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Interpolating and dominating sequences

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In this thesis we will be working with dominating and interpolating sequences. We worked with a geometric approach and used pseudohyperbolic translated to the Euclidean disc in order to show that a sequence within a certain radius of a dominating sequence is dominating as well.

DEDICATION

This Thesis is dedicated to my mother, who without her constant support I would almost certainly not be in such a position to even attempt a master's thesis. She should be recognized just as well for any academic successes I have achieved.

ACKNOWLEDGEMENTS

Thank you to Dr. Smith for working with me through this thesis as well as all of my teachers who have helped me throughout my academic career. Thank you for instilling in me a desire to never stop learning.

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CHAPTER I

BACKGROUND KNOWLEDGE

History

Dominating Sequences

Our topic of discussion for this thesis, dominating and interpolating sequences, is a relatively young subject in mathematics. What later came to be called dominating sequences, where a sequence, $\{z_n\}_{n=1}^{\infty}$, is said to be dominating if given the function

$$R: H^{\infty} \rightarrow \ell^{\infty} \text{ by } (Rf)(n) = f(z_n) \tag{1.1}$$

$$\|f\|_{\infty} = sup\{f(z_n): n \in \mathbb{N}\} \ \forall\ f \in H^{\infty} \tag{1.2}$$

The term would eventually be coined in the mid 1960's in a paper by Leon Brown, Allen Shields, and Karl Zeller by covering topics "On Absolutely Convergent Exponential Sums" in which they show that these sequences on the unit disc have the property that "Every boundary point $p = e^{i\theta}$ may be approached non-tangentially (inside of some angle with vertex at p) by points of S." in which they defined $\mathrm{S} = \{\alpha_\mathrm{n}\}$ $(\mathrm{n} = 1,\ 2,\ \ldots)$ as a sequence with distinct points in the region with no interior limit points. This use of non-tangential limits will be the basis for our work. Since our work will be focusing primarily on dominating sequences it would behoove us to go into greater detail regarding them. The authors paper proved the following theorem which is what we used to define a dominating sequence. The proof is as follows:

Proof

We want to show that $sup\left|f(e^{i\theta})\right| = \|f\| \ \forall f \in H^{\infty} \ \Rightarrow$ almost every boundary point $p = e^{i\theta}$ may be approached non-tangentially (inside of some angle with vertex at p) by the points of S with S as defined above. The authors started by assuming that the condition that is implied fails and they will go on to show that the condition $sup\left|f(e^{i\theta})\right| = \|f\|$ fails as well.

Since the implied condition is false, then there is a set E of positive measure on the unit circumference (i.e. $|z| = 1$) such that no point of E can be approached non-tangentially by the sequence α_n. This of course means that any angle with a vertex on a point of E can contain only a finite number of α_n. In this case it is true for a right angle, placed so that the radius to the point bisects the angle.

This will then imply that at each point $p = e^{i\theta} \in E$ there is a right triangle, denoted as $\varDelta_\theta$, with the right angle vertex at the point p and the other two vertices inside of the unit circle, having the radius p as an axis of symmetry and containing none of the α_n. There will then exist a number $b > 0$ and a closed subset $E_1 \subset E$ of positive measure such that at each point of E_1 the altitude of $\varDelta_\theta$ measured from the vertex p has length $\geq b$. The authors then chose a closed arc, I, that has endpoints that are in E_1 for which $|E_1 \cap I| > 0$ and $|I| < b$ where the vertical bars denote Lebesgue measure. We then let G be the complement of E_1 with respect to the arc I. Then G is the union of a set of open arcs $\{I_n\}$. Then, take one of the arcs, say I_j with endpoints $e^{i\alpha}$ and $e^{i\beta}$ and draw the two triangles $\varDelta_\alpha$ and $\varDelta_\beta$. We then easily can see that the sides of the triangles cross over the interval I_j to form a smaller triangle, denoted by T_j, one side of which is the arc I_j. Then if t represents a point of I_j, then any of the α_n sufficiently near to t must lie within T_j.

We will now let $k(\varphi)$ be the characteristic function of the set G, and define $f(z)$ by:

$$f(z) = exp\left\{\frac{1}{2\pi}\int_0^{2\pi} k(\varphi)\frac{z+e^{i\varphi}}{z-e^{i\varphi}}d\varphi\right\} \tag{1.3}$$

Then

$$|f(z)| = exp\left\{\frac{-1}{2\pi}\int_0^{2\pi} k(\varphi)\frac{1-r^2}{|z-e^{i\theta}|^2}d\varphi\right\} \tag{1.4}$$

for $z = re^{i\theta}$, and so $|f(z)| \leq 1$ for $|z| < 1$.

The authors then employ a well-known property of the Poisson integral that

$$\lim\left|f(re^{i\theta})\right| = 1 \tag{1.5}$$

at almost all points $e^{i\theta}$ of E_1.

We now want to show that

$$|f(z)| \leq e^{\frac{-1}{2}} \qquad\qquad (z \in T_j, j = 1, 2, \cdots) \tag{1.6}$$

Now we will consider the arc I_j with endpoints as defined above. Then we get $|f(z)| = \prod|f_n(z)|$ where

$$f_n(z) = exp\left\{\frac{1}{2\pi}\int_{I_n} \frac{z+e^{i\varphi}}{z-e^{i\varphi}}d\varphi\right\} \tag{1.7}$$

since each $|f_n(z)| \leq 1$ for $|z| < 1$ this means that $|f(z)| \leq |f_j(z)|$ so then the previous equation will be proven if it can be shown that

$$\frac{1}{2\pi}\int_{\alpha}^{\beta} \frac{1-|z|^2}{|z-e^{i\varphi}|^2}\,d\varphi \geq \frac{1}{2} \qquad (z \in T_j) \tag{1.8}$$

The authors then use the fact that the previous integral has a well-known and simple geometric interpretation. One can extend the line segment from $e^{i\alpha}$ to z until it meets the boundary of the unit circle at some point w_1. The same can be done in a similar fashion, extending the line segment from $e^{i\beta}$ to z until it meets the boundary at a point w_2. Then, the integral is equal to the arc length from w_1 to w_2 in the counterclockwise direction divided by 2π. Using this, we can then get that the minimum of the integral for $z \in T_j$ is reached at the interior vertex of T_j and that the minimum value is $\frac{1}{2} + \frac{(\beta-\alpha)}{2\pi}$. This then proves the prior integral, which of course will prove the integral that directly preceded it.

Now, we will let $t = e^{i\theta}$ be a point of E_1, interior to the arc I, at which the previous limit holds. To make things simple it is assumed that $t = 1$. Then there is an $\varepsilon > 0$ such that if $Re(\alpha_n) > 1 - \varepsilon$, then α_n is in one of the triangles T_j. We then get that $|f(\alpha_n)| \leq e^{-1/2}$ at all such points α_n. Then, let $g(z) = f(z)e^z$. $g \in H^\infty$ and $\|g\| = e$ but

$$|g(\alpha_n)| \leq e^{\frac{1}{2}} \qquad for\ Re(\alpha_n) > 1 - \varepsilon \tag{1.9}$$

$$|g(\alpha_n)| \leq e^{1-\varepsilon} \qquad for\ Re(\alpha_n) \leq 1 - \varepsilon \tag{1.10}$$

and so $sup|g(\alpha_n)| < \|g\|$. This of course tells us that the initial condition is not satisfied. Because our implied condition was false this then shows that that $sup\left|f(e^{i\theta})\right| = \|f\| \; \forall f \in H^{\infty}$ $\Rightarrow$ almost every boundary point $p = e^{i\theta}$ may be approached non-tangentially. ■

The below image shows the geometric layout of the proof in the paper by Brown, Shields, and Zeller.

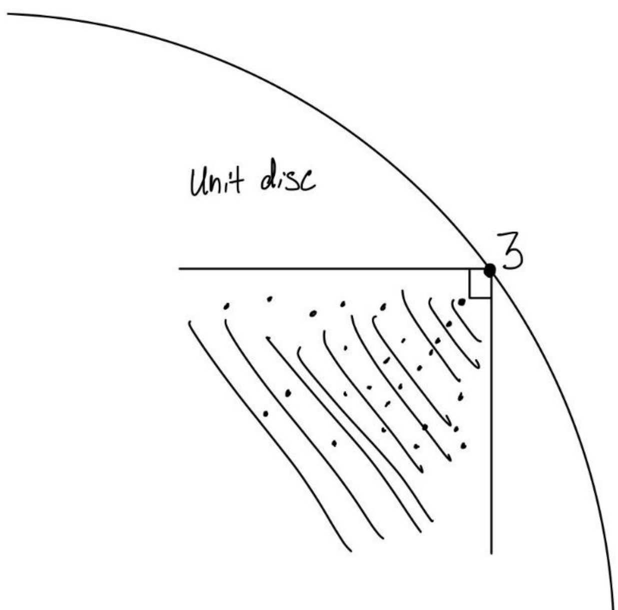

Figure 1.1 Non-tangential limit as described by Brown, Shields, and Zeller

Interpolating Sequences

Lennart Carleson, in the late 1950's, went on to characterize an interpolating sequence, on the unit disc in his paper "An interpolation problem for bounded analytic functions". A paper by Shapiro and Shields later simplified an interpolating sequence to the following definition:

they proved that if $\{a_n\} \in \ell^\infty$, then there is an $f \in H^\infty$ that interpolates the values a_n at z_n i.e. $f(z_n) = a_n \ \forall n$ then

$$\mathrm{inf_n} \prod_{j \neq n} \rho(z_j, z_n) > 0 \tag{1.11}$$

$$\text{where } \rho(z, w) = \left|\frac{z - w}{1 - \bar{w}z}\right| \tag{1.12}$$

is defined as the pseudo-hyperbolic metric on $\mathbb{D}$ and This notion of taking advantage of the pseudo-hyperbolic metric will be expanded upon in the coming chapter.

Daniel Luecking

Perhaps most fascinating, given the definitions of interpolating and dominating sequences above, Daniel Luecking showed in the early 1980's that if we have a sequence $z_n \in \mathbb{D}$. Define

$$\mathrm{R}\colon \mathrm{H}^\infty \to \ell^\infty \text{ via } (\mathrm{Rf})(\mathrm{n}) = \mathrm{f}(\mathrm{z_n}) \tag{1.13}$$

then R has a closed range if and only if $\{z_n\}_{n=1}^{\infty}$ is either a dominating sequence or an interpolating sequence. The Open Mapping Theorem will imply that there exists some constant $C > 0$ such that $\forall g \in H^\infty, \exists\, f \in H^\infty$ such that $RF = Rg$ and $\|f\|_\infty \leq C\|Rg\|_\infty$.

Given our knowledge of these two types of sequences the differences between the outcome of such a function just from the assumption of a closed range is striking and will be expanded upon further. The backwards direction of the if and only if statement is clear while the forward direction is where we want to spend the bulk of our time, and as mentioned before will produce two very distinct outcomes.

Case 1: R is 1-1 (Dominating)

In this case Luecking showed that if R is 1-1 then the sequence $\{z_n\}_{n=1}^{\infty}$ is a dominating sequence with the following proof:

Let $f \in H^{\infty}$, then $\|f\|_{\infty} \leq C\|Rf\|_{\infty}. \forall n \in \mathbb{N}, \|f\|_{\infty}^{n} = \|f^n\|_{\infty} \leq C\|Rf^n\|_{\infty} = C\|Rf\|_{\infty}^{n}$. Thus, $\|f\|_{\infty} \leq C^{1/n}\|Rf\|_{\infty}$. By letting $n \to \infty$ we get $\|f\|_{\infty} \leq \|Rf\|_{\infty}$. We can clearly see that $\|Rf\|_{\infty} \leq \|f\|_{\infty}$ hence from the following inequalities we get that $\|Rf\|_{\infty} = \|f\|_{\infty}$. $Thus$ $\{z_n\}_{n=1}^{\infty}$ is a dominating sequence. ■

Case 2 R is not 1-1 (Interpolating)

In this case Luecking used the proof as follows:

We start by letting $x \in \ell^{\infty}$, then there exists $F \in H^{\infty}$such that $RF = 0$ yet $F \not\equiv 0\ on\ \mathbb{D}$. $\forall n \in \mathbb{N}\ \exists\ m_n \in \mathbb{N}$ and $\tau_n \in \mathcal{H}(\mathbb{D})$ such that $\tau_n(z_n) \neq 0$ and $F(z) = (z - z_n)^{m_n}\tau_n(z)$. Clearly then every τ_n is bounded on $\mathbb{D}$.

Now, for $n \in \mathbb{N}$ put

$$g_n = \sum_{k=1}^{n} \frac{x(k)\tau_n}{\tau_n(z_k)} \tag{1.14}$$

Then $g_n \in H^{\infty}$and $$Rg_n = \begin{cases} x(k)\ if\ 1 \leq k \leq n \\ 0\ if\ k > n \end{cases} \tag{1.15}$$

There then exists $f_n \in H^\infty$ such that $Rf_n = Rg_n$ and $\|f_n\|_\infty \leq C\|Rg_n\|_\infty \leq C\|x\|_\infty$. Now, $\{f_n: n \in \mathbb{N}\}$ is a normal family on $\mathbb{D}$ so there exists an $f \in \mathcal{H}(\mathbb{D})$ and a subsequence of $f_n, \left\{f_{n_j}\right\}_{j=1}^\infty$ such that $f_{n_j} \to f$ uniformly on each compact set in $\mathbb{D}$ as $j \to \infty$. For any $z \in \mathbb{D}$,

$$|f(z)| = \lim_{j\to\infty} \left|f_{n_j}(z)\right| \leq C\|x\|_\infty \; so \; f \in H^\infty. \tag{1.16}$$

Now, let $n \in \mathbb{N}$. If $j \geq n$, then $n_j \geq n$ so

$$Rf(n) = f(z_n) = \lim_{j\to\infty} \left|f_{n_j}(z_n)\right| = x(n). \text{ Thus, } Rf = x. \tag{1.17}$$

The previous equality agrees with the definition of an interpolating sequence hence when R is not 1-1 the sequence $\{z_n\}_{n=1}^\infty$ is interpolating. ■

Summary and additional content

It can be easy and even tempting to fall into the trap of assuming that such distinct outcomes in terms of sequences must be completely unrelated. Yet as was discussed earlier in the section by Luecking, both interpolating and dominating sequences can be produced from a sequence mapped onto the unit disc. With the only difference between the two being whether the function R is 1-1 or not. The ultimate reason for this comes down to the fact that R has closed range.

A further, deeper, result was shown by Carleson and Garnett in which they proved that given that $\{z_n: n \in \mathbb{N}\}^{-w^*}$ is homeomorphic to $\beta\mathbb{N}$ given that the function R from before is defined as:

$$R^*|_{\beta\mathbb{N}}: \beta\mathbb{N} \to (H^\infty)^* \tag{1.18}$$

Even under this restriction Carleson and Garnett were able to show that $R(H^\infty) = \ell^\infty$ meaning that we still receive a dominating sequence. This is a very deep and complex result and will not be expanded upon further. However, this serves as a good example of the level of complexity in which these topics can attain.

CHAPTER II

RESULTS

Comparison of $\boldsymbol{\rho_{\mathbb{D}}}$ to disc to $\boldsymbol{\partial\mathbb{D}}$

In this section we will prove that if $\{z_n\}_{n=1}^{\infty}$ is dominating for $\mathbb{D}$ with $0 < r < 1$ and $\rho_{\mathbb{D}}(z_n, w_n) < r$ for all $n \in \mathbb{N}$ such that $\rho_{\mathbb{D}}$ is the pseudo-hyperbolic metric on the unit disc given by

$$\rho_{\mathbb{D}}(\mathrm{z}, \mathrm{w}) = \left|\frac{\mathrm{z} - \mathrm{w}}{1 - \bar{\mathrm{w}}\mathrm{z}}\right| \tag{2.1}$$

then the sequence $\{w_n\}_{n=1}^{\infty}$ is dominating for $\mathbb{D}$.

Proof

We start with the assumption that $0 < r < 1$ and choose some $z_0 \in \mathbb{D}$ then

$$\psi_{\mathrm{z}_0} = \frac{z_0 - \mathrm{B}(0;\mathrm{r})}{1 - \bar{z}_0\mathrm{B}(0;\mathrm{r})} \tag{2.2}$$

$$D(\mathrm{z}_0; r) = \{z: \rho(z, \mathrm{z}_0) < r\} \tag{2.3}$$

$$= \left\{\mathrm{z}: \left|\psi_{\mathrm{z}_0}(\mathrm{z})\right| < \mathrm{r}\right\} \tag{2.4}$$

$$= \psi_{z_0}^{-1}(\mathrm{B}(0;\mathrm{r})) = \psi_{z_0}(\mathrm{B}(0;\mathrm{r})) \tag{2.5}$$

such that $B(z;r)$ is the ball centered at z with radius $r > 0$ and $\psi_{z_0} \in \mathcal{M}$ with $\mathcal{M}$ being the Mobius transformation implies that $D(z_0, r)$ is a Euclidean disc. Thus, we can take advantage of the rotation-invariance property. So, assume now that $0 < x_0 < 1$ then the image below can be used to represent the Euclidean disc. With this in mind this is what we will perform the bulk of our mathematics on. We then perform the following:

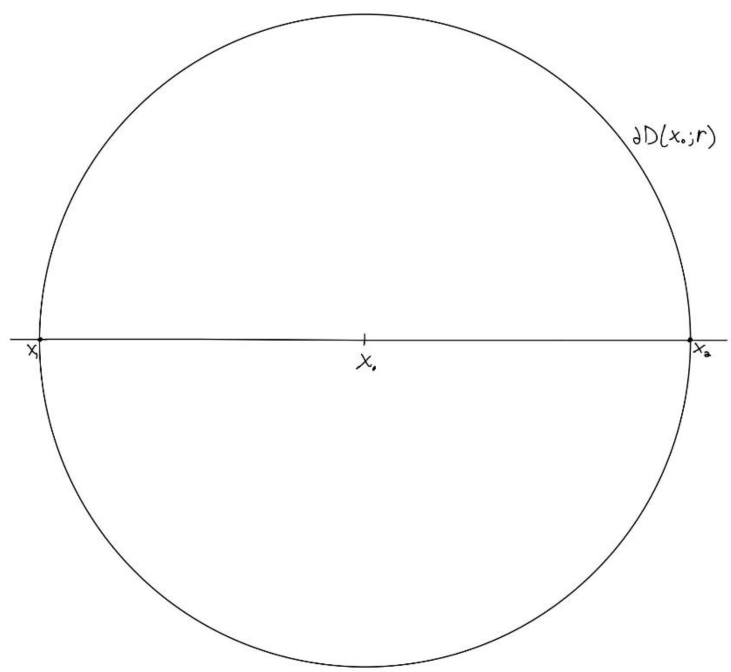

Figure 2.1 The Euclidean disc with pseudohyperbolic center x_0, x_1at the left, and x_2 at the right

$$\frac{x_0 - x_1}{1 - x_0 x_1} = \left|\frac{x_0 - x_1}{1 - x_0 x_1}\right| = \rho(x_1, x_0) = r \tag{2.6}$$

$$\frac{x_2 - x_0}{1 - x_0 x_2} = \left|\frac{x_0 - x_2}{1 - x_0 x_2}\right| = \rho(x_2, x_0) = r \tag{2.7}$$

where $x_1, x_2 \in \partial D(z_0, r)$. We now desire to find the Euclidean center. We start by solving for x_1

$$\begin{aligned} x_0 - x_1 &= r - r x_0 x_1 \\ x_0 - r &= (1 - r x_0) x_1 \\ x_1 &= \frac{x_0 - r}{1 - r x_0} \end{aligned} \tag{2.8}$$

taking the same approach for x_2, we get that

$$x_2 = \frac{x_0 + r}{1 + r x_0} \tag{2.9}$$

Now, using the formula for Euclidean center and performing some careful algebra

$$\begin{aligned} \frac{x_1 + x_2}{2} &= \frac{1}{2} \frac{(x_0 - r)(1 + r x_0) + (x_0 + r)(1 - r x_0)}{1 - r^2 x_0^2} \\ &= \frac{1}{2} \frac{x_0 - r + r x_0^2 - r^2 x_0 + x_0 + r - r x_0^2 - r^2 x_0}{1 - r^2 x_0^2} \\ &= \left(\frac{1 - r^2}{1 - r^2 x_0^2}\right) x_0 \end{aligned} \tag{2.10}$$

Lastly, we can calculate the Euclidian radius

$$
\begin{aligned}
\frac{x_2 - x_1}{2} &= \frac{1}{2}\frac{(x_0 + r)(1 - rx_0) - (x_0 - r)(1 + rx_0)}{1 - r^2x_0^2} \\
&= \frac{1}{2}\frac{x_0 + r - rx_0^2 - r^2x_0 - x_0 + r - rx_0^2 + r^2x_0}{1 - r^2x_0^2} \\
&= \left(\frac{1 - x_0^2}{1 - r^2x_0^2}\right)r
\end{aligned} \tag{2.11}
$$

In general, now, for $z_0 \in \mathbb{D}$ and $0 < r < 1$ we get

$$
D(z_0; r) = B\left(\left(\frac{1 - r^2}{1 - r^2|z_0|^2}\right)z_0;\ \left(\frac{1 - |z_0|^2}{1 - r^2|z_0|^2}\right)r\right) \tag{2.12}
$$

We now want to show that the sequence $\{w_n\}_{n=1}^{\infty}$ satisfies the criterion from Brown, Shields, and Zeller. Then, we can prove that the sequence is a dominating sequence. The rest of this proof will be shown via figures. We will first consider the case where the aperture of the non-tangential limit does not have to be widened to encompass the sequence $\{w_n\}_{n=1}^{\infty}$. As illustrated in the following figures.

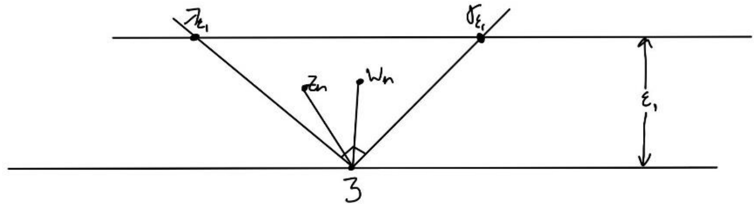

Figure 2.2 The resulting triangle with our sequences inside given the first iteration of epsilon.

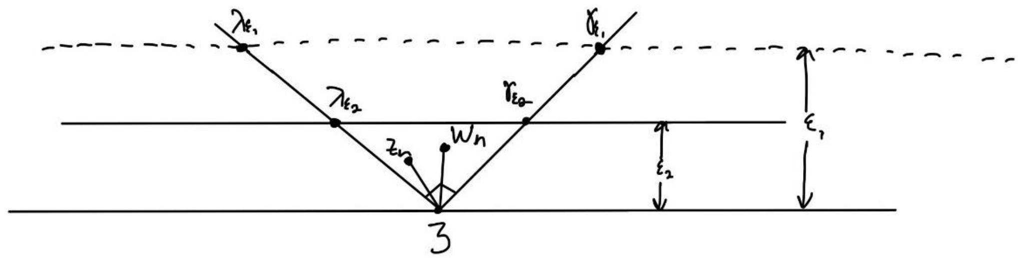

Figure 2.3 The resulting triangle with our sequences inside given the second iteration of epsilon.

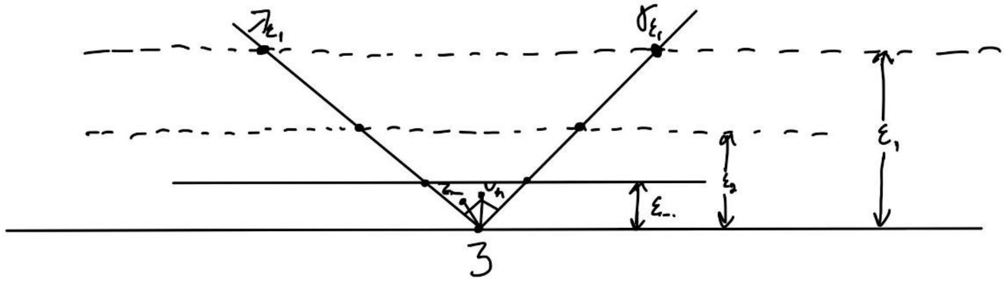

Figure 2.4 The resulting triangle with our sequences inside given the third iteration of epsilon. It’s at this point that we show that this trend of shrinking epsilon will continue.

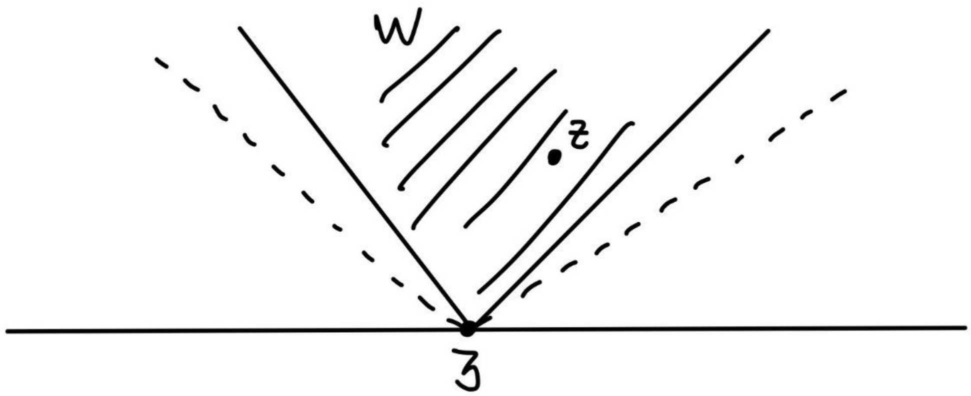

Figure 2.5 Non-tangential angle capturing the dominating sequence along with the region W with a wider aperture to ensure the sequence w is within a non-tangential angle.

What the previous figures show is that as the distance between epsilon and the unit circle gets smaller the line segments from λ_{ε_1} to ζ and γ_{ε_1} to ζ get smaller so do the distances from z_n to ζ and w_n to ζ. The process of shrinking the distance between epsilon and the unit circle is repeated through the last two figures. With the last figure showing the continuation of the process. The concept of the shrinking distances is continued as well. If we keep in mind the geometry of a triangle, we know that any point inside the triangle, in this case the points we care about most are z_n and w_n, will be a shorter distance away from our ζ point than the distances between λ_ε to ζ and γ_ε to ζ. With this in mind, we can "squeeze" our triangle down to a smaller and smaller size. And since z_n is dominating hence approaching the point ζ and w_n is within the distance of z_n using the Euclidean metric above it follows that w_n will approach to the point ζ as well. Hence, w_n is also a dominating sequence on $\mathbb{D}$.

Now, what was stated above can be easily shown using a more geometric approach, that is of course that the sequence w_n approaches ζ along with z_n is obvious given the metric that was shown above. What we want to take a closer look at now and what is the more interesting of the two scenarios is when w_n is outside of the non-tangential angle in which the sequence z_n is inside of. This scenario is illustrated in figure 2.2 in which the dashed lines represent the widened aperture to ensure w_n falls within in it, this is what we will discuss further. Now, we will define the region W (as shown in the figure above) with

$$|z-\zeta| < C(1-|z|^2) \text{ for some constant C} \tag{2.13}$$

We now want to show that there is then a constant B such that

$$|z_n - \zeta| < C(1 - |z_n|^2) \Rightarrow |w_n - \zeta| < B(1 - |w_n|^2) \tag{2.14}$$

Performing some careful algebra, we put

$$B = \frac{8}{(2-r)(1-r)} + \frac{2C(1+r)}{1-r} \tag{2.15}$$

Now, if $|z_n| > \frac{1}{2}$ and $|z_n - \zeta| < C(1 - |z_n|^2)$ we will get $|w_n - \zeta| < B(1 - |w_n|^2)$ as desired. Lastly, we can say that as z_n approaches ζ then w_n will approach it as well. Since this is within a non-tangential angle and $\zeta \in \partial\mathbb{D}$ it suffices to say that if z_n is a dominating sequence then w_n is also a dominating sequence on $\mathbb{D}$. ■

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