There is a pattern to the seemingly unrelated ratios of the scale above: when we divide the second ratio by the first, third by second, fourth by third and so on, we see that the intervals are

$$\frac{9}{8} \quad \frac{9}{8} \quad \frac{256}{243} \quad \frac{9}{8} \quad \frac{9}{8} \quad \frac{9}{8} \quad \frac{256}{243}$$

The interval of $\frac{9}{8}$ was called the **whole tone**, having a slightly larger value than the **half tone** of $\frac{256}{243}$ (decimal representation: 1.125 > 1.053…), [11]. Thus, the Pythagorean diatonic scale has a construction of whole, whole, half, whole, whole, whole, half, denoted *W W H W W W H* throughout the rest of the work. Below is a depiction of how the intervals line up between the frequency ratios of the scale.

		W		W		H		W		W		W		H	
Intervals:		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{9}{8}$		$\frac{256}{243}$	
Frequency Ratios:	1		$\frac{9}{8}$		$\frac{81}{64}$		$\frac{4}{3}$		$\frac{3}{2}$		$\frac{27}{16}$		$\frac{243}{128}$		2
	I		II		III		IV		V		VI		VII		I’

Figure 1.2 The frequency ratios of the Pythagorean diatonic scale

The intervals are spaced appropriately between the ratios of the scale. The Roman numerals are the note names.

For centuries to come, the Pythagorean diatonic scale was the standard for tuning. Today, however, the standard tuning is that of equal temperament. But why is Pythagorean tuning no longer acceptable tuning practice? A simple explanation invokes the use of the circles of fifths. The **circle of fifths** is a sequence of notes in which a starting note is played and followed by successive fifths until the starting note is reached again albeit several octaves higher. This is the pattern that Pythagoras (roughly) followed when dividing the monochord string length – he was continuously taking fifths. (This was alluded to on the previous page). All the notes in the Pythagorean diatonic scale can be found by making use of this. In Figure 4, the notes written as $E\#/F$ or $C\#/D\flat$ are called **enharmonics**, notes with the same pitch but multiple notations.

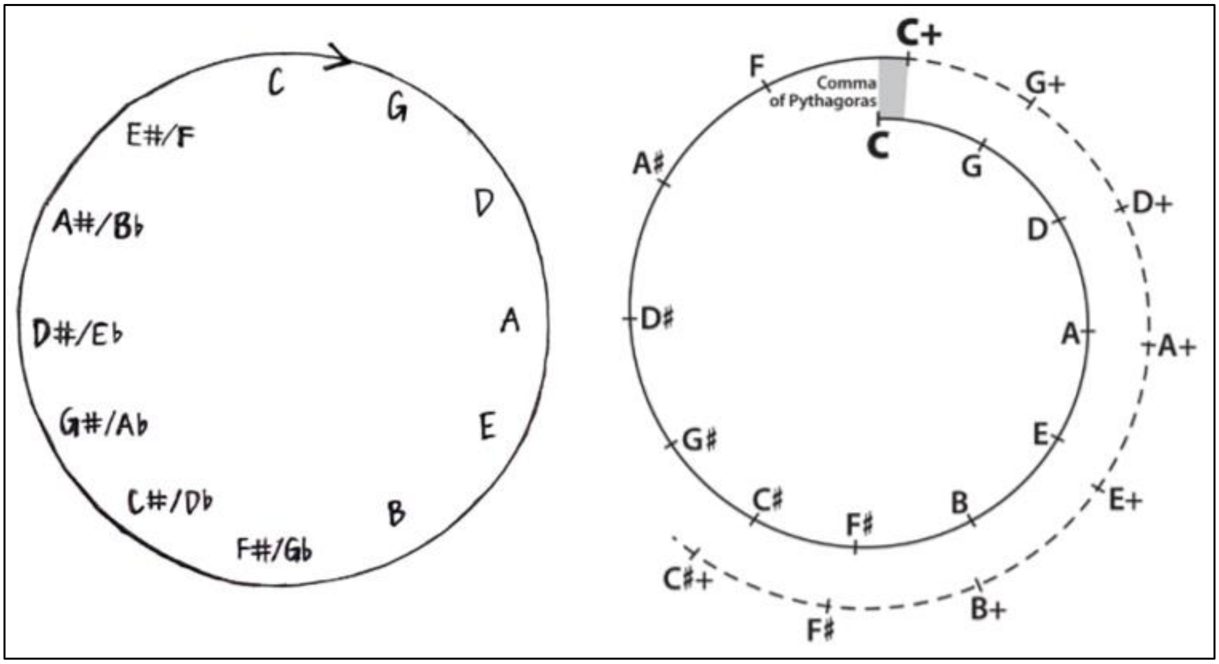

Figure 1.3 Comparison of the true circle of fifths against a Pythagorean circle of fifths [3]

The circle on the left shows a true circle of fifths in which the octaves line up; the “circle” on the right accounts for the Pythagorean comma and shows the discrepancy in ratio as the octaves increase.

This is fairly simple to represent mathematically. The final note of the circle of fifths should be a power of 2 since it is an octave representation of the initial note; specifically, it is seven octaves higher. If we are going by the Pythagorean scale, it should also be a power of $\frac{3}{2}$ since that is the frequency ratio of the fifth. However, we will never find a positive integer n to satisfy $\left(\frac{3}{2}\right)^m = 2^7$; more generally, we will never find positive integer *m* or *n* to satisfy $\left(\frac{3}{2}\right)^m = 2^n$, [11]. Mathematically, this discrepancy is shown as follows:

Pythagorean Representation:	$\left(\frac{3}{2}\right)^{12} \cong 129.75$
Seven Octaves above Original Note:	$2^7 = 128$

Figure 1.3 Decimal discrepancy between the Pythagorean method and the circle of fifths

For the Pythagorean representation, $\left(\frac{3}{2}\right)^{12}$ is the final note of the circle of fifths. Since the octave is built on powers of 2 and the final note in the circle of fifths is seven octaves higher than the initial note, 2^7 should equal $\left(\frac{3}{2}\right)^{12}$, but it clearly does not. This discrepancy in the frequency ratios is known as the **Pythagorean Comma**, and its numerical value is $\frac{129.75}{128} \cong 1.013643\ Hz$ [6]. The Pythagorean Comma proves that Pythagoras' tuning system was faulty, only allowing for changes in mode rather than true modulations in pieces. **Modes** are variations of the diatonic scale that do not require a new root or **tonic** note. They are obtained by a "circular shift" on the tonic, keeping the same notes of the original key but choosing a starting note other than the tonic [12]. There are several different modes, but an example of a mode would be the Aeolian mode, shifting so that the sixth of the scale was placed as the first note, consisting of the intervals $W\ H\ W\ W\ H\ W\ W$. By not having to change the tonic, the tuning remains consistent throughout the mode. On the other hand, **modulation** is when piece of music or song has transitioned from one key into another key, meaning that the tonic of the piece has changed from whatever it was originally. This requires the instruments that are playing in Pythagorean tuning to be re-tuned to the new key because of the Pythagorean comma. If the instruments were left untuned after a modulation, the new key would not sound harmonious and pleasing like the first one because the intervals would not maintain the correct spacing.

Another major error in the Pythagorean tuning system was that since it was purely mathematical in origin, it did not account for the natural overtones of the notes [11]. **Overtones**, also known as **harmonics**, are the (typically) unheard tones that accompany the heard tone. They contribute to the **timbre** of the tone — quality or distinct sound that each tone has apart from its pitch. Overtones would not be discovered until the 18th century by a man named Joseph Sauveur, but they play a critical role in the development of the scale. The Pythagorean scale, by virtue of its construction, left out important tones like the major and minor third; their respective ratios of $\frac{5}{4}$ and $\frac{6}{5}$ could not be obtained from $\frac{3}{2}$ [11]. The fault in the Pythagorean scale of the untrue octave neglected the overtones of notes and will be discussed later in more detail. Evidently, numbers do not rule the universe, at least in the way that Pythagoras thought.

1.3 The Just Intonation Scale

The next major tuning system was based on the **just intonation scale**. Just like its predecessor, it was also flawed in its tuning, and so it was not in practice but a few centuries; compared to how long the Pythagorean scale was used, the just intonation scale was the blink of an eye. In Eli Maor's book, the construction of this scale is attributed to Gioseffo Zarlino in 1558, but other resources leave the designer unnamed and focus solely on the mathematics [11]. The important quality of this scale is how its construction deals with the Pythagorean comma – by spreading it "evenly among all the fifths" while still maintaining small ratios [12]. Some of the frequency ratios may appear familiar as from the Pythagorean construction above, but the mathematical construction of this scale varies. Similar to that of the Pythagorean scale, the just intonation scale is also built on the fifth, but instead of being based on the octave, fourth, and fifth to build the scale, the just intonation scale is built using the octave, major third, and fifth, which mimics the modern-day arpeggio. This can be done by using the geometric, harmonic, and arithmetic means to find the different ratios of the scale (the geometric mean is used mainly for philosophical considerations that are not discussed in this work) [12]. The means are defined as follows, where a and b are two frequency ratios:

$$\boldsymbol{Arithmetic\ mean}: A(a,b) = \frac{a+b}{2}$$

$$\boldsymbol{Geometric\ mean}: G(a,b) = \sqrt{ab}$$

$$\boldsymbol{Harmonic\ mean}: H(a,b) = \frac{2ab}{a+b}$$

With the knowledge that the octave has the ratio of $\frac{2}{1}$, we can use the arithmetic mean to find the fifth; that is,

$$A(1,2) = \frac{1+2}{2} = \frac{3}{2}$$

From here, the major third can be found using the arithmetic mean of the root and the fifth:

$$A\left(1,\frac{3}{2}\right) = \frac{1+\frac{3}{2}}{2} = \frac{\left(\frac{5}{2}\right)}{2} = \frac{5}{4}$$

By a similar process, we can find all the intervals of the just intonation scale. (The entire process is detailed in “Proportion of Musical Scales | Sacred Geometry,” pages 13 and 14). They are listed below by their mathematical origin and then by their frequency ratio:

1	$H\left(1,\frac{5}{4}\right)$	$A\left(1,\frac{3}{2}\right)$	$H(1,2)$	$A(1,2)$	$A\left(\frac{4}{3},2\right)$	$A\left(\frac{7}{4},2\right)$	2
1	$\frac{10}{9}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

Figure 1.4 The frequency ratios of the just intonation scale and their mathematical origin

For comparison, here is the Pythagorean scale again:

$$1 \quad \frac{9}{8} \quad \frac{81}{64} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{27}{16} \quad \frac{243}{128} \quad 2$$

Notice that the second, third, sixth, and seventh of the just intonation scale are slightly different from that of the Pythagorean. The intervals for the just intonation scale are also inconsistent with that of the Pythagorean scale:

$$\frac{9}{8} \quad \frac{10}{9} \quad \frac{16}{15} \quad \frac{9}{8} \quad \frac{10}{9} \quad \frac{9}{8} \quad \frac{16}{15}$$

There is an obvious error with this construction. Logically, a whole note should have a consistent interval throughout the scale, but that is not the case here. As noted when discussing the Pythagorean intervals, the scale has an interval pattern of $W\ W\ H\ W\ W\ W\ H$ with one interval assigned the whole tone and one assigned to the half tone. However, here we have both $\frac{9}{8}$ *and* $\frac{10}{9}$ acting as whole notes, even though there is a decimal discrepancy of $\approx .0138888\ Hz$ between the two ratios [11]. Just as it did in the Pythagorean scale, this discrepancy caused issues with modulation in pieces and was quickly phased out of use by most. Some musicians and scholars argue that just intonation produces the purest sound, but overall, the cons outweigh the pros for most musicians.

1.4 The Equal Temperament Scale

The last and most pervasive tuning system is that of **equal temperament**. It is called equal-temperament rather than equal tuning because tuning implies that everything is actually in tune, which is not the case with equal-temperament. It gets its name from the *equal* distance between each half tone [6]. After studying both the Pythagorean diatonic scale and the just intonation scale, it is clear that this new scale diverges from the “numbers rule the universe” theory. In an age of number theory and religion, how did this scale triumph? An answer to this will require some history and knowledge of beats and overtones, or harmonics.

Father Marin Mersenne (1588-1648), a French monk, was an unlikely candidate for a mathematician or scientist, but he happened to be both. He is most famous for the Mersenne prime numbers, which can be defined as a prime number that is one less than a power of two. They are represented by the following equation: $M_n = 2^n - 1$ [11]. However, his main interest was in acoustics, although they were not known as such yet. He was the first person to measure the actual frequency of notes, and he helped popularize the usage of equal temperament tuning [6]. He did this by using the monochord to sound the first pitch and then doubling the strength length while maintaining the tension of the string, thereby obtaining lower and lower octaves of the original pitch, until he could visibly see individual vibrations slow enough to count them [11]. The measure of frequency eventually led to the main method of relative tuning (adjusting the tuning relative to the notes being sounded) – beat elimination. What are beats? **Beats** are a pulsing rhythm produced by notes that are out of tune with each other when sounded at the same time. As Maor puts it, they are an “undulation in intensity of sound emitted by two sources slightly out of tune” [9]. The further away from being in tune two notes are, the faster the beats or pulsing; the closer to being in tune two notes are, the slower the beats become. Beats are virtually imperceptible if the notes are in tune with each other. This discovery was Mersenne's main contribution to music theory and history, although he is also credited with "[giving] us the first complete account of the equal-tempered scale," and composing several pieces, most lost to the world for lack of record [11].

The discovery of beats was paramount to the advancement of the inner workings of music theory and mathematics. About half a century after Mersenne, Joseph Sauveur (1653-1716), a man of little renown and (oddly enough) severely hearing impaired, made music his life's work. He coined the term "acoustics," and he proved the existence of overtones, or harmonics, within notes [11]. By placing pieces of paper on a string at several points, he noticed that different parts of the string were vibrating independently from the rest. In combination with the phenomenon of beats, he showed that the portions that vibrated independently were integer multiples of the fundamental frequency, or frequency of the main note being sounded. This may not sound meaningful, but it had huge implications on the rest of string theory and subsequent discoveries in both math and music. In music, the term **harmony** is often used to describe a note(s) that accompanies(y) or supports the main note being sounded. Within a piece or song, harmonies accompany the **melody**, which is the main musical theme (think of the hummable portion of a song). How does this relate to overtones? The overtones of a note determine harmonies that are "agreeable" to the ear. (What is "agreeable" to the ear is highly subjective, but for the purposes of this work, it refers to the consonant sounds). If one note is sounded and another note played with it has an integer-multiple frequency of the first note, they will sound harmonious. The overtone sequence is as follows:

$$1f \qquad 2f \qquad 3f \qquad 4f \qquad 5f \ldots$$

After understanding how overtones work mathematically, it is easier to see why the Pythagorean and just-tuning scales were faulty: they did not have a perfect octave. Since their octaves were not equal to powers of 2 of the original frequency, this did not allow for integer multiples of original frequency and therefore no true overtones. The proof of overtones also greatly affected mathematics and string theory – it showed that a single wave could be represented as the sum of multiple waves. This spurred mathematicians on into finding an equation to accurately represent this reality. Eventually, Joseph Fourier came up with the equations of the Fourier series through his research in heat transfer (not even music!), which, simply put, is the mathematical representation of the sum of waves. This will be discussed in greater depth in the next chapter of the work.

But how do these two concepts lead to the development of equal-temperament tuning? Think back to the method of beat-elimination. Its goal is to eliminate the beats amongst the notes being played together; it is relative to the chord being played at a particular moment. Similarly, equal temperament tunes each note relative to its neighboring half tone. The scale sequence of $W\,W\,H\,W\,W\,W\,H$ is transformed into twelve half tones: $H\,H\,H\,H\,H\,H\,H\,H\,H\,H\,H\,H$, two half tones for each whole tone, in which the interval is equal for every half tone. Musically, we can hear this in the fact that every octave has the frequency ratio of exactly $\frac{2}{1}$ (Every octave sounds a perfectly equal distance apart. No beats can be heard.), whereas with Pythagorean and just tuning, the distance between the octaves increase with the interval. Since the foundation of overtones is the octave, equal temperament also ensures that overtone properties within a note are satisfied. The intervals for equal temperament have a frequency ratio of $\sqrt[12]{2}:1\ =\ 1.509$.... How can we definitively say that this ratio is correct? Consider this short proof:

We know there are 12 half steps within the octave and that the octave has a frequency ratio of $\frac{2}{1}$. Choose the starting point for our octave to be middle *C*, and let $\frac{C\#}{C} = h$ be the half step between the two notes and $\frac{C'}{C} = \frac{2}{1}$ for the octave. Then we can also say that $C\# = C * h$.

If $C\# = C * h$, then $D = C\# * h = (C * h) * h = C * h^2$

If $D = C * h^2$, then $D\# = D * h = (C * h^2) * h = C * h^3$

If $D\# = C * h^3$, then $E = D\# * h = (C * h^3) * h = C * h^4$ ...

This continues in a similar manner until we reach *C'*, the *C* an octave above middle *C*. Thus, we obtain $C' = C * h^{12}$. By solving for *h*, we find that $h = \sqrt[12]{\frac{C'}{C}}$. Since we know that $\frac{C'}{C} = \frac{2}{1}$, this gives $h = \sqrt[12]{2}$. Therefore, the interval of the half step is $h = \sqrt[12]{2}$. ■. (For original proof, see [6]).

This method would have been quite atrocious to Pythagoras because it utilizes a class of numbers other than rational numbers, namely the irrational number. As hinted at in the name, **irrational numbers** are numbers that cannot be represented by a simple fraction or ratio. The equal-temperament method dealt with the problem of tuning and modulation by essentially making everything except the octaves slightly out of tune. It hid the Pythagorean comma in between every interval [12]. In order to have a perfect-sounding and mathematically-accurate octave, the intervals between the octaves could not be represented by clean, small ratios like Pythagoras postulated. Instead, irrational numbers came into the picture and equally divided the octave into twelve intervals, shirking aside the long-held ideology that numbers ruled the universe.

Although this position is up for debate amongst musicians, equal-temperament or the **12-tone equal tempered scale (12-TET)** is the most advantageous method of tuning. As mentioned earlier, it takes care of the problem of modulation that is present in other tuning systems. By making every half-tone equidistant from the next, the octaves are in tune with each other and allow for easy key changes without re-tuning. It also favors the fifth of the scale in any given key, similar to the Pythagorean and just intonation scales. Although not perfectly in tune, the frequency interval of the fifth is extremely close to the rational frequency ratio of $\frac{3}{2}$ and is therefore more heavily emphasized than the other notes of the scale. From a more practical standpoint, having only 12 intervals within an octave provides a nice layout for the standard keyboard. Pianists can easily play an octave with one hand and have the option to play at least ten of the twelve notes in one chord if they choose to do so. Additionally, the 12-TET scale allows for 13! (= 622702080) dyads (chords of two notes) of different roots to be played [2]. The options seem endless, although there are a few drawbacks to a seemingly perfect system. The main issue that scholars and musicians have with the 12-TET scale is the fact that the intonation is not quite as harmonious as the just intonation or Pythagorean scale. By purposefully being slightly out of tune, the irrational frequency ratios present in equal-tempered scale take away some of the natural beauty that the rational frequency ratios produce. Instead of hearing the purity of perfectly tune chords, beats, however slight, will always be present in the chords tuned to equal temperament. [2].

1.5 A Brief Discussion on the Effects of Different Tuning Systems on the Listener

"Agreeable consonances are pairs of tones which strike the ear with a certain regularity; this regularity consists in the fact that the pulses delivered by the two tones, in the same interval

of time, shall be commensurable in number, so as not keep the eardrum in perpetual torment, bending in two different directions in order to yield to the ever discordant impulses."
– Galileo [12]. The discord and consonance of chords is a puzzle that has stood the tests of time, causing much disagreement among musical scholars about the benefits and detriments of certain tuning systems. What makes a chord agreeable or disagreeable? Is it something inherent to the nature of the music or something decided by the listener? As much as we have detailed the mathematical aspects of tuning, there is a philosophical aspect to it as well. Scholars have often contended that the difference between Pythagorean or just intonation and equal temperament is the difference between a light, comfortable attitude and one of hostility and irritability. Maria Renold, a concert violist and violinist that has done extensive research in the area of tuning, noted that after performing a concert in the which the piano was tuned in equal temperament, there was a "spiteful atmosphere [that] developed amongst the people present". She compared this to a performance in which the piano was tuned to just intonation, and she said that "everyone present was delighted at the splendid sound and felt sustained by a harmonic mood that left people free" [12]. To the trained ear, there is a significant difference in how the music is perceived depending on the tuning system that is used. The beats within an out-of-tune chords are painfully obvious to a trained ear, whereas the beauty of a perfectly tuned chord resounds within their being. It is difficult to research in this area since so much of the material is subjective to the listener, but it would certainly be interesting to see the results of a controlled group of people, both trained and untrained, listen to the same concert performed twice under different tuning systems. What would the atmosphere be? How would people respond to the open chords versus chords riddled with beats? These are questions yet to be answered.

CHAPTER II

INSTRUMENTAL TUNING FROM A MATHEMATICAL PERSPECTIVE

2.1 Introduction to Fourier Series and Sound Waves

The Fourier Series are one among many great renowned mathematical results. They have remained largely unchanged since their invention in the late 1700s, and their representation and application has been transformed and adapted to solve various problems, the most prominent being the Fourier Transform, which transforms time into frequency for the analysis of sound. A common modern-day application is the computer algorithm called the Fast Fourier Transform. This adaptation has allowed for the Fourier Series (and Transforms) to contribute to a variety of others academic and research fields, quite possibly more so than any other mathematical tool. Fourier Series are also the best way to gain a deeper understanding of sound waves and frequency. However, the focus of this work is limited to the construction of the Fourier Series and their involvement with the equation of the sound wave.

Despite their usefulness in many areas, the conception of the Fourier Series was from an unexpected area of research. Jean Baptiste Joseph Fourier (1768-1830) was a successful French mathematician with several other notable results. He took an interest in math at an early age and was later able to study some of the most famous mathematicians of all time – Lagrange and Laplace – at École Normale in Paris. He spent part of his career as a military scientific advisor in Egypt. Fourier invented the Fourier Series while working on a heat transfer problem. He had first derived an equation outlining the conduction of heat in solids, and within seven years, he had fully developed the Fourier Transform [14]. This will be discussed in greater depth later in this chapter, but at the time, Fourier's proposition that functions could be represented as a combination of sinusoidal waves was unfathomable and far beyond what anyone had considered. Lagrange exclaimed that Fourier's proposition was "nothing short of impossible" [14]. Contemporary mathematicians of Fourier such as Euler, Bernoulli, and Lagrange, had also been working on this same problem but through the lens of string theory. Their goal was to find a mathematical representation of periodic waves, thus making it ironic that Fourier discovered the representation while working on a completely unrelated problem [11].

Before going into how the Fourier Series was constructed, it is crucial to mention why they are relevant to this work: frequency and sound waves. As mentioned briefly in Chapter 1, frequency can be simply defined as cycles per unit of time, and it determines the pitch of the sound that we hear. This can be mathematically represented in another form:

$$f = \frac{1}{T} \tag{2.1}$$

where *T* is the period, that is, "the amount of time it takes to go through one full cycle" [11]. (A physical way to think about the period *T* is the string length; recall from chapter 1 that string length and frequency are inversely proportional.) This exactly fits within the basic formula of the sound wave:

$$f(t) = A\sin(\omega\, t) \tag{2.2}$$

where *A* is the amplitude of the wave, ω is the angular frequency, and *t* is time. This equation represents the sound of a pure tone; they are called **pure tones** due to the simplicity of the equation – they can be represented by a single trigonometric function. These cannot be produced in a natural setting, only via a computer. All other tones are **complex tones**, being made up of the summation of pure tones [4]. Angular frequency, as is in the name, can be written in terms of a frequency f by using Equation (2.1):

$$\omega = 2\pi f \Longrightarrow \omega = \frac{2\pi}{T} \tag{2.3}$$

Now we can write the sound wave with the substitution of Equation (2.3):

$$f(t) = A\sin\left(\frac{2\pi t}{T}\right) \tag{2.4}$$

We will see this exact representation appear in the Fourier Series later.

For a more practical example of how this works within the context of music, imagine someone plays a C_4 (the middle C on a keyboard). It is tuned to 261.62 Hz – this is its frequency. Thus, we can find its period and angular velocity by using Equation 2.1 and 2.3, respectively:

$$T = \frac{1}{f} = \frac{1}{261.62} = .00382 \text{ seconds}$$

$$\omega = 2\pi f = 2\pi * 261.62 = 1643.807 \text{ radians/second}$$

2.2 Fourier Series

The Fourier Series can be summarized as a mathematical way to represent the summation of sine and cosine waves. As mentioned in the previous section, they are particularly useful in acoustics and harmonic analysis. In order to fully understand the construction of the Fourier Series, we must review some concepts from linear algebra. For the following definitions, let $f(t), g(t) \in C(\mathbb{R})$.

Vector Space. A vector space (linear space) on a field $\mathbb{F}$ is defined to be as a nonempty set V with two operations, called addition and scalar multiplication, with the following properties:

a) $\forall\, x, y \in V, x + y \in V$;

b) $x + y = y + x, \quad \forall\, x, y \in V$;

c) $(x + y) + z = x + (y + z), \forall\, x, y, z \in V$;

d) $x + 0 = 0 + x, \quad \forall\, x \in V$, where 0 is the zero vector;

e) $x + (-x) = (-x) + x = 0, \quad \forall\, x \in V$, where 0 is the zero vector;

f) $\alpha(\beta x) = (\alpha\beta)x, \forall \alpha, \beta \in \mathbb{F}, \forall\, x \in V$;

g) $(\alpha + \beta)x = \alpha x + \beta x, \forall \alpha, \beta \in \mathbb{F}, \forall\, x \in V$;

h) $\alpha(x + y) = \alpha x + \alpha y$;

i) $1x = x$.

Remark 1. The elements of set V are vectors and the elements of $\mathbb{F}$ are scalars. $\mathbb{F}$ can be the real ($\mathbb{R}$) or complex numbers ($\mathbb{C}$) .

Remark 2. $\mathbb{R}^N$ and $\mathbb{C}^N$ are finite-dimensional vector spaces.

Definition. Function spaces. Let Ω an open subset of $\mathbb{R}^N$. The following function spaces are defined as follows:

$C(\Omega) =$ The space of all continuous real (or complex) functions on Ω.

$C^k(\Omega) =$ The space of all continuous real (or complex) functions on Ω, with continuous partial derivatives of order k, where $k \in \mathbb{N}$ or $k = \infty$.

Remark 3. $C[a, b]$ and the above-mentioned function spaces are infinite dimensional vector spaces on $\mathbb{R}$ (or $\mathbb{C}$).

Normed Spaces. A norm, $\|\cdot\|$, on a vector space is a function from a vector space V into satisfying the following properties:

a) $\|v\| \geq 0, \forall\, v \in V$

b) $\|cv\| = |c|\|v\|,\ \forall\, v \in V, \forall\, c \in \mathbb{R}$

c) $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|, \forall\, v_1, v_2 \in V$

d) $\|v\| = 0 \Leftrightarrow v = 0$

Inner Product Spaces. The inner product on a vector space V, is a function from $V \times V$ into $\mathbb{R}$, satisfying the following properties:

a) $\langle x, y\rangle = \langle y, x\rangle,\ \forall\, x, y \in V$;

b) $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle,\ \forall\, x, y, z \in V$;

c) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle, \forall\, x, y \in V$ and $\forall \alpha \in \mathbb{R}$;

d) $\langle x, x\rangle \geq 0\ \ \forall\, x \in V$;

e) $\langle x, x\rangle = 0 \ \Leftrightarrow\ x = 0.$

Remark. For the function space $C[a, b]$, the inner product is defined as

$$\langle f(t), g(t)\rangle = \int_a^b f(t)g(t)\, dt \tag{2.5}$$

Definition. Two functions $f(t)$ and $g(t)$ are said to be **orthogonal** on the interval $[a, b]$ if

$$\langle f(t), g(t)\rangle = \int_a^b f(t)g(t)\, dt = 0 \tag{2.6}$$

Definition. A set of functions $\{\phi_n(t)\}_{n=1}^{\infty}$ is said to be an **orthogonal set** on the interval $[a, b]$ if

$$\langle \phi_m(t), \phi_n(t)\rangle = \int_a^b \phi_m(t), \phi_n(t)\, dt = 0\, , \text{for } m \neq n \tag{2.7}$$

Remark. It is important to note that the inner product induces a natural norm, that is, the L_2-norm. The L_2-norm of a function over the interval $[a, b]$ is defined as:

$$\|f\|_{L_2} = \left(\int_a^b |f(x)|^2 dt\right)^{\frac{1}{2}} = \langle f(t), f(t)\rangle^{\frac{1}{2}} \tag{2.8}$$

Definition. An **orthonormal set** is an orthogonal set that is normalized by the norm.

2.2.1 Generalized Fourier Series

These concepts introduced a problem that mathematicians attempted to solve for years: given a function $f(t)$ and an orthogonal set $\{\phi_n(t)\}_{n=1}^{\infty}$, where $\phi_n(t) \in C[a, b]$, is it possible to find orthogonal basis in $C[a, b]$, such that

$$f(t) = c_1\phi_1(t) + c_2\phi_2(t) + c_3\phi_3(t) + \cdots + c_n\phi_n(t) + \cdots = \sum_{i=1}^{\infty} c_i\,\phi_i(t) \tag{2.9}$$

where the c_i's are real constants? In other words, is it possible to approximate a function $f(t)$ an as a linear combination of orthonormal functions? This is the problem of writing a Fourier series. Equation (2.9) is called the generalized Fourier series of $f(t)$. The core of the problem of Fourier series is in the notion of Banach, Hilbert spaces and the completeness of orthogonal sets $\{\phi_n(t)\}_{n=1}^{\infty}$, which are described below.

Definition. A normed vector space (linear space) V is complete if every Cauchy sequence in V has a limit in V.

Definition. A complete normed linear space is called a Banach space and a complete inner product space in its natural norm is called a Hilbert space.

Remark. $C[a, b]$ is not a complete space, i.e., the limit of the Cauchy sequences in $C[a, b]$ can be functions, which are not continuous and their integrals are not defined by using classical Riemann integral [9]. By using Lebesgue integral and defining the L_2-norm with this integral, we can complete the space of $C[a,b]$, that is, $L_2[a,b]$, which is a Banach and a Hilbert space.

Now for problem of writing a Fourier series, we have the following observations: if for a function $f(t) \in L_2(a,b)$, there is a generalized Fourier series with respect to the orthogonal set,

$$f(t) = \sum_{i=1}^{\infty} c_i \phi_i(t)$$

Observe that

$$\langle f(t), \phi_j(t) \rangle = \int_a^b \left(\sum_{i=1}^{\infty} c_i \phi_i(t) \right) \phi_j(t)\, dt = c_i \sum_{i=1}^{\infty} \int_a^b \phi_i(t) \phi_j(t) dt \tag{2.10}$$

But by orthogonality, we see that

$$\int_a^b \phi_i(t)\phi_j(t)dt = \begin{cases} 0 & \text{if } i \neq j \\ \|\phi_i\|^2 & \text{if } i = j \end{cases}$$

Therefore, we conclude that

$$c_i = \frac{\int_a^b f(t)\phi_i(t)\, dt}{\int_a^b \phi_i^2(t)\, dt} = \frac{\langle f(t), \phi_i(t) \rangle}{\|\phi_i(t)\|^2}, i = 1, 2, 3, \ldots \tag{2.11}$$

Definition. An orthonormal set $S = \{\phi_n(t)\}_{n=1}^{\infty}$, is a complete orthonormal set if for all $f \in L_2$,

$$\sum_{n=1}^{\infty} \langle f, \phi_n \rangle \phi_n = f$$

Theorem. The following are equivalent.

1) An orthonormal system $S = \{\phi_n(t)\}_{n=1}^{\infty}$ is complete.
2) For every $f \in L_2$, we have the Parseval's identity:

$$\sum_{n=1}^{\infty} c_n^2 = \|f\|^2$$

3) If $\langle f(t), \phi_n(t)\rangle = 0$, for all n, then $f \equiv 0$.

Proof. See [9].

2.2.2 Classical Fourier Series

All of the above concepts can be applied to simple trigonometric functions. Consider the following set of functions:

$$S = \left\{1, \cos\left(\frac{n\pi t}{l}\right), \sin\left(\frac{n\pi t}{l}\right)\right\}_{n=1}^{\infty} \tag{2.12}$$

To show that the entire set is orthogonal, we show that all the elements in the set are orthogonal with one another as long as the two elements are different.

$$\langle 1, \cos\left(\frac{n\pi t}{l}\right)\rangle = \int_{-l}^{l} \cos\left(\frac{n\pi t}{l}\right) dt = 0$$

Likewise, by properties of even, and odd functions,

$$\langle 1, \sin\left(\frac{n\pi t}{l}\right)\rangle = 0$$

$$\langle \cos\left(\frac{m\pi t}{l}\right), \sin\left(\frac{n\pi t}{l}\right)\rangle = 0, m \neq n$$

The last two checks are a bit more involved:

$$\begin{aligned}\langle \cos\left(\frac{n\pi t}{l}\right), \cos\left(\frac{m\pi t}{l}\right)\rangle \\ &= \int_{-l}^{l} \cos\left(\frac{n\pi t}{l}\right)\cos\left(\frac{m\pi t}{l}\right) dt = \frac{1}{2}\int_{-l}^{l} [\cos(n+m)t + \cos(n-m)\,t]\, dt \\ &= \frac{1}{2}\left[\frac{\sin(n+m)\,t}{n+m} + \frac{\sin(n-m)\,t}{n-m}\right]\Bigg|_{-l}^{l} = 0\end{aligned}$$

$\langle \sin\left(\frac{n\pi t}{l}\right), \sin\left(\frac{m\pi t}{l}\right)\rangle = 0, n \neq m$ can be shown in a similar fashion.

2.2.2.1 Theorem: Classical Fourier Series

If f is piecewise smooth on $[-l, l]$, and $2l$-periodic otherwise, then its **Fourier series** is the trigonometric series given by the following formula. The series is convergent pointwise for all $t \in \mathbb{R}$ to the value $f(t)$ if f is continuous at t and is convergent to $\frac{f(t^+)+f(t^-)}{2}$, if f is discontinuous at t.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{l}\right) + b_n \sin\left(\frac{n\pi t}{l}\right)\right) \tag{2.13}$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(t)\, dt$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(t)\cos(nt)\, dt$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(t)\sin(nt)\, dt$$

Proof. see [5].

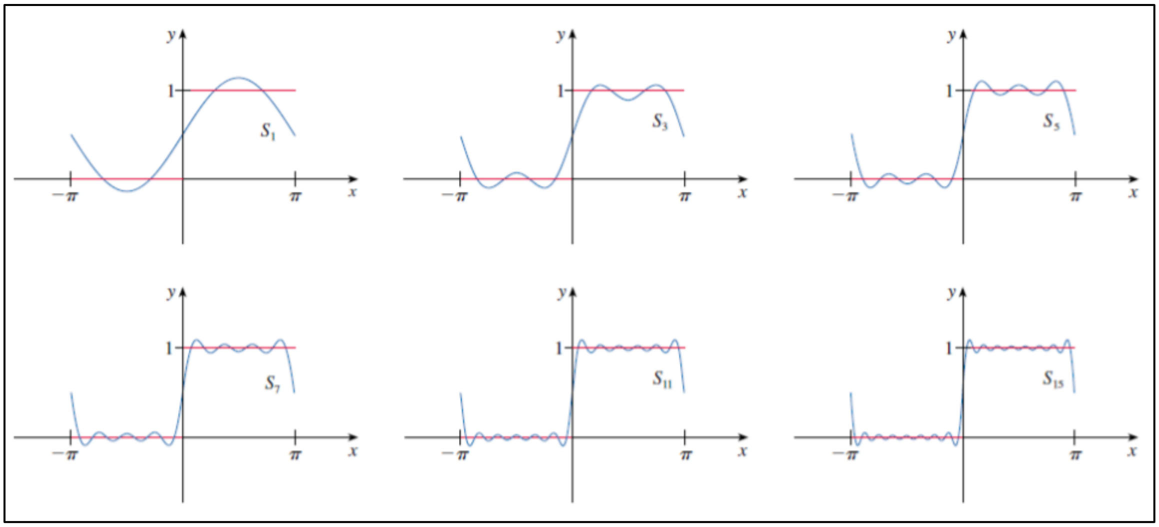

Figure 2.1 A Fourier Series representation for the function $f(x) = \begin{cases} 1 \; for \; 0 < x < \pi \\ 0 \; for - \pi < x < 0 \end{cases}$

"As $n \to \infty$, the Fourier Series for *f(x)* will have enough terms that it will converge to the function" [5].

Below is the derivation of a_0, a_n, and b_n respectively for the functions $f: [-l, l] \to \mathbb{R}$. We will use Equation (2.11) to find the Fourier coefficients.

$$\frac{a_0}{2} = \frac{\int_{-l}^{l} f(t) * 1 \, dt}{\int_{-l}^{l} 1^2 dt} = \frac{\int_{-l}^{l} f(t) * 1 \, dt}{2l} \Longrightarrow a_0 = \frac{1}{l} \int_{-l}^{l} f(t) \, dt$$

$$a_n = \frac{\int_{-l}^{l} f(t) \cos\left(\frac{n\pi t}{l}\right) dt}{\int_{-l}^{l} \cos^2\left(\frac{n\pi t}{l}\right) dt} = \frac{1}{l} \int_{-l}^{l} f(t) \cos\left(\frac{n\pi t}{l}\right) dt$$

$$b_n = \frac{\int_{-l}^{l} f(t) \sin\left(\frac{n\pi t}{l}\right) dt}{\int_{-l}^{l} \sin^2\left(\frac{n\pi t}{l}\right) dt} = \frac{1}{l} \int_{-l}^{l} f(t) \sin\left(\frac{n\pi t}{l}\right) dt$$

Now, notice the second term of the Fourier Series summation, $b_n \sin\left(\frac{n\pi t}{l}\right)$. It should look familiar, as it is almost the exact same as the equation for the sound wave, $A \sin\left(\frac{2\pi t}{T}\right)$. The sound equation is built into the Fourier Series, but an obvious question arises out of this: how do the other parts and terms of the Fourier Series affect the sound produced by the sound equation? The answer is in its quality, or **timbre**. The other trigonometric terms change how the sound is audibly perceived by the listener in qualities such as brightness and purity of sound. This is why the flute, piano, and trumpet are all distinguishable when they play together – they each have a distinct timbre. Another aspect that affects sound are the Fourier coefficients. Relating the sound wave equation to the Fourier Series, we see that the amplitude $A = b_n$. As mentioned in Chapter 1, the amplitude of the sound wave determines the loudness of the sound, and the amplitude of the waves within the Fourier Series are the coefficients $a_0, a_n,$ and b_n.

2.3 Overtones

The summation of waves not only affects the timbre of the instrument, but also provides the basis for the **overtones** that were mentioned in Chapter 1. Recall that overtones are the unheard frequencies that are contained within the heard note, and mathematically, they are represented by integers multiples of the fundamental frequency, where $1f$ is the fundamental:

$$1f \qquad 2f \qquad 3f \qquad 4f \qquad 5f \ldots$$

Angular frequency, ω, is another aspect of the sound wave equation that is present in the Fourier Series, hidden within the expression inside each trigonometric function. We know that $\omega = 2\pi f$, and from this and the Fourier Series equations we can conclude that $f = \frac{n}{2l}$. The fundamental frequency in the case of the Fourier Series is $f = \frac{1}{2l}$. By nature of the summation operation, we will step through each *n*, thereby increasing the frequency *f* by integers multiples and producing new sound waves every time. This makes the overtone sequence $\left\{\frac{1}{2l}, \frac{2}{2l}, \frac{3}{2l}, \ldots\right\}$ quite apparent within the context of the Fourier Series [1].

The relationship of the overtone sequence to the tuning process can now be detailed in more explicit terms. Recall from Chapter 1 that the main method of tuning between instruments is beat elimination. Mathematically, this means that between two notes that are sounded simultaneously, the goal is to match wavelengths of their individual Fourier Series. The closer to equal that the wavelengths become, the slower the beats get until they eventually dissipate as the wavelengths are matched. Practically, this looks like a piano tuner adjusting a bolt to lengthen or tighten the string length to match the sound produced from a tuning fork. For wind instruments, this could either be adjusting the length of a tuning slide or adjust their embouchure (the shape of the mouth) to raise or lower the pitch of the note they are producing.

2.4 The Wave Equation and its solution: vibration of strings

Think back to chapter one and Pythagoras' use of the monochord. There is a partial differential equation that perfectly represents the vibrations and sounds produced by a plucked string. For some string tightly fixed at two ends, $x = 0$ and $x = L$, having length L, the wave equation "is a model that governs the vertical displacement" from equilibrium at each point in the interval $[0, L]$ [9].

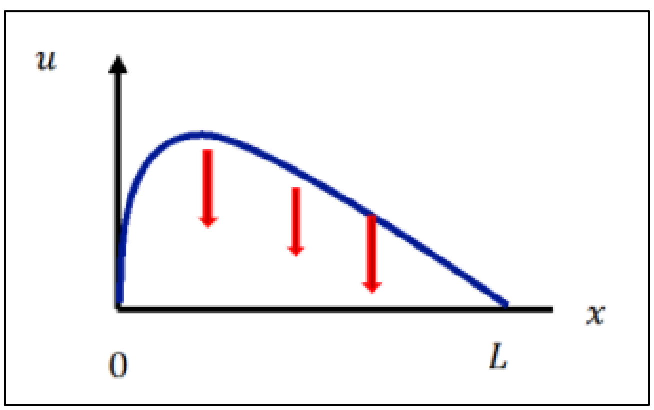

Figure 2.2 Vibration of a string

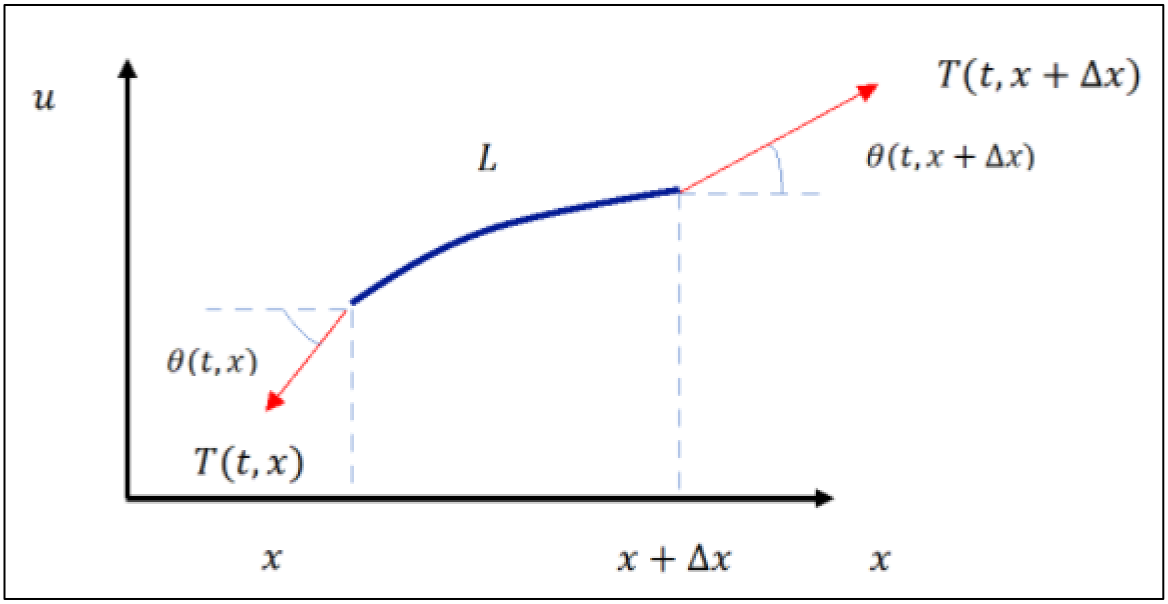

Figure 2.3 Derivation of the wave equation

If T represents the tension of the string and ρ the linear density of it, then according to the figure, on the small segment of the string, we have

$$\tan\theta(x) = \frac{\partial u(x)}{\partial x}$$

$$\tan\theta(x + \Delta x) = \frac{\partial u(x + \Delta x)}{\partial x}$$

The mass of the small segment is $\rho\Delta x$, and the vibration happens vertically, therefore by applying Newton's law, we get

$$\mathrm{T}\cos\theta(x + \Delta x) = T\cos\theta(x)$$

$$\mathrm{T}\sin\theta(x + \Delta x) - T\sin\theta(x) = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

Dividing the two equations:

$$\mathrm{T}\tan\theta(x + \Delta x) - T\tan\theta(x) = T\left[\frac{\partial u(x + \Delta x)}{\partial x} - \frac{\partial u(x)}{\partial x}\right] = \rho\Delta x\frac{\partial^2 u}{\partial t^2}$$

Now, by dividing both sides by Δx, and evaluating the limit, when $\Delta x \to 0$, we get

$$u_{tt} - c^2 u_{xx} = 0 \tag{2.14}$$

where $c^2 = \frac{T}{\rho}$, T being the constant tension and ρ being the linear density (mass per length) of the string.

This partial differential equation is accompanied with the boundary and initial conditions as follows:

$$\begin{cases} u_{tt} = c^2 u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

In this equation, the initial conditions, $f(x)$ and $g(x)$, represent the initial displacement, $u(x,0)$, of the string from equilibrium (the height of the string at the point from which it is released), and the initial velocity of the string, $u_t(x,0)$, respectively.

2.4.2 Fourier Series Solution to the Wave Equation

It is possible to find a Fourier series solution to the wave equation by the method of separation of variables. Assuming $u(x,t) = X(x)T(t)$, results in $XT'' = c^2 X'' T$, and therefore

$$\frac{X''}{X} = \frac{T''}{c^2 T} = \lambda, \quad \lambda\text{:constant}$$

From this we get following eigenvalue-eigenfunction problems:

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X(0) = X(L) = 0 \end{cases}, \quad T'' - \lambda c^2 T = 0$$

This will result in

$$\lambda_n = -n^2,\ X_n(x) = \sin\left(\frac{n\pi x}{L}\right),\ \ T_n(t) = a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)$$

$$u_n(x,t) = X_n(x)T_n(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n u_n(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right)\right] \sin\left(\frac{n\pi x}{L}\right)$$

By applying the initial conditions:

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$g(x) = u_t(x,0) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

Therefore, the coefficients are evaluated as follows:

$$a_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

2.4.3 D'Alembert Solution to the Wave equation

The solution of Equation (2.14) can be written in terms of two arbitrary functions *f* and g.

$$u(x,t) = f(x+ct) + g(x-ct) \tag{2.15}$$

This solution is the "superposition of right- and left-traveling waves moving at speed *c*" [9]. The arbitrary functions are made explicit by the initial displacement and velocity conditions [9].

2.5 Brief Discussion of the Application of Fourier Series to Music

Now, notice how the Fourier Series is a function of one variable *t* i.e., time. When we think of a sound wave, the Fourier Series considers the height of the wave at a specific moment in time *t*. or the amplitude of the sound wave. As previously stated, the iteration of *n* changes the frequency of the sound at that point in time. What if there was another perspective by which to study the sound wave? The answer to this question lies in the Fourier Transform – a transform acting on the function *f(t)*, changing it from a function of time to a function of frequency. How does this work? Without showing the derivation of the transform itself, let us think practically. When we listen to sound, it is a function of time – we are hearing music or talking or noise in time. But by using the Fourier Transform, we can analyze the "intensities of specific pitches" heard within the sound [8]. Mathematically, this is represented by the formula below:

$$\mathcal{F}\{f(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\omega t} dt = \mathcal{F}(\omega) \tag{2.16}$$

Perhaps the most famous example of Fourier analysis in music is Dr. Jason Brown's work in analyzing the opening chord of *A Hard Day's Night* by the Beatles. This chord puzzled musicians for years, having "a distinct *chang*" to it, something that no matter how close one came to the correct notes, no one could ever exactly reproduce [5]. This was the case until Dr. Brown used the Fourier Transform to perform an analysis on it. According to Dr. Brown, when a CD recording is made, the amplitude is sampled 44,000 times a second to capture the sound, and within that, individual frequencies can be picked out when analyzed using a Fourier Transform. To do this, he took a one-second clip from the middle of the opening chord, ran it through a Mathematica program, and the results were immense – it returned over 29,000 frequencies! In order to narrow this down, he considered only the largest 48 frequencies with the knowledge that the larger amplitude determine which tones are sounded; more specifically, he chose frequencies that had an "amplitude of .02 or larger" [4].

Freq. (Hz)	Ampl.	Freq. (Hz)	Ampl.	Freq. (Hz)	Ampl.	Freq. (Hz)	Ampl.
110.34	0.0600967	299.494	0.0298296	1050.86	0.0687151	2368.93	0.0221358
145.619	0.025485	392.57	0.0309716	1185.97	0.0372155	2371.19	0.0212846
148.621	0.0264278	438.358	0.0286329	1286.55	0.0231789	2371.94	0.0436633
149.372	0.0656018	524.678	0.0680974	1314.32	0.03819	2372.69	0.036042
150.123	0.175149	587.73	0.020613	1320.33	0.0223535	2637.65	0.0261839
174.142	0.0275547	588.48	0.0310337	1321.08	0.0494908	2638.4	0.0237794
174.893	0.0380282	589.231	0.0231753	1488.47	0.0241328	2754.	0.020001
175.643	0.0407103	785.141	0.0323532	1632.58	0.0205742	2763.76	0.0493617
195.159	0.0405164	786.642	0.0251928	1750.43	0.0234704	3083.52	0.0332062
218.428	0.0448308	787.393	0.0268553	2359.93	0.0366079	3147.32	0.0293723
261.964	0.0302402	960.784	0.0228509	2367.43	0.0267098	3148.07	0.0418507
262.714	0.0234502	981.801	0.02242	2368.18	0.0755327	3158.58	0.0285631

Figure 1.5 The 48 largest frequencies and their amplitudes chosen by Dr. Brown [4]

The rest of the frequencies comprised the unheard overtones that are present in the chord. From here, he rounded the numerical frequencies of the forty-eight largest frequencies to the nearest half tones by comparing them against A 220 Hz; this gave him a list of notes that composed the chord [5]. With the knowledge of the instruments that were being played and their physical ranges, Brown was able to reconstruct the correct instrumentation of the chord that had confounded musicians for years.

2.6 The Future of Mathematical, Musical Research

The interplay between mathematics and other subjects, particularly music, is vast. As mentioned earlier, there is always more research that can be done on how the differences in tuning systems qualitatively affect the listener. This work just provided a basis for understanding how the sound wave in present in the Fourier Series. Their involvement could be developed further into a workable problem such as playing a single chord using each of the three tuning systems and numerically and graphically tracking their similarities and differences. Additionally, abstract concepts like the Golden Ratio have been found in Mozart and Bach compositions; measure theory has been used to explain equal-temperament tuning; graph theory has connections to the dodecaphonic scale construction; and the list goes on [10]. The research topics are innumerable in which one could take to further develop an understanding of how mathematics is involved with music.

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