Investigation and control of Görtler vortices in high-speed flows

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Investigation and control of Görtler vortices in high-speed flows

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in Engineering
in the Department of Aerospace Engineering

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High-amplitude freestream turbulence and surface roughness elements can excite a laminar boundary-layer flow sufficiently enough to cause streamwise-oriented vortices to develop. These vortices resemble elongated streaks having alternate spanwise variations of the streamwise velocity. Following the transient growth phase, the fully developed vortex structures downstream undergo an inviscid secondary instability mechanism and, ultimately, transition to turbulence. This mechanism becomes much more complicated in high-speed boundary layer flows due to compressibility and thermal effects, which become more significant for higher Mach numbers. In this research, we formulate and test an optimal control algorithm to suppress the growth rate of the aforementioned streamwise vortex system. The derivation of the optimal control algorithm follows two stages.

In the first stage, to optimize the computational cost of the analysis, the study develops an efficient numerical algorithm based on the nonlinear boundary region equations (NBREs), a reduced form of the compressible Navier-Stokes equations in a high-Reynolds-number asymptotic framework. The NBREs algorithm results agree well with direct numerical simulation (DNS)
results. The numerical simulations are substantially less computationally costly than a full DNS and have a more rigorous theoretical foundation than parabolized stability equation (PSE) based models. The substantial reduction in computational time required to predict the full development of a Görtler vortex system in high-speed flows allows investigation into feedback control in reasonable total computational time, which is the focus of the second part of the study.

In the second stage, the method of Lagrange multipliers is utilized – via an appropriate transformation of the original constrained optimization problem into an unconstrained form – to obtain the adjoint compressible boundary-region equations (ACBREs) and corresponding optimality conditions, which constitute the basis of the optimal control approach. Numerical solutions for high-supersonic and hypersonic flows reveal a significant decrease in the kinetic energy and wall shear stress for all configurations considered. Streamwise velocity contour plots illustrate the qualitative effect of the optimal control iterations, demonstrating a significant decrease in the amplitude of the primary vortex instabilities.
DEDICATION

In loving memory of my dearest mother, Atika Bahammou, whose love, guidance, and unwavering belief in me have shaped the man I am today. Though you are no longer here to witness this milestone in my academic journey, I carry your love and wisdom in my heart. This dissertation is a tribute to your memory, a testament to your enduring influence in my life, and a small token of my gratitude for all you have done for me.

With all my love,

Your Son Omar.
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CHAPTER I
INTRODUCTION

1.1 Motivation

Flow control techniques are an integral part of fluid mechanics research as they offer possible optimization and improvement in the performance and operation of mechanical systems involving fluids. Automobiles, aircraft, and marine vehicles offer possible avenues to implement such control methods to determine the optimal configuration of wings, surface/edge geometry, engine inlet, nozzle design, and many other components to improve design metrics. The (mathematically) optimized design could potentially increase the lift-to-drag ratio on an aerodynamic surface, delay/accelerate the transition to turbulence, postpone flow separation, enhance mixing, or alter the morphology of the vortices embedded within the turbulence field.

Research into controlling transitional or fully-developed turbulent boundary layers is crucial because it has several practical applications that can benefit various industries. Turbulent boundary layers are responsible for considerable drag on different types of vehicles, including ships, aircraft, and cars. By understanding and controlling these boundary layers, we can increase the efficiency of these vehicles, which can lead to reduced fuel consumption and emissions.

Turbulent boundary layers can also pose safety risks, particularly in aircraft. Thus, understanding and controlling these layers can improve the safety of various vehicles, making them more reliable and trustworthy.
Furthermore, turbulence is still not fully understood and is considered one of the most significant challenges in fluid mechanics. By studying turbulent boundary layers and developing ways to control them, researchers can gain insights into the physics of turbulence, which can have implications in many other fields.

Lastly, understanding and controlling turbulent flows have significant industrial applications, including chemical reactors, heat exchangers, and pipelines. By developing control strategies for turbulent boundary layers, industries can improve the efficiency and safety of these processes.

In conclusion, research into controlling transitional or fully-developed turbulent boundary layers is essential because it can lead to practical applications that can benefit various industries, increase the efficiency and safety of vehicles, and improve our understanding of turbulence.

1.2 Problem summary

Görtler vortices refer to streamwise elongated streaks in a boundary layer flow over a concave surface. They develop due to the imbalance between the centrifugal forces and the wall-imposed pressure gradient (e.g., Görtler [26], Hall [30, 31, 32], Swearingen & Blackwelder [82], Malik & Hussaini [56], Saric [71], Li & Malik [48], Boiko et al. [4], Wu et al. [90], or Sescu et al. [73, 74], Es-Sahli et al. [19], Ren & Fu [66], Dempsey et al. [15], Xu et al. [93]). The higher the wall curvature, the more rapid the vortex formation is, and the quicker the transition occurs. Low to medium wall curvature also induce vortex formation, altering the primary flow and causing the laminar flow to break down.

For boundary layers over flat plates or wings, such streaky flows occur when the upstream roughness height exceeds a critical value (e.g., Choudhari & Fischer [9], White [88], White et
al. [89], Goldstein et al. [22, 23, 24], and Wu & Choudhari [91]) or when the amplitude of the freestream disturbances surpasses a certain threshold. (e.g., Kendall [40], Westin et al. [87], Matsubara & Alfredsson [58], Leib et al. [47], Zaki & Durbin [95], Goldstein & Sescu [25], and Ricco et al. [67]). As shown in figure 1.1, Görtler vortices appear in the form of counter-rotating streamwise elongated streaks originating in the near-wall region giving rise to the so-called mushroom structures. The streaks are characterized by adjacent regions of acceleration (high-speed) and deceleration (low-speed) of fluid particles (Kendall [40], Matsubara & Alfredsson [58], or Landahl [45]).

![Direction of flow](Image)

Figure 1.1

The formation of Görtler vortices on a concave wall [29].
Studies in the literature treating and discussing Görtler vortices evolving in boundary layer flows are abundant; the reader is encouraged to consult the studies mentioned here and the many others in their respective reference lists.

1.3 Research goals

In this dissertation research, we tackle Görtler vortices, a particular type of flow structure in turbulent boundary layers. The first goal of the dissertation is to develop an efficient algorithm based on the nonlinear boundary region equations (NBREs) – a simplified version of the Navier-Stokes equations – capable of accurately and efficiently predicting the development of Görtler vortices in high-speed flows while not requiring high-performance computing, that is, all simulation results presented herein were obtained using a local workstation machine (Intel Core i7 processor).

The optimized computational cost allows the implementation of an optimal control approach based on the method of Lagrange multipliers to suppress the growth of these structures in high-speed flows, which is the second goal of the dissertation. The optimal control approach uses transpiration velocity as the control parameter. The optimally controlled blowing and suction pores on the wall surface allow for active control of the centrifugal structures in the boundary layer, making the method optimal for multiple engineering applications – aircraft wings, turbofan-, jet-, and scramjet engines, wind tunnels, and many other engineering applications involving boundary layer flows.
1.4 Contributions

The present work has accomplished the following:

- Derived a version of nonlinear compressible boundary region equations NCBREs based on asymptotic analysis.

- Developed a robust and efficient NCBRE model capable of timely predicting the full development of Görtler vortices in high-speed flows without requiring large computational resources.

- Validated the method against DNS results and subsequently used the model to conduct parametric studies for boundary layer and curved free shear layer flows.

- Employed the method of Lagrange multipliers to transform the constrained optimization problem to the unconstrained form.

- Derived the adjoint compressible boundary region equations ACBREs and the optimality condition for the transpiration velocity control case.

- Implemented a Lagrange-multiplier-based optimal control algorithm to suppress the growth of Görtler vortices in high-speed flows.
CHAPTER II
OVERVIEW OF GÖRTLER VORTICES

The first to study this problem was the German mathematician Henry Görtler [26], who obtained solutions to the disturbance equations in the form of streamwise, counter-rotating vortices. Although, in reality, the flow over concave surfaces is not parallel, Görtler made a local parallel flow assumption and employed normal-mode analysis to show that the instabilities may occur when a specific dimensionless parameter, $Re \sqrt{\delta/R}$ (where $Re$ is Reynolds number in terms of freestream velocity, $\delta$ is the boundary layer thickness, and $R$ is the radius of the curvature) exceeds a critical value. Gregory & Walker [27] performed the first set of experimental measurements using china-clay surface visualization to produce Görtler vortex structure, while Tani [83] later used hot-wire measurements of the flow inside the boundary layer for the detection method. These experiments, and others that followed, showed that the type of instability is convective and develops along the streamwise direction.

Other early studies relied on the so-called normal-mode analysis approach to reduce the governing Navier-Stokes equations to a set of ordinary differential equations (assuming a constant growth rate along the streamwise direction), which enables solving the Görtler problem using eigenvalue analysis. Hall [31] was the first to consider a non-parallel formulation of the Görtler problem, wherein he obtained the solution to the resulting partial differential equations by a marching tech-
nique. He found that the stability depends on the location and form of the initial disturbance and concluded that it was impossible to determine a unique neutral stability curve for the Görtler problem. However, Day et al. [14] found modest differences between the normal-mode analysis and the marching scheme in their investigations, concluding the possible use of the normal-mode analysis for engineering studies via appropriate empirical corrections. Boiko et al. [4], theoretically and experimentally, studied the steady and unsteady linear Görtler instability. They carried out calculations based on locally parallel and nonlocal/non-parallel linear-stability theories and compared the results with experimental data.

Many theoretical and numerical studies investigated centrifugal instabilities on curved incompressible free shear layers. Plesniak et al. [64, 65] conducted extensive experimental measurements to study curved two-stream mixing layers to show how centrifugal effects yield streamwise vortices. The untripped case within this suite of experiments exhibited organized streamwise vorticity, while the tripped ones did not. They explained this by the existence of spatially-stationary streamwise vortices, which provide extra-entrainment (Bell & Mehta [2] showed this for a plane mixing layer). Hu et al. [36] and Liou [51] focused on the effect of the curvature on the inflectional Rayleigh modes. Their results showed that the curvature effect is small, although it excites an unstable three-dimensional disturbance with the amplitude increasing as the streamwise wavenumber decreases.

The analytical and numerical study of Otto et al. [60] showed that the unstable modes largely depend on surface curvature. They also employed numerical simulations to solve the parabolic equations, assuming that the wavenumber and Görtler number are both of order one. They found that as the difference between the freestream speeds increased, the layer became more susceptible to centrifugal instabilities.
Studies aiming to establish the correlation between the transition Reynolds number and freestream turbulence (FST) level demonstrated that the FST level (Dryden [16], Schneider [72]) and surface roughness (Pate [63]) significantly shift the position where the transition takes place. Yet, only a few investigations cover the detailed physics underlying such correlation. The experiments of Kendall [39] provide much information concerning supersonic boundary-layer transition under the influence of high-level FST. A salient feature is that fluctuations over a wide frequency range experience substantial growth within the boundary layer. Sufficiently downstream, a spectral peak emerges corresponding to the Mack-I mode in the low-Mach-number supersonic range \( M < 4.5 \) (Mack [55]). For \( M > 4.5 \), a secondary, less pronounced peak representing the Mack-II instability appears. These results indicate that some receptivity mechanism operates to generate instability waves in a nominally flat plate.

Studies of Görtler vortices in compressible boundary layer flows are motivated by the abundance of engineering applications in which these instabilities occur. These applications include high-speed flow in engine intakes, flows over the concave surfaces of turbomachinery blades, or flows over the walls of supersonic nozzles. For instance, Kobayashi and Kohama [42] investigated the instability using parallel flow theory. While El-Hady & Verma [17], Hall & Fu [33], and Hall & Malik [34] focused on the non-parallel effects. Spall & Malik [80] further improved the parallel eigenvalue framework by adding initial conditions to the partial differential equations, assuming zero amplitude perturbations in the external boundary layer (see slight modifications of this approach in Wadey [85] or Dando & Seddougui [12]). The number of experiments involving Görtler vortices developing in compressible boundary layers is not as large as the number of
experiments performed in the incompressible regime. Worth mentioning are the experiments of De Luca et al. [13], Ciolkosz and Spina [10], or Wang et al. [86].

While the above theoretical and numerical studies are relatively old, there has been a resurgence of interest in Görtler vortices evolving in high-speed boundary layers in recent years. For instance, Li et al. [49] studied the linear and nonlinear growth of Görtler vortices in hypersonic boundary layers using the parabolized stability equations (PSE), linear stability analysis, and direct numerical simulations (DNS). They identified multiple sets of unstable secondary instability modes and investigated their linear and nonlinear spatial development. They used DNS to explore the transition onset to determine the physical mechanisms associated with high-speed boundary layers.

Ren and Fu [66] conducted a series of numerical computations to investigate the fundamental, subharmonic, and detuned secondary instabilities of Görtler vortices in high-speed boundary layer flows with a focus on the Mach number effect. They found that the growth rate associated with Görtler vortices decreases with the Mach number and contributes to the appearance of the trapped-layer mode in the primary instability. Chen et al. [8] employed DNS and linear stability analysis to explore the transition of stationary Görtler vortices in high-speed boundary layer flows by exciting the instability using steady blowing and suction on the wall, similar to the disturbance employed in this work. They showed that, depending on its frequency and wavelength, the first Mack mode transforms into either a varicose or sinuous mode streak instability, while the second Mack mode only transforms into a varicose instability mode.

Li et al. [50] utilized DNS and linear secondary instability to study Görtler vortices and their associated secondary instabilities in the hypersonic boundary layer flow over a cone featuring a concave body. Results of this study showed that sinuous modes, concentrating in the wall-
normal internal shear layer in the lower portion of the mushroom-like structures, predominantly characterize the secondary instability.

Ricco & Wu [67] extended the incompressible analysis by Leib et al. [47] to the compressible regime and explained the formation and growth of thermal streaks thought to play a significant role in the secondary instability process (see also Ricco [68]). In a later study, Viaro and Ricco [84] implemented a form of compressible boundary region equations to study the evolution of Görtler vortices in compressible boundary layer flows using the approach derived by Ricco & Wu [67]. Their results showed that the developed formalism correctly accounted for both the effect of the initial conditions and the boundary condition at the top of the boundary layer. Ricco, Tran & Ye [69] and Ricco, Shah & Hicks [70] further studied the influence of wall heat transfer and wall suction on the thermal streaks, respectively.

The work of Song et al. [79] detailed the evolution of first and second Mack modes inside a compressible boundary layer involving Görtler vortices via DNS and PSE. Nevertheless, the DNS-based models fall short of computation efficiency for large Reynolds numbers, whereas PSE-based approaches remain largely ad-hoc and suffer from convergence problems [1]. The methods’ disadvantages motivate investigation into a more efficient numerical solution based on a more robust theoretical foundation.

Xu et al. [94] investigated the influence of the curvature, turbulence level, and pressure gradient on the development of streaks and Görtler vortices in a boundary layer over a flat or concave wall. They found that an adverse/favorable pressure gradient causes the Görtler vortices to saturate sooner/later, but at a lower/higher amplitude than that in the case of a zero-pressure-gradient. On the other hand, for the same pressure gradient and at low levels of the free stream vortical
disturbances (FSVD), the vortices saturate earlier and at a higher amplitude as Görtler number increases. Raising FSVD intensity reduces the effects of the pressure gradient and curvature. At a high FSVD level of 14 %, the curvature does not impact the vortices, while the pressure gradient only influences the saturation intensity.

Chen et al. [8] used linear stability theory (LST) and direct numerical simulation (DNS) to investigate the stability and transition of Görtler vortices in a hypersonic boundary layer at \( M = 6.5 \). They used blowing and suction to excite Görtler vortices with different spanwise wavelengths \( L_3, L_6, \) and \( L_9 \) (3, 6, and 9 mm, respectively). Their results showed, among other findings, that for \( L_3 \), the sinuous-mode instability dominates the breakdown process, whereas, for \( L_6 \), the varicose-mode instability is most dangerous. For \( L_9 \), sinuous and varicose instability modes appear equally important, with the former controlling the near-wall region and the latter appearing on the top of the mushroom structures. Subsequently, we will show that our optimal control method effectively controls the near-wall sinuous-mode instability, making it very suitable for small and large spanwise wavelength cases where this instability is dominant.

Analogous to other instabilities, streamwise-oriented vortices (and their accompanying streaks) occur in various engineering applications, such as the flow over turbomachinery blades and flow close to the walls of wind tunnels or turbofan engine intakes. Not only do these instabilities contribute to transition, but they also induce a significant increase in noise in supersonic and hypersonic wind tunnels, which can cause interference with the measurements in the test section (Schneider [72]). Thus, comparing and validating wind tunnel measurements to real-flight conditions becomes challenging. The objective function for a control algorithm in this context would be to reduce the streamwise vortex energy, which would delay nonlinear breakdown and the transition
onset. However, since the transient part of the primary instability is responsible for the growth of three-dimensional disturbances (and their subsequent breakdown), the applied control strategy must restrict the growth of transient modes.

Wall transpiration control techniques of boundary layers can be applied via localized suction and blowing regions underneath low- and high-velocity streaks. The net result is a decrease in the spanwise variation of the streamwise velocity and, therefore, a corresponding reduction in the number and strength of the bursting events.

A more efficient way of achieving these results is by applying active wall control methods. In the recent literature, researchers utilized such methods in turbulent channel flow as a technique to reduce skin friction drag (see Choi et al. [7]). Choi et al. [7] conducted direct numerical simulations of an active-control approach based on wall transpiration velocity by placing sensors in a sectional plane parallel to the wall. This technique achieved approximately a 25% reduction in frictional drag. However, from a practical standpoint, installing the sensors in the flow is challenging, if not impossible, as they could amplify existing disturbances or create new ones. Thus, to avoid interference issues, Choi et al. [7] investigated the same control algorithm with sensors placed at the wall. The sensors feed into the leading term in the Taylor series expansion of the vertical velocity component close to the wall surface. This approach resulted only in a 6% reduction. Koumoutsakos [43, 44] implemented a similar feedback control algorithm informed by flow quantities at the wall. Inserting the vorticity flux components as inputs to the control algorithm resulted in a significant drop in skin friction (approximately 40%).

Lee et al. [46] derived new suboptimal feedback control laws based on blowing and suction to manipulate the flow structures in the proximity to the wall using surface pressure or shear stress
distribution (the reduction in the frictional drag was in the range of 16-20%). Observing that the opposition control technique is more effective at low Reynolds number turbulent wall flows, Pamies et al. [61] proposed the utilization of blowing only at high Reynolds numbers, and by doing so, they obtained a significant reduction in the skin-friction drag for these flows. Recently, Stroch et al. [81] compared the opposition control applied in the framework of turbulent channel flow to a spatially developing turbulent boundary layer. They found that the rates of frictional drag reduction are approximately similar in both cases. The paper by Kim [41] reviews the technical issues and limitations of the opposition control type.

In a linear, full-state optimal control theory framework, Hogberg et al. [35] reported the first successful re-laminarization of a $Re_\tau = 100$ turbulent channel flow by applying zero mass flux blowing and suction at the wall. They showed that the information in the linearized equations is sufficient to construct linear controllers able to re-laminarize a turbulent wall flow, but this may be limited to low Reynolds number flows.

In addition to studies based on numerical methods, several experimental studies used blowing and suction to control the disturbances in laminar or turbulent boundary layer flows. We briefly mention a few of these here. Gad-el-Hak & Blackwelder [21] used continuous or intermittent suction to eliminate artificially generated instabilities in a flat-plate boundary layer flow. Experiments of Myose & Blackwelder [59] used the same idea to delay the breakdown of Görtler vortices. Jacobson & Reynolds [37] developed a new type of actuation based on a vortex generator to control disturbances induced by a cylinder along the wall normal-direction and unsteady boundary layer streaks generated using pulsed suction. As for the latter, the actuation significantly reduced the spanwise gradients of the streamwise velocity, which are one of the main driving forces of sec-
ondary instabilities (see Swearingen & Blackwelder [82]). Experiments of Lundell & Alfredsson [54] controlled streamwise velocity streaks in a channel flow by using localized suction regions downstream of the inflow. This approach delays secondary instabilities and, ultimately, the onset of transition.

While several studies employed optimal control for laminar or turbulent boundary layer flows, others considered the implementation of optimal control for shear flows; see the work of Gunzburger [28], or the more recent review by Luchini & Bottaro [53], although the latter is in a slightly different context. Numerous other studies aimed at controlling disturbances developing in laminar or turbulent boundary layers; (see Bewley & Moin [3], Joslin et al. [38], Cathalifaud & Luchini [5], Corbet & Bottaro [11], Hogberg et al. [35], Zuccher et al. [96], Cherubini et al [6], Lu et al. [52], Sescu & Afsar [77], Sescu et al. [78]).

Bewley & Moin [3] applied an optimal control method based on blowing and suction to turbulent channel flows. They reported a 17% frictional drag reduction as a result of this scheme. Cathalifaud & Luchini [5] used the same optimal control approach to reduce the energy of disturbances in a boundary layer flow over a flat plate and a concave surface. The study of Zuccher et al. [96] discussed and tested an optimal and robust control strategy in the framework of steady three-dimensional disturbances in the form of streaks that formed in a boundary layer flow over a flat plate. They established the optimal control method on an adjoint-based optimization technique to initially determine the optimal state for a given set of initial conditions and then find the worst initial conditions, amongst that set, for the optimal control.

Lu et al. [52] derived an optimal control algorithm within the linearized unsteady boundary region equations. As we shall see in chapter 2, these equations are the asymptotic reduced form of
the Navier-Stokes equations assuming low frequency and low streamwise wavelength. Their study aimed at controlling both streaks developing in flat-plate boundary layer flows and Görtler vortices evolving along concave surfaces. Cherubini et al. [6] applied a nonlinear optimal control strategy with blowing and suction, starting with the full Navier-Stokes equations and using the method of Lagrange multipliers to determine the maximum decrease of the disturbance energy.

Papadakis, Lu & Ricco [62] derived a closed-loop optimal control technique based on wall transpiration in the framework of a flat plate laminar boundary layer excited by freestream disturbances. They split the optimal control into two sequences obtained by marching the corresponding equations in the forward and backward directions, respectively. They found that the feedback sequence is more effective than the feed-forward sequence. Xiao & Papadakis [92] derived an optimal control algorithm in the framework of the full Navier-Stokes equations and Lagrange multipliers. The study aimed at delaying transition in a flat plate boundary layer excited by freestream vortical disturbances based on blowing and suction.

Although researchers have made significant progress in the last decades toward understanding the phenomenology behind the formation and influence of streamwise vortices and streaks in incompressible boundary layers, research into the compressible regime remains modest. Moreover, bypass transition at high upstream flow speeds remains largely unexplored. In this paper, we consider the control of streamwise vortices in boundary layers by conducting a parametric study of various flow parameters within a self-consistent asymptotic mathematical framework.

The first part of the present study focuses on investigating the streamwise vortices and the associated Görtler vortices developing in high-speed boundary layers over concave surfaces using an efficient model based on the nonlinear compressible boundary region equations (NCBREs).
The NCBREs represent the high Reynolds number asymptotic limit of the Navier-Stokes (N-S) equations under the assumption that the streamwise wavenumbers of the disturbances are much smaller than those associated with the crossflow disturbances. This set of equations is parabolic in the streamwise direction allowing for a straightforward marching procedure. The model imposes upstream conditions to trigger the instabilities by a low-amplitude disturbance at the wall in the form of wall transpiration. At the upstream boundary, the numerical algorithm uses mean flow profiles from a similarity solution applied to the compressible boundary layer equations (equivalent to the Blasius solution for an incompressible flow). As an effect of the wall’s concavity, the disturbances take the form of Görtler instabilities featuring counter-rotating pairs of vortices and associated streaks, with crossflow streamwise velocity contours resembling mushroom-like shapes. The first stage of the study analyzes and quantifies the evolution of these streaks via contour plots of velocity and temperature in crossflow planes as well as vortex energy, wall shear stress, and wall heat flow distributions versus the streamwise coordinate. As reported in the results section, the numerical algorithm applied to the NCBREs is very efficient, making it suitable for timely parametric studies.

The second part of the study extends the incompressible flow analysis of Sescu & Afsar [77] into the compressible regime. They implemented an optimal control approach to limit the growth of Görtler vortices developing in an incompressible laminar boundary layer flow over a concave wall. They based the adjoint boundary region equations and the associate optimality condition on the incompressible boundary region equations. They implemented the optimal control algorithm using wall deformation and velocity transpiration as flow control parameters. They showed that the optimal control algorithm effectively reduces the amplitude of the Görtler vortices, especially for the control based on wall displacement. On the other hand, this study derives the ACBRE and
the optimality condition for the NCBRE. The numerical algorithm uses wall transpiration velocity as the control variable with wall shear stress as the cost functional.
CHAPTER III
INVESTIGATION OF GÖRLTER VORTICES IN HIGH-SPEED FLOWS

3.1 Governing Equations

We consider a compressible boundary layer flow developing over a flat plate with the $x^*$ coordinate aligned with the streamwise direction, the $y^*$ coordinate in the wall-normal direction, and the $z^*$ coordinate in the spanwise/lateral direction. The freestream conditions are represented by: density $\rho^*_{\infty}$, pressure $p^*_{\infty}$, velocity $U^*_{\infty}$, temperature $T^*_{\infty}$, dynamic viscosity $\mu^*_{\infty}$, and thermal conductivity $k^*_{\infty}$.

3.1.1 Scalings

All dimensional spatial coordinates ($x^*, y^*, z^*$) are normalized by the spanwise wavelength of the freestream disturbance $\lambda^*$ set to 0.5cm, while the dependent variables by their respective freestream values. The pressure field is normalized by the dynamic pressure. Hence:

$$\bar{t} = \frac{t^*}{\lambda^*/U^*_\infty}; \quad \bar{x} = \frac{x^*}{\lambda^*}; \quad \bar{y} = \frac{y^*}{\lambda^*}; \quad \bar{z} = \frac{z^*}{\lambda^*}$$

$$\bar{u} = \frac{u^*}{U^*_\infty}; \quad \bar{v} = \frac{v^*}{U^*_\infty}; \quad \bar{w} = \frac{w^*}{U^*_\infty}; \quad \bar{\rho} = \frac{\rho^*}{\rho^*_{\infty}}$$

$$\bar{p} = \frac{p^* - p^*_{\infty}}{\rho^*_{\infty} V^2_{\infty}}; \quad \bar{T} = \frac{T^*}{T^*_{\infty}}; \quad \bar{\mu} = \frac{\mu^*}{\mu^*_{\infty}}; \quad \bar{k} = \frac{k^*}{k^*_{\infty}}$$
where $u^*$, $v^*$, and $w^*$ are the velocity components in the streamwise, wall-normal, and spanwise directions, respectively. $\rho^*$, $p^*$, $T^*$, $\mu^*$, and $k^*$ represent the dimensional density, pressure, temperature, dynamic viscosity, and thermal conductivity, respectively.

The Reynolds number is defined based on the spanwise wavelength, the Mach number, and the Prandtl number as

$$R_A = \frac{\rho^*_\infty U^*_\infty \lambda^*}{\mu^*_\infty}, \quad M_\infty = \frac{U^*_\infty}{c^*_\infty}, \quad Pr = \frac{\mu^*_\infty C_p}{k^*_\infty},$$

where $C_p$ is the specific heat at constant pressure and $\mu^*_\infty$, $c^*_\infty$, and $k^*_\infty$ are the freestream dynamic viscosity, speed of sound, and thermal conductivity, respectively.

For boundary layer flows over curved surfaces, the $O(1)$ global Görtler number is given by

$$G_A = \frac{R_A^2 \lambda^*}{r^*}$$

where $r^*$ is the radius of the curvature (set to 1.5m).

### 3.1.2 Compressible Navier-Stokes Equations (N-S)

For a full compressible flow, the primitive form of the Navier-Stokes equations in non-dimensional variables are as follows

$$\frac{D\tilde{\rho}}{Dt} + \tilde{\rho}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}} + \frac{\partial \tilde{w}}{\partial \tilde{z}}\right) = 0 \quad (3.1)$$

$$\frac{\tilde{\rho} D\tilde{u}}{Dt} = -\frac{\partial \tilde{\rho}}{\partial \tilde{x}} + \frac{1}{R_A} \frac{\partial}{\partial \tilde{x}} \left[ 2\tilde{\rho}\left(\frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{v}}{\partial \tilde{y}} - \frac{\partial \tilde{w}}{\partial \tilde{z}}\right) + \frac{\partial}{\partial \tilde{y}} \left(\mu \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) + \frac{\partial}{\partial \tilde{z}} \left(\mu \left(\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}}\right)\right)\right) \right] \quad (3.2)$$

$$\frac{\tilde{\rho} D\tilde{v}}{Dt} = -\frac{\partial \tilde{\rho}}{\partial \tilde{y}} + \frac{1}{R_A} \frac{\partial}{\partial \tilde{y}} \left[ 2\tilde{\rho}\left(\frac{\partial \tilde{v}}{\partial \tilde{y}} - \frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{w}}{\partial \tilde{z}}\right) + \frac{\partial}{\partial \tilde{x}} \left(\mu \left(\frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{\partial \tilde{u}}{\partial \tilde{y}}\right) + \frac{\partial}{\partial \tilde{z}} \left(\mu \left(\frac{\partial \tilde{v}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}}\right)\right)\right) \right] \quad (3.3)$$

$$\frac{\tilde{\rho} D\tilde{w}}{Dt} = -\frac{\partial \tilde{\rho}}{\partial \tilde{z}} + \frac{1}{R_A} \frac{\partial}{\partial \tilde{z}} \left[ 2\tilde{\rho}\left(\frac{\partial \tilde{w}}{\partial \tilde{z}} - \frac{\partial \tilde{u}}{\partial \tilde{x}} - \frac{\partial \tilde{v}}{\partial \tilde{y}}\right) + \frac{\partial}{\partial \tilde{x}} \left(\mu \left(\frac{\partial \tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{u}}{\partial \tilde{y}}\right) + \frac{\partial}{\partial \tilde{y}} \left(\mu \left(\frac{\partial \tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{v}}{\partial \tilde{y}}\right)\right)\right) \right] \quad (3.4)$$
\[
\begin{align*}
\hat{\rho} \frac{D \hat{T}}{D \hat{t}} &= \frac{1}{Pr_{R_1}} \left[ \frac{\partial}{\partial \hat{x}} \left( k \frac{\partial \hat{T}}{\partial \hat{x}} \right) + \frac{\partial}{\partial \hat{y}} \left( k \frac{\partial \hat{T}}{\partial \hat{y}} \right) + \frac{\partial}{\partial \hat{z}} \left( k \frac{\partial \hat{T}}{\partial \hat{z}} \right) \right] \\
&- (\gamma - 1) M_{\infty}^2 \left[ \frac{p}{\hat{P}} \left( \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right) - 2 \mu \left( \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right)^2 \right] \\
&+ (\gamma - 1) M_{\infty}^2 \frac{\mu}{R_1} \left[ 2 \left( \frac{\partial \hat{u}}{\partial \hat{x}} \right)^2 + 2 \left( \frac{\partial \hat{v}}{\partial \hat{y}} \right)^2 + 2 \left( \frac{\partial \hat{w}}{\partial \hat{z}} \right)^2 + \left( \frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} \right)^2 \right] 
\end{align*}
\]

where

\[
\frac{D}{D \hat{t}} = \frac{\partial}{\partial \hat{t}} + \hat{u} \frac{\partial}{\partial \hat{x}} + \hat{v} \frac{\partial}{\partial \hat{y}} + \hat{w} \frac{\partial}{\partial \hat{z}}
\]

is the substantial derivative (for what follows, the steady-state case for the N-S equations is considered, i.e. \( \partial / \partial \hat{t} = 0 \)). The pressure \( p \), the temperature \( T \) and the density \( \rho \) of the fluid are combined in the equation of state in non-dimensional form, \( \hat{p} = \hat{\rho} \hat{T} / \gamma M_{\infty}^2 \), under the assumption of non-chemically-reacting flows.

Other notations include the dynamic viscosity \( \mu \), the thermal conductivity \( k \), and the free-stream Mach number \( M_{\infty} = U_{\infty}/c_{\infty}^* \), which is otherwise \( O(1) \). \( \mu \) and \( k \) are expressed as a function of the temperature using the power law of viscosity in the form

\[
\begin{align*}
\mu &= T^b; \\
k &= \frac{C_p H}{Pr}
\end{align*}
\]

where \( b = 0.76 \) (Ricco & Wu [68]), \( C_1 = 1.458 \times 10^{-6} \), \( C_2 = 110.4 \), \( C_p = \gamma R / (\gamma - 1) \), \( \gamma = 1.4 \), and \( Pr = 0.72 \) for air.

### 3.1.3 Nonlinear Compressible Boundary-Region Equations (NCBREs)

We consider a compressible flow of uniform velocity \( U_{\infty}^* \) and temperature \( T_{\infty}^* \) past a flat or curved surface. The air is treated as a perfect gas so that the sound speed in the freestream \( c_{\infty}^* = \sqrt{\gamma RT_{\infty}} \), where \( \gamma = 1.4 \) is the ratio of the specific heats, and \( R = 287.05 Nm/(kgK) \) is the universal gas constant. The Mach number is assumed to be of order one. The asymptotic structure of the flow in figure 3.1 is composed of four regions as in Leib et al. [47], Ricco & Wu [67] or Marensi et al. [57].
Region I is close to the leading edge and outside the boundary layer. In this region, the flow is inviscid, and the disturbances are small perturbations of the base flow. Region II is the boundary layer close to the leading edge with a thickness much smaller than the freestream disturbances spanwise wavelength, $\lambda$.

The linearized boundary region equations now govern the perturbation field solutions in which the spanwise diffusion is of the same order of magnitude as that in the wall-normal direction. Region III is the fully viscous region governed by the nonlinear compressible boundary region equations (NCBRE) where the boundary layer thickness is of the same order of magnitude as $\lambda$. Region IV is inviscid since the viscous effects are negligible. In the latter region, the displacement effect due to the increased thickness of the viscous layer influences the flow at leading order.

We derive the NCBRE from the full steady-state compressible N-S equations. Thanks to the characteristics of the flow in region III, we re-scale the streamwise distance and time coordinate at which the vortex system forms by the following $O(1)$ variables: $x = \bar{x}/R_\lambda$, and the time as $t = \bar{t}/R_\lambda$. Note that the distance
in the wall-normal and spanwise directions are the same, \( y = \bar{y}, \ z = \bar{z} \). Another thing to mention is that, in this region, the crossflow velocity component is small compared to the streamwise velocity component, and pressure variations are negligible. Appropriate dominant balance considerations suggest that the dependent variables in this region must also re-scale as follows:

\[
\begin{align*}
    u &= \bar{u}; & v &= \bar{v}/R_\lambda; & w &= \bar{w}/R_\lambda; & \rho &= \bar{\rho}; \\
    p &= \bar{p}/R^2_\lambda; & T &= \bar{T}; & \mu &= \bar{\mu}; & k &= \bar{k}.
\end{align*}
\]

Inserting the above re-scaled dependent and independent variables into the full unsteady continuity (3.1), Navier Stokes (3.2)-(3.4) and energy equation (3.5) gives

\[
\begin{align*}
    \mathbf{V} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{V} &= 0 \quad \text{(3.8)} \\
    \rho \mathbf{V} \cdot \nabla u &= \nabla_c \cdot (\mu \nabla_c u) \quad \text{(3.9)} \\
    \rho \mathbf{V} \cdot \nabla v + G_\lambda u^2 &= -\frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left[ \frac{2}{3} \mu \left( \frac{3}{2} \frac{\partial v}{\partial y} - \nabla \cdot \mathbf{V} \right) \right] + \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left[ \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \quad \text{(3.10)} \\
    \rho \mathbf{V} \cdot \nabla w &= -\frac{\partial p}{\partial z} + \frac{\partial}{\partial z} \left[ \frac{2}{3} \mu \left( \frac{3}{2} \frac{\partial w}{\partial z} - \nabla \cdot \mathbf{V} \right) \right] + \frac{\partial}{\partial x} \left( \mu \frac{\partial u}{\partial z} \right) + \frac{\partial}{\partial y} \left[ \mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right] \quad \text{(3.11)} \\
    \rho \mathbf{V} \cdot \nabla T &= \frac{1}{Pr} \nabla_c \cdot (k \nabla_c T) + (\gamma - 1)M^2_{\infty} \mu \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] \quad \text{(3.12)}
\end{align*}
\]

where \( \mathbf{V} \) is the velocity vector and \( \nabla_c \) is the crossflow nabla operator:

\[
\begin{align*}
    \mathbf{V} &= u \hat{i} + v \hat{j} + w \hat{k}; & \nabla_c &= \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k},
\end{align*}
\]

and the temporal and streamwise derivatives are absent because they remain \( O(1/R_\lambda) \) and are therefore asymptotically small for the vortex flow in region III. The effect of the wall curvature is contained in the term involving the global Görtler number, \( G_\lambda = O(1) \).

This set of equations is parabolic in the streamwise direction and elliptic in the spanwise direction. Appropriate initial/upstream and boundary conditions are necessary to close the problem; these conditions
The numerical algorithm developed in Es-Sahli et al. [20] is used to solve the NCBRE.

### 3.2 Flow Domain and Numerical Algorithm

The three-dimensional flow domain of region III begins at the initial streamwise location $X_0$ and extends $50 \times \lambda^*$ in the streamwise direction, $10 \times \lambda^*$ in the wall-normal direction, and $1 \times \lambda^*$ in the spanwise direction, forming a rectangular prism shape. The effect of the wall curvature is introduced by the global Görtler number term, $G_{\lambda}$, in equation 3.10.

As for the numerical setup of the simulations, the model employs second and fourth-order finite-difference schemes to discretize the spatial derivatives in the wall-normal and spanwise directions, respectively. Periodic conditions are implemented in the spanwise direction to avoid compromising stability, and a staggered arrangement is employed in the wall-normal direction to prevent decoupling between the velocity and pressure. This arrangement visually extends the domain to $2 \times \lambda^*$ in the spanwise direction, as demonstrated in subsequent figures in the results section. The model applies a first-order finite-difference marching scheme in the streamwise direction and converges the equations using a nonlinear time relaxation method.

Appropriate initial/upstream and boundary conditions are necessary to close the problem. In this work, the model initiates the perturbations in the boundary layer using a small amplitude transpiration velocity ($v_t$) at the wall in the form:

$$v_t = A \sin \left( \pi \frac{(x - x_s)}{(x_e - x_s)} \right)^2 \cos \left( \pi \frac{z}{\Lambda} \right)$$

(3.14)

Here, $A$ is the amplitude of the perturbation, $x_s = 1.5 \lambda^*$ and $x_e = 4.5 \lambda^*$ are the start and end streamwise locations of the perturbation, respectively. The initial streamwise location of region III, denoted by $X_0$, is set to $0.8 m$ from the leading edge of the boundary layer. The streamwise location of the control starting
The NCBREs are numerically solved using an algorithm similar to the one employed by Sescu & Thompson [74] in the incompressible regime. In the compressible regime, the continuity equation does not result in the usual divergence-free condition, thus making the compressible set of equations numerically stiff. Instead, it is an equation for density. Therefore, the same numerical algorithm becomes more efficient when applied to the compressible regime. We impose vanishing gradients for all dependent variables along the top boundary.

We generate the mean inflow condition from a similarity solution obtained using the Dorodnitsyn-Howarth coordinate transformation; \( \bar{Y}(x, y) = \int_0^y \rho(x, \tilde{y}) d\tilde{y} \). We define the similarity variable as \( \eta = \tilde{Y} \left( Re_x / 2x \right)^{1/2} \), where \( Re_x \) is the Reynolds number calculated based on the freestream velocity and the distance from the leading edge \( X_0 \)
We express the base velocity and temperature as follows;

\[ U = F'(\eta), \quad V = (2xRe_{x})^{-1/2}(\eta_{c}TF' - TF), \quad T = T(\eta) \quad (3.15) \]

where the prime superscript represents differentiation with respect to \( \eta \), and \( \eta_{c} = 1/T \int_{0}^{\eta} T(\tilde{\eta})d\tilde{\eta} \). \( F \) and \( T \) satisfy the following coupled equations

\[
\left( \frac{\mu}{T} F'' \right)' + FF'' = 0, \\
\frac{1}{Pr} \left( \frac{\mu}{T} T' \right)' + FT' + (\gamma - 1)M^2 \frac{\mu}{T} F'' = 0, \quad (3.16)
\]

subject to the boundary conditions \( F(0) = F'(0) = 0, \quad T'(0) = 0, \quad F' \rightarrow 1, \quad T \rightarrow 1 \rightarrow \text{as} \ \eta \rightarrow \infty \).

We numerically solve equations (3.16) to determine \( F \) and \( T \), which we then use to obtain the mean inflow condition using equations (3.15). Figure 3.3 shows this solution for a Mach number of 3 and both isothermal and adiabatic wall conditions.

Figure 3.3
Base flow solution.
3.3 NCBREs Model Validation

We validate the accuracy of our method by comparing our results to the DNS results of Song et al. [79]. We use a disturbance similar to that employed in their study with an amplitude of 0.5% for the transpiration velocity; note, however, that they imposed the disturbance at the inlet boundary, whereas in our simulation, we impose it at the wall. In Song et al., the freestream Mach number is 6.5 and the radius of curvature is 1.6 m, the same as in our parametric study. In figure 3.4, we plot the scaled amplitude, \( A = \max_{y,z}(T') \), based on the temperature disturbance calculated from each \((y, z)\)-plane at every fixed streamwise location \( (A_0 \text{ in figure 3.4 is the same amplitude calculated at the } x\text{-coordinate of the imposed disturbance}) \). Our results in figure 3.4 compare very well with the DNS in Song et al. (note that the streamwise coordinate is scaled to match the DNS range).

![Figure 3.4](image)

Comparison to DNS results of Song et al. [79].

3.4 Results: High-speed Boundary Layer Flow

We consider Görtler vortices developing in high-speed boundary layers, with the Mach number ranging from supersonic, \( M_{\infty} = 2 \), to hypersonic conditions, \( M_{\infty} = 6 \). We neglect chemical reactions inside the boundary layer at hypersonic speeds. We vary the spanwise separation of the vortices between 0.3 cm and
0.7 cm. We calculate the similarity solution imposed at the upstream boundary using the Reynolds number based on the distance from the leading edge, which is maintained constant for a given Mach number (note that the Reynolds number based on the spanwise separation differs for each $\lambda$). The radius of curvature of the concave surface is 1.6 m for all cases involved in the parametric study. The boundary layer flow is excited by a small disturbance applied to the vertical velocity imposed at the wall, with an amplitude of 0.2% of the freestream velocity (the form of this disturbance is given in equation (3.14)). Figure 3.2 shows a sketch of the flow domain, with region III highlighted in blue. We impose the similarity type velocity and temperature profiles for a compressible boundary layer at the upstream boundary, located on the left-hand side of the sketch. At the wall, we impose the no-slip boundary condition for the velocity and either an isothermal or adiabatic boundary condition for the temperature field. We impose vanishing gradients at the top boundary, and a symmetry condition along the spanwise direction since we only simulate one streamwise vortex (corresponding to half of the mushroom shape), belonging to the pair of counter-rotating vortices.

3.4.1 Effects of Mach Number and Spanwise Separation

We focus on the parametric study by varying the Mach number and the spanwise separation. In figures 3.5 and 3.6, we show temperature contour plots in consecutive cross-stream planes, for a Mach number $M_\infty = 3$ and a spanwise separation $\lambda = 0.4$ cm. The contour plots in figure 3.5 correspond to the adiabatic wall condition, while those in figure 3.6 to the isothermal condition. They both illustrate the streamwise development of the velocity magnitude and the temperature field of the Görtler vortices, which display the mushroom-shaped characteristic structures. For both wall conditions, blue and green colors in the velocity magnitude contours correspond to the low-speed streaks, while the red color regions are associated with high-speed streaks. The temperature contour plots of the adiabatic wall condition indicate that the low-speed streaks are associated with an increase in temperature (in red or yellow). The high-speed streaks are characterized by a lower temperature level (shown here in green). For the isothermal wall-condition case,
however, it appears as if the high-temperature region of the structures is sandwiched between two cooler regions belonging to both low and high-speed streaks. This ‘sandwich’ phenomenon leads to a significant difference between the two wall conditions in terms of the height that these mushroom-shaped structures take. In the adiabatic wall-condition case, for example, a significant increase in temperature at the wall generates more convection, contributing to the growth of ‘thermals’ inside the structures. In contrast, the cooler regions at the wall result in less convection and therefore smaller vortex (i.e. mushroom) structures in the isothermal case.

Figure 3.5

Contour plots of velocity magnitude (a) and temperature (b) in crossflow planes for \( M_\infty = 3 \), \( \lambda = 0.4 \), and adiabatic wall condition.
Figures 3.7 and 3.8 show selected velocity and temperature profiles along the wall-normal direction at the spanwise location corresponding to the center of the mushroom-shape structures and the streamwise location corresponding to the highest vortex energy (see figure 3.9). In figure 3.7, we plot the profiles at different Mach numbers (keeping the spanwise separation constant at $\lambda = 0.4$ cm), whereas in figure 3.8, we repeat this plot at different spanwise separations (keeping the Mach number constant at $M_\infty = 2$). As we increase the Mach number, the ‘height’ of the streamwise vortices (i.e the size of the mushroom-shape structures) increases (figs. 3.7a and 3.7c) as a result of the increase in the temperature close to the wall (figs. 3.7b and 3.7d). The increase in size is more evident in the adiabatic case, confirming the aforementioned
observations. For the highest Mach number ($M_{\infty} = 6$) the temperature near the wall increases by a factor of 6 for the adiabatic case and 2.5 in isothermal conditions, compared to the lowest considered Mach number of $M_{\infty} = 2$. It is evident from figure 3.8 that as we increase the spanwise separation, the vertical size of the mushroom-shape structures decreases (almost linearly in fact) for both streamwise velocity and temperature fields.
Velocity and temperature boundary layer profiles in $z = 0$, for different Mach numbers: (a) velocity, adiabatic wall; (b) temperature, adiabatic wall; (c) velocity, isothermal wall; (d) temperature, isothermal wall.

Figure 3.7
Velocity and temperature boundary layer profiles in $z = 0$, for different spanwise separations: a) velocity, adiabatic wall; b) temperature, adiabatic wall; c) velocity, isothermal wall; d) temperature, isothermal wall.

3.4.1.1 Streamwise Vortex Kinetic Energy

We quantify the vortex energy as

$$E(x) = \int_{z_2}^{z_1} \int_0^{\infty} \left[ |u(x, y, z) - u_m(x, y)|^2 + |v(x, y, z) - v_m(x, y)|^2 + |w(x, y, z) - w_m(x, y)|^2 \right] \, dz \, dy,$$

(3.17)
where $u_m(x, y)$, $v_m(x, y)$, and $w_m(x, y)$ are the spanwise mean components of velocity, and $z_1$ and $z_2$ are the coordinates of the spanwise domain boundaries.

In figure 3.9, we plot the vortex energy from equation (13) against the streamwise coordinate, $x$. For each sub-figure, we fix the Mach number at a constant value and vary the spanwise separation, $\lambda$. At all Mach numbers, we find that, as we increase the spanwise separation, there is a noticeable reduction in the scaled energy for both isothermal (in black) and adiabatic (in red) wall conditions; the energy saturation location moves downstream as the spanwise separation increases. The drop in the vortex energy associated with the increase in the spanwise separation becomes less evident as the Mach number increases, particularly in the adiabatic wall condition as the curves of the different $\lambda$ values coalesce until they almost fall on top of each other at the highest Mach number (figure 3.9e). One possible explanation of this observation is that the reduction in the vortex energy caused by increasing $\lambda$ diminishes when the convection levels are increased at the wall, particularly for high Mach numbers (note that as the Mach number increases, the temperature in proximity to the wall increases considerably; see figure 3.7b). Another interesting aspect that we can extract from figure 3.9 is the effect that an adiabatic wall condition has on the vortex energy development: i.e., it appears that the energy growth is delayed in the adiabatic case, while the peak energy appears to be fixed at the level for the isothermal case in relatively low supersonic Mach number conditions (figures 3.9a and 3.9b). On the other hand, this level is slightly higher at high supersonic or hypersonic flow conditions (figures 3.9c, 3.9d and 3.9e).
Vortex energy distribution along the streamwise direction for both isothermal (in black) and adiabatic (in red) wall conditions: a) $M_\infty = 2.0$; b) $M_\infty = 3.0$; c) $M_\infty = 4.0$; d) $M_\infty = 5.0$; e) $M_\infty = 6.0$

3.4.1.2 Spanwise Averaged Wall Shear Stress

The spanwise averaged wall shear stress is evaluated using the integral
\[
\tau_w(x) = \frac{1}{(z_2 - z_1)} \int_{z_1}^{z_2} \frac{\partial u}{\partial y} \Big|_{y=0} (x, 0, z) \, dz 
\]  \hspace{1cm} (3.18)

is plotted in figure 3.10 against the streamwise coordinate for both isothermal and adiabatic conditions. For each case, we fix the Mach number at a constant value and vary the spanwise separation. The results show that, in all considered cases, the wall shear stress increases as the spanwise separation increases. Also, as expected, the wall shear stress of the adiabatic wall condition is lower than that of the isothermal case as a result of the high level of heating in proximity to the wall (this was observed in previous studies, such as Spall and Malik [80], Elliot [18], Sescu et al. [75], and Es-Sahli et al. [19] etc.). The jump in the wall shear stress coincides (approximately) with the same location associated with the energy saturation initiation point. The effect of increasing \( \lambda \) on the shear stress becomes less apparent in the case of the highest Mach number (i.e. \( M_\infty = 6 \)) in adiabatic wall conditions.
Figure 3.10

Spanwise averaged wall shear stress distribution along the streamwise direction: a) $M_\infty = 2.0$; b) $M_\infty = 3.0$; c) $M_\infty = 4.0$; d) $M_\infty = 5.0$; e) $M_\infty = 6.0$; isothermal (in black) and adiabatic (in red) wall conditions
3.4.1.3 Spanwise Averaged Wall Heat Flux

The spanwise averaged wall heat flux is calculated according to

\[ q_w(x) = -\frac{1}{(z_2 - z_1)} \int_{z_1}^{z_2} \frac{\partial T}{\partial y} \bigg|_{y=0} (x, 0, z) \, dz, \]  

(3.19)

which is plotted in figure 3.11 for the isothermal wall condition (the wall heat flux is zero for the adiabatic counterpart) as a function of the streamwise coordinate. It is evident from this figure that the wall heat flux decreases as the spanwise separation increases. Moreover, we notice a decay of the wall heat flux in the streamwise coordinate range where the energy saturation takes place.
Spanwise averaged wall heat flux distribution along the streamwise direction: a) \( M_\infty = 2.0 \); b) \( M_\infty = 3.0 \); c) \( M_\infty = 4.0 \); d) \( M_\infty = 5.0 \); e) \( M_\infty = 6.0 \); isothermal (in black) and adiabatic (in red) wall conditions

### 3.4.1.4 CPU Time

The parabolic character of the NCBRE framework allows the solution to be determined efficiently by a marching procedure in the streamwise direction. This makes the numerical algorithm very fast, proving it suitable for parametric studies that can be conducted in a timely manner. In table 3.1, we show the CPU
time for different cases for a grid resolution of 600 points in the streamwise direction, 201 points along $y$, and 81 points along $z$; the spanwise length is limited to the spanwise separation of the vortices, while the in the wall-normal direction the domain size is 5 times the spanwise separation, with the grid stretched toward the upper boundary. In addition, the method does not require large computational resources.

Table 3.1
CPU time for different cases

<table>
<thead>
<tr>
<th>Mach number</th>
<th>CPU time (isothermal)</th>
<th>CPU time (adiabatic)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.9 min</td>
<td>2.1 min</td>
</tr>
<tr>
<td>3.0</td>
<td>2.2 min</td>
<td>2.3 min</td>
</tr>
<tr>
<td>4.0</td>
<td>2.6 min</td>
<td>2.7 min</td>
</tr>
<tr>
<td>5.0</td>
<td>2.7 min</td>
<td>2.9 min</td>
</tr>
<tr>
<td>6.0</td>
<td>2.9 min</td>
<td>3.1 min</td>
</tr>
</tbody>
</table>

3.4.2 Effect of wall cooling or heating

We conduct a parametric study examining the effect of varying the Mach number and wall temperature on the development of secondary instabilities. Figures 3.12 and 3.13 display contour plots of temperature in consecutive cross-flow planes for different wall temperatures at $M = 2$ and $M = 5$, respectively. The contours illustrate the streamwise development of the temperature field of the Görtler vortices characterized by the mushroom-shaped structures. Although, from a practical standpoint, it makes more sense to only consider cooling for high Mach numbers, especially in the hypersonic regime since the flow temperature can become higher than the applied wall temperatures considered in this study, we sought to include all results for consistency. In the contour plots, blue and green correspond to the low-temperature streaks whereas red regions are associated with high-temperature streaks. Es-Sahli et al. [20] reported that the high- and low-temperature streaks are respectively associated with low- and high-velocity streaks. In figure 3.12 ($M = 2$), contour plots in (a), (b), and (c) correspond to cooling while (d), (e), and (f) to heating, see
figure 3.15a, whereas in figure 3.13 ($M = 5$), only (f) correspond to heating and the rest of the contour plots correspond to cooling, see figure 3.15d.
Contour plots of temperature in crossflow planes for different wall-temperatures: a) 100 K, b) 200 K, c) 300 K, d) 600 K, e) 900 K, and f) 1500 K for a Mach number M=2.0
For the supersonic Mach number case, figure 3.12, the size (height and width) of the mushroom-shaped structures corresponding to the heating cases are significantly larger than the ones corresponding to the cooling cases. For the heating condition cases, the size of the mushroom-shaped structures is directly proportional to the wall temperature due to the growth of thermals inside the structures. The thermals are created by the upward flow forced by the juxtaposed counter-rotating vortices. They grow as the wall temperature increases (with respect to the adiabatic wall temperature). As the wall temperature is decreased, however, these thermals reduce in size resulting in smaller structures and consequently a smaller boundary layer thickness (although the structures corresponding to the 200 \( K \) wall temperature appear narrower than the 100 \( K \) cases, which is an interesting observation suggesting further examination). On the other hand, contour plots of the hypersonic Mach number case, figure 3.13, show a steady trend of the mushroom-shape structures gradually increasing in size as the wall temperature is increased with the heating case boundary layer thickness being the largest.
Figure 3.13

Contour plots of temperature in crossflow planes for different wall-temperatures: a) 100 $K$, b) 200 $K$, c) 300 $K$, d) 600 $K$, e) 900 $K$, and f) 1500 $K$ for a Mach number $M=5.0$
In figures 3.14 and 3.15, we plot selected profiles of velocity and temperature at different Mach numbers in the normal direction at the streamwise and spanwise coordinates corresponding, respectively, to the maximum vortex energy (see figure 3.16) and the center of the domain. For all Mach numbers, the velocity profiles indicate a reduction in the boundary layer thickness corresponding to the cooling cases (< adiabatic wall temperature) and an increase in the boundary layer thickness for cases corresponding to the heating cases (> adiabatic wall temperature). This quantitative result agrees with and supports the qualitative observations of the contour plots. However, the profiles start coalescing and the difference (increase/decrease) between the various wall-temperature cases becomes less significant as the Mach number increases.
Figure 3.14
Streamwise velocity profiles for wall-temperatures; 100 \( K \), 200 \( K \), 300 \( K \), 600 \( K \), 900 \( K \), 1200 \( K \), and 1500 \( K \) at different freestream Mach number conditions: a) \( M = 2.0 \), b) \( M = 3.0 \), c) \( M = 4.0 \), d) \( M = 5.0 \), e) \( M = 6.0 \)
3.4.2.1 Streamwise Vortex Kinetic Energy

In figures 3.16 and 3.17, we plot the vortex energy from equation 3.17 and the energy growth \((1/x \times dE/dx)\) against the streamwise coordinate, \(x\). We notice the same trend here as the cooling condition depicts
less vortex energy and energy growth levels compared to heating. As for the adiabatic case, the vortex energy as well as the energy growth level increase significantly as the streamwise Mach number is increased. We also notice a delay in both the vortex energy and energy growth associated with the adiabatic case for the supersonic Mach number cases (a), (b), and (c) as opposed to the hypersonic cases (d) and (e).
Vortex energy profiles for wall-temperatures; 100 $K$, 200 $K$, 300 $K$, 600 $K$, 900 $K$, 1200 $K$, and 1500 $K$ at different freestream Mach number conditions: a) $M = 2.0$, b) $M = 3.0$, c) $M = 4.0$, d) $M = 5.0$, e) $M = 6.0$
3.4.2.2 Spanwise Averaged Wall Shear Stress

We calculate the spanwise averaged wall shear stress using equation 3.18 and plotted against $x$ in figure 3.18. Here the trend is inverted: considering each Mach number case individually, the shear stress associated
with the cooling and heating conditions is, respectively, higher and lower than that of the adiabatic case. However, as the streamwise Mach number is increased, the adiabatic flow temperature increases resulting in a higher convection level at the wall, thus causing the shear stress to further decrease compared to the other isothermal cases. The shear stress gradually increases for all cases. The jump in the wall shear stress coincides (approximately) with the location at which the energy saturation initiates.
Figure 3.18

Spanwise averaged wall shear stress profiles for wall-temperatures; 100 $K$, 200 $K$, 300 $K$, 600 $K$, 900 $K$, 1200 $K$, and 1500 $K$ at different freestream Mach number conditions: a) $M = 2.0$, b) $M = 3.0$, c) $M = 4.0$, d) $M = 5.0$, e) $M = 6.0$

3.4.2.3 Spanwise Averaged Wall Heat Flux

We use equation 3.19 to calculate the spanwise averaged wall heat flux and plot it in figure 3.19 for the cooling and heating wall conditions as a function of the streamwise coordinate (the null adiabatic heat
flux is illustrated by the black dotted line). As expected, the cooling and heating cases respectively result in negative and positive heat flux values. The increase in the freestream Mach number results in a drop in the heat flux regardless of the wall-temperature condition, however, the reduction in the cooling cases is more significant.
Spanwise averaged wall heat flux profiles for wall-temperatures; 100 K, 200 K, 300 K, 600 K, 900 K, 1200 K, and 1500 K at different freestream Mach number conditions: a) $M = 2.0$, b) $M = 3.0$, c) $M = 4.0$, d) $M = 5.0$, e) $M = 6.0$

### 3.5 Results: High-speed Curved Free-Shear Layer Flow

In this section, we present the results of the curved free shear layer numerical simulations. The flow domain is split into ‘fast’ and ‘slow’ streams both having velocities $V_f$ and $V_s$, which we define as; $V_f = V_{\infty}$
and $V_s = (1 - \Delta V)V_\infty$, where $\Delta V$ (20\%, 30\%, 40\%, and 50\%) is the relative velocity difference. We set the fast stream flow Mach number to 2.0, 4.0, and 6.0. We consider three values for the shear layer thickness $\delta$; 0.2, 0.3, and 0.4. The velocity difference and the shear layer thickness are set at the inlet boundary. The Reynolds number (per unit length) and the global Görtler number are commensurate with the freestream velocity. Figure 3.20 shows the flow domain and the grid; we use grid stretching in the radial direction, towards the far-field boundaries, and equally spaced grid points in the streamwise and spanwise directions. Note that there is no grid in the streamwise direction since the equations are parabolic and the solution advances in a marching procedure, but the spatial step along the streamwise marching direction is constant. Centrifugal instabilities are excited by the non-dimensional artificial disturbance in equation 3.14 imposed on the base flow at the inflow boundary. Similar to the boundary layer case, the effect of the wall curvature is introduced by the global Görtler number term, $G_A$, in equation 3.10.
3.5.1 Effects of Mach Number and Spanwise Separation

In figure 3.21, we plot vortex kinetic energy distribution from equation 3.17 for different values of $\Delta V$, $\delta$, and $A$ at $M = 2$. Results show that the kinetic energy of the vortex system is directly proportional to $\Delta V$ as it is highest for $\Delta V = 50\%$ and decreases as $\Delta V$ is reduced for all considered cases. Moreover, the streamwise location of the energy saturation (the point at which the energy starts to level off) moves farther downstream as $\Delta V$ decreases. On the other hand, it is inversely proportional to $\delta$ due to viscous effects, which are more prominent in thicker shear layers. However, the energy reduction due to the increase in the shear layer thickness is very insignificant, if not negligible, as a 100% increase in $\delta$ results in less than 1% drop in $E$, see figure 3.24. Increasing the magnitude of the disturbance amplitude $A$ results in an increase in the kinetic energy, although not significant as a 100% increase in $A$ results in less than 1% increase in $E$. Increasing $A$
also shrinks the gap between the energy curves of the considered $\Delta V$ values, which indicates that the effect of $\Delta V$ becomes less important as the amplitude of the disturbance $A$ increases. Similar observations are seen for the $M_\infty = 4$ and $M_\infty = 6$ cases in figures 3.22 and 3.23, see also figures 3.25 and 3.26.

Figure 3.21

Vortex energy distribution of different parametric settings for the $M = 2$ case.
Figure 3.22

Vortex energy distribution of different parametric settings for the $M = 4$ case.
Figure 3.23

Vortex energy distribution of different parametric settings for the $M = 6$ case.
Effect of the shear layer thickness $\delta$ variation on the vortex energy distribution for the $M = 2$ case.

Effect of the shear layer thickness $\delta$ variation on the vortex energy distribution for the $M = 4$ case.
Figures 3.27-3.29 present consecutive contour plots depicting the magnitude of crossflow velocity for different Mach numbers and disturbance amplitudes, with $\Delta V$ set at 30%. The color scheme represents the flow velocity, with dark red indicating the fast stream and light blue representing the slow stream. To accurately characterize the shape of the centrifugal instabilities, we identify two types of structures: "primary" and "secondary." The primary flow structure refers to the mushroom-like formation that evolves as the main instability, as exemplified by figure 3.27-d. On the other hand, the secondary structures are elongated features that emerge from the edges of the primary flow structure, as seen in figure 3.27-l.

Increasing the disturbance amplitude amplifies the amplitude of the mushroom-like primary structures and makes the secondary structures more prominent, implying enhanced mixing. Comparing the centrifugal instabilities at the same streamwise coordinate across various Mach numbers, we observe a delay in the growth of the mushroom-like structures as $M_\infty$ increases. For instance, in figure 3.27-e, with $M_\infty = 2$, the elongated secondary structures have already begun to form, whereas in figure 3.29-e, at $M_\infty = 6$, only the primary flow structure is visible without any secondary structures. This finding suggests that, for high Mach

\footnote{To avoid repetition, we only present one value for $\Delta V$. However, it is worth noting that similar development patterns of the centrifugal instabilities are observed for other $\Delta V$ values as well.}
number free shear layer flows, the same mixing efficiency is achieved further downstream. From a numerical perspective, simulating higher Mach number cases necessitates a longer computational domain (larger grid size) to attain comparable mixing efficiency, thereby increasing the computational cost.

Figure 3.27

Contour plots of the streamwise velocity $u$ at different streamwise locations for the $M = 2$ case; top row $A = 0.02$, bottom row $A = 0.04$
Contour plots of the streamwise velocity $u$ at different streamwise locations for the $M = 4$ case; top row $A = 0.02$, bottom row $A = 0.04$
Figure 3.29

Contour plots of the streamwise velocity $u$ at different streamwise locations for the $M = 6$ case; top row $A = 0.02$, bottom row $A = 0.04$
CHAPTER IV
OPTIMAL CONTROL PROBLEM IN THE NONLINEAR REGIME

While it is common to formulate an optimal flow control problem in the framework of a dynamical system described by a set of equations that are parabolic in time, in the present case, the time direction is replaced by the streamwise direction due to the parabolic nature of the NCBREs in the streamwise direction. This requires streamwise marching to obtain a solution.

4.1 Generic Optimal Control Formalism

For brevity, equations (3.8)-(3.12) are written in a more generic and compact form:

\[ \mathcal{G}(\mathbf{q}) = 0, \]  
\[ \text{(4.1)} \]

with initial and boundary conditions

\[ \mathbf{q}(0, y, z) = \mathbf{q}_0(y, z) \]  
\[ \text{(4.2)} \]

\[ \mathbf{q}(x, 0, z) = \phi, \quad \lim_{y \to \infty} \mathbf{q}(x, y, z) = \mathbf{q}_B(x, z), \]  
\[ \text{(4.3)} \]

along the wall-normal direction \( y \), and periodic or symmetry boundary conditions in the spanwise direction \( z \).

In equation (4.1), the differential operator \( \mathcal{G}() \) represents the NCBREs in abstract notation, \( \mathbf{q} = (\rho, u, v, w, T) \) is the vector of state variables, \( \phi \) represents the control variable associated with the boundary conditions (such as the transpiration velocity at the wall, \( v_w \)), \( \mathbf{q}_0(y, z) \) represents the upstream or initial condition at
\( x = 0 \), and \( \mathbf{q}_\infty \) is a given function that specifies the boundary condition at infinity. An objective (or cost) functional is defined as follows:

\[
\mathcal{J}(\mathbf{q}, \phi) = \mathcal{E}(\mathbf{q}) + \sigma \left( \| \phi_x \|^{\beta_2} + \| \phi \|^{\beta_2} \right),
\]

where \( \mathcal{E}(\mathbf{q}) \) is a specified target function to be minimized (e.g., the energy of the disturbance, or the wall shear stress; the latter is considered in this study), the second term on the right-hand side of (4.4) is a penalization term depending on the norm of the control variable (usually, this quantity place a constraint on the magnitude of the admissible control variable, since it cannot increase or decrease indefinitely), \( \sigma \) and \( \beta \) are given constants, and the subscript \( x \) denotes derivative with respect to \( x \). The norm \( \| \| \) in equation (4.4) is now associated with an appropriate inner product of two complex functions, \( f \) and \( g \), defined as

\[
\langle f, g \rangle = \int_0^{X_f} f^* g dX
\]

in the space \([0, X_f]\), with \( X_f \) being the terminal streamwise location (the star in (4.5) denotes complex conjugate).

One common technique to convert a (nonlinear) constrained optimization problem into an unconstrained problem is to utilize the method of Lagrange multipliers. This method entails the creation of a new function called the Lagrangian, which is a combination of the objective function and constraints. The benefit of this technique is that it enables the use of standard unconstrained optimization algorithms, which often be more efficient and accurate than constrained optimization methods. Implementing the Lagrange multipliers approach streamlines the optimization process and can lead to superior results in a more efficient manner (see, for example, Joslin et al. [38], Gunzburger [28], Zuccher et al. [96]). With this in mind, the following Lagrangian is considered

\[
\mathcal{L}(\mathbf{q}, \phi, \mathbf{q}^a) = \mathcal{J}(\mathbf{q}, \phi) - \langle \mathcal{G}(\mathbf{q}), \mathbf{q}^a \rangle,
\]

\( 65 \)
where $\mathbf{q}^a$ is the vector of Lagrange multipliers $(\rho^a, u^a, v^a, w^a, T^a)$, also known as the adjoint vector. In other words, the Lagrange multipliers are introduced in order to transform the minimization of $\mathcal{J}(\mathbf{q}, \phi)$ under the constraint $\mathcal{G}(\mathbf{q}) = 0$ into the unconstrained minimization of $\mathcal{L}(\mathbf{q}, \phi, \mathbf{q}^a)$. The unconstrained optimization problem can be formulated as:

*Find the control variable $\phi$, the state variables $\mathbf{q}$, and the adjoint variables $\mathbf{q}^a$ such that the Lagrangian $\mathcal{L}(\mathbf{q}, \phi, \mathbf{q}^a)$ is a stationary function, that is*

$$
\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}} \delta \mathbf{q} + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial \mathbf{q}^a} \delta \mathbf{q}^a = 0 \quad (4.7)
$$

where

$$
\frac{\partial \mathcal{L}}{\partial a} \delta a = \frac{\mathcal{L}(a + \epsilon \delta a) - \mathcal{L}(a)}{\epsilon} \quad (4.8)
$$

represents directional differentiation in the generic direction $\delta a$. All directional derivatives in (4.7) must vanish, providing different sets of equations:

- adjoint BRE equations are obtained by taking the derivative with respect to $\mathbf{q}$,

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0 \quad \Rightarrow \quad \mathcal{G}^a(\mathbf{q}^a) = 0 \quad (4.9)
$$

- optimality conditions are obtained by taking the derivatives with respect to $\phi$,

$$
\frac{\partial \mathcal{L}}{\partial \phi} = 0 \quad \Rightarrow \quad \mathcal{O}(\mathbf{q}^a, \mathbf{q}, \phi) = 0 \quad (4.10)
$$

- the original BRE equations are obtained by taking the derivative with respect to $\mathbf{q}^a$,

$$
\frac{\partial \mathcal{L}}{\partial \mathbf{q}^a} = 0 \quad \Rightarrow \quad \mathcal{G}(\mathbf{q}, \phi) = 0. \quad (4.11)
$$

Equations (4.9)-(4.11) form the optimal control system that can be utilized to determine the optimal states and the control variables. One can note that the stationarity of the Lagrangian with respect to the adjoint variables $\mathbf{q}^a = (\rho^a, u^a, v^a, w^a, T^a)$ essentially yields the original state equations, while the stationarity
with respect to the state variables \( q = (\rho, u, v, w, T) \) yields the adjoint equations that depend on the state variables. The relationship between the state variables and the adjoint variables can be expressed by the following adjoint identity:

\[
\langle G(q), q^a \rangle = \langle q, G^a(q^a) \rangle + B(\phi)
\]  

(4.12)

where the last term, \( B \), represents a residual from the boundary conditions.

4.2 Governing Equations

4.2.1 Adjoint Nonlinear Compressible Boundary Region Equations

In the particular case of a boundary layer flow over a flat or concave surface with wall transpiration, the ACBREs are derived starting with the integral

\[
\mathcal{L}(\rho, u, v, w, T, p, \mu, k, \nu, \rho^a, u^a, v^a, w^a, T^a, p^a, \mu^a, k^a, s) = \sigma \int_{X_0}^{X_1} \int_G \left[ \frac{\partial v_w}{\partial X} \right]^2 d\Gamma dX + \int_{X_0}^{X_1} \int_G s v d\Gamma dX - \int_{X_0}^{X_1} \int_G s (v - v_w) d\Gamma dX - \int_{X_0}^{X_1} \int_G s v d\Gamma dX
\]

(4.13)
The optimal control is only applied in the specified interval \([X_0, X_1]\). \(\Omega\) is the cross-section domain \([0, \infty] \times [z_1, z_2]\) ranging from the wall \((y = 0)\) to infinity and from \(z_1\) to \(z_2\) in the spanwise direction and \(\Gamma\) is the wall boundary line for a given \(X\), \(\tau_w\) is the wall shear stress, \(\tau_0\) is a target shear stress (equal to the value corresponding to the similarity solution), and \([X_{s0}, X_{s1}]\) is the interval where the cost function is defined.

If the directional derivative of the Lagrangian is taken with respect to \(\rho\), the result is obtained.

\[
\int_0^{X_t} \int_{\Omega} \rho \left( \mathbf{V} \cdot \nabla \delta \rho + \delta \rho \nabla \cdot \mathbf{V} \right) + \rho \mathbf{V} \cdot (u^a \nabla u + v^a \nabla v + w^a \nabla w + T^a \nabla T) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \, \mathrm{d}\Omega \, \mathrm{d}X
\]

\[
+ \frac{TR^2}{\gamma M^2_{\infty}} \left( \frac{v^a}{\gamma M^2_{\infty}} \frac{\partial \rho}{\partial \gamma} \frac{\partial \rho}{\partial y} + w^a \frac{\partial \delta \rho}{\partial z} \right) + \frac{\delta \rho R^2_{e \lambda}}{\gamma M^2_{\infty}} \left( \frac{v^a}{\gamma M^2_{\infty}} \frac{\partial \rho}{\partial \gamma} + w^a \frac{\partial \rho}{\partial z} \right) - p^a \frac{\delta \rho T}{\gamma M^2_{\infty}} R^2_{e \lambda} \, \mathrm{d}\Omega \, \mathrm{d}X = 0 \quad (4.14)
\]

We obtain the first adjoint equation by using integration by parts in \([0, X_t]\) and in \(\Omega\). It is assumed that there are arbitrary variations of \(\delta \rho\) in \([0, X_t] \times \Omega\), with \(\delta \rho|\Gamma = 0\) (spanwise variation) and \(\delta \rho|0^{X_t} = 0\) (streamwise variation).

\[
\mathbf{V} \cdot (u^a \nabla u + v^a \nabla v + w^a \nabla w + T^a \nabla T) - p^a \frac{TR^2_{e \lambda}}{\gamma M^2_{\infty}} \mathbf{V} \cdot \rho^a - \frac{R^2_{e \lambda}}{\gamma M^2_{\infty}} T \nabla c \cdot \mathbf{V}^a = 0 \quad \text{on} \quad [0, X_t] \times \Omega
\]

\[
(4.15)
\]

where \(\mathbf{V}^a = u^a \mathbf{i} + v^a \mathbf{j} + w^a \mathbf{k}\).

The ACBREs corresponding to the rest of the state variables are obtained in a similar fashion by taking the directional derivative of the Lagrangian with respect to \(u\), \(v\), \(w\), and \(T\), respectively, as follows:

\[
\rho \left( \mathbf{V}^a \cdot \frac{\partial \mathbf{V}^a}{\partial x} + T^a \frac{\partial T}{\partial x} \right) + 2G_{\lambda} u v^a - \rho \frac{\partial \rho^a}{\partial x} - \rho \mathbf{V} \cdot \nabla u^a - \left( \mu \nabla_c \cdot (\nabla_c u^a) + \nabla_c \mu \cdot \nabla_c u^a \right)
\]

\[
+ \frac{2}{3} \frac{\partial \mu}{\partial x} (\nabla_c \cdot \mathbf{V}^a) - \left( \frac{1}{3} \mu \nabla_c \cdot \frac{\partial \mathbf{V}^a}{\partial x} + \nabla_c \mu \cdot \frac{\partial \mathbf{V}^a}{\partial x} \right)
\]

\[
+ 2(\gamma - 1) \left( T^a \mu \nabla_c \cdot \nabla_c u + \mu \nabla_c u \cdot \nabla_c T^a + T \nabla_c u \cdot \nabla_c \mu \right) = 0 \quad \text{on} \quad [0, X_t] \times \Omega
\]

\[
(4.16)
\]
\[
\rho \left( V^a \cdot \frac{\partial V}{\partial y} + T^a \frac{\partial T}{\partial y} \right) - \rho \frac{\partial \rho^a}{\partial y} - \rho V \cdot \nabla v^a = \left[ \frac{4}{3} \left( \frac{\partial \mu}{\partial y} + \frac{\partial v^a}{\partial y^2} \right) + \frac{\partial \mu}{\partial y} + \mu \frac{\partial^2 v^a}{\partial y^2} \right] + \frac{2}{3} \frac{\partial \mu}{\partial y} \frac{\partial w^a}{\partial y} - \frac{2}{3} \frac{\partial \mu}{\partial y} \frac{\partial w^a}{\partial y} = 0 \text{ on } [0, X_t] \times \Omega
\]

The ACBREs (4.15)-(4.19) are linear and parabolic and can be solved via a marching procedure in the backward direction, starting from the terminal streamwise location, \(X_t\), towards the initial streamwise location \(X_0\). The state variables \((\rho, u, v, w, T)\) in the ABREs are determined from equations (3.8)-(3.12). Here, \(\alpha\) is a constant pre-factor that controls the penalization of the wall shear stress. The values of the constant factors \(\alpha\) and \(\sigma\) are set to 1 and 0.1, respectively.

The non time-dependent adjoint variables \(p^a, \mu^a, k^a\) are determined in a similar fashion as follows:

\[
p^a = \nabla_c \cdot V^a
\]

\[
k^a = -\frac{1}{P_r} \left( \nabla_e T^a \cdot \nabla_e T \right)
\]

\[
\mu^a = k^a + T^a (\gamma - 1) M_\infty^2 \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial u}{\partial z} \right)^2 \right] - \left[ \nabla_c u^a \cdot \nabla_e u + \frac{2}{3} \frac{\partial v^a}{\partial y} \left( \frac{3}{\partial y} \nabla \cdot V \right) \right] + \frac{2}{3} \frac{\partial w^a}{\partial z} \left( \frac{3}{\partial z} \nabla \cdot V \right) + \nabla_c u^a \cdot \frac{\partial V^a}{\partial x} + \left( \frac{\partial v^a}{\partial z} + \frac{\partial w^a}{\partial y} \right) \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)
\]
4.2.2 Optimality Condition

The control variables are updated using the optimality conditions given by:

\[ \sigma \left( v_w - \frac{\partial^2 v_w}{\partial x^2} \right) + s = 0 \] (4.23)

where \( s \) is the Lagrange multiplier corresponding to the transpiration condition. It is obtained from the derivation of the third adjoint equation (4.17) as result of the integration by parts

\[ s = -\rho^a \rho - v^a \rho (u + v + w) + \mu \left( \frac{4}{3} \frac{\partial v^a}{\partial y} + \frac{\partial v^a}{\partial z} - \frac{2}{3} \frac{\partial w^a}{\partial z} + \frac{\partial w^a}{\partial y} \right) \] (4.24)

4.2.3 Initial and Boundary Conditions

The initial and boundary conditions associated with the ACBREs are

\[ (\rho^a, u^a, v^a, w^a, T^a)|_{X=X_t} = (0, 0, 0, 0, 0) \text{ in } \Omega, \] (4.25)

\[ (u^a, v^a, w^a)|_{\Gamma} = \begin{cases} 
(\alpha (\tau_w - \tau_0), 0, 0) & \text{for } X \in [X_{s0}, X_{s1}] \\
(0, 0, 0) & \text{otherwise}
\end{cases} \] (4.26)

and

\[ (\rho^a, u^a, v^a, w^a, T^a)|_{Y=\infty} = (0, 0, 0, 0) \] (4.27)

4.3 Optimal Control Algorithm

The control algorithm is composed of several steps. First, the NCBREs for the uncontrolled boundary layer are solved, followed by the solution to the ACBREs, which depends on the NCBREs solution. The difference between the wall shear stress and the original laminar wall shear stress is then compared to a desired value. If the difference exceeds a specified threshold, the model utilizes...
the steepest descent method to determine the new wall transpiration velocity \( v_w \), and the loop starts over. The process repeats for a total of 10 control iterations.

The ACBREs (4.15)-(4.19) and their associated initial and boundary conditions (4.25)-(4.27) are numerically solved on the same grid as the original NCBRE state equations (3.8)-(3.12) using the same numerical algorithm, but with backward marching starting from the terminal streamwise location. The computational cost of the control algorithm is reduced thanks to the robust mathematical model of the NCBREs, and the associated numerical algorithm previously developed by Es-Sahli et al.[20].

4.4 Results

We present the numerical simulation results of the boundary layer optimal control approach. We assess the efficiency of the control approach across different flow conditions, including two high-supersonic cases \( (M_\infty = 3 \text{ and } M_\infty = 4) \) and two hypersonic cases \( (M_\infty = 5 \text{ and } M_\infty = 6) \), by examining two main criteria: the spanwise averaged wall shear stress and the vortex kinetic energy. Since we are using wall transpiration to control the flow, the cumulative blowing or suction at the wall is employed such that a zero mass flow rate is maintained to avoid injecting or absorbing mass into/from the flow.

4.4.1 Effect of the Control Starting Point

We evaluate the spanwise averaged wall shear stress and the vortex kinetic energy distribution using equations 3.18 and 3.17, respectively. We test three different \( X_c \) locations to determine the optimal streamwise location to apply the transpiration velocity control; \( X_{c1} = 4 \times \lambda^* \), \( X_{c2} = 8 \times \lambda^* \), and \( X_{c3} = 12 \times \lambda^* \). In figures 4.1 and 4.2, we compare the wall shear stress and vortex energy of the
uncontrolled flow with that of the last (10th) control iteration of each $X_c$ for all considered Mach numbers. We notice that as we move the control streamwise location starting point farther from $X_0$ (as $X_c$ increases), the control approach effect on the wall shear stress and vortex kinetic energy weakens due to the shrinkage of the control surface. Therefore, for all Mach numbers, starting the control mechanism at $X_{c1}$ results in the highest reduction in the wall shear stress and the vortex kinetic energy of the centrifugal instabilities.

Figure 4.1

Effect of the optimal control starting point on the wall shear stress for $M_\infty = 3 \ (a)$, $M_\infty = 4 \ (b)$, $M_\infty = 5 \ (c)$, and $M_\infty = 6 \ (d)$
Effect of the optimal control starting point on the vortex energy for \( M_\infty = 3 \) (a), \( M_\infty = 4 \) (b), \( M_\infty = 5 \) (c), and \( M_\infty = 6 \) (d)

We quantitatively assess the drop in the wall shear stress \( \tau_w \) and vortex kinetic energy \( E \) due to the control method by calculating the decrease percentage at the terminal streamwise location \( X_t \) of the 10\(^{th}\) control iteration with respect to the uncontrolled boundary layer flow parameters (considering the similarity solution obtained using the Dorodnitsyn-Howarth coordinate transformation as a base reference) as

\[
\text{decrease}(\%) = \left| \frac{a(x_{\text{max}}) - a_u(x_{\text{max}})}{a_u(x_{\text{max}}) - a_s(x_{\text{max}})} \right| \times 100
\]

where \( a \) is a generic variable representing \( \tau_w \) or \( E \), and the subscripts \( u \) and \( s \) denote the uncontrolled flow and the similarity solution parameters, respectively. Results are summarized in table 4.1.

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Table 4.1

Percentage drop in wall shear stress and vortex kinetic energy due to the control approach at the control locations $X_{c1}$, $X_{c2}$, and $X_{c3}$ for all considered Mach numbers.

<table>
<thead>
<tr>
<th>Mach number $M_{\infty}$</th>
<th>Wall shear stress $\tau_w$</th>
<th>Vortex energy $E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{\infty} = 3$</td>
<td>47.7 45.28 36.46 72.31</td>
<td>68.61 58.65</td>
</tr>
<tr>
<td>$M_{\infty} = 4$</td>
<td>36.64 33.41 29.77 72.38</td>
<td>69.52 63.98</td>
</tr>
<tr>
<td>$M_{\infty} = 5$</td>
<td>29.56 25.80 20.27 73.27</td>
<td>71.24 60.42</td>
</tr>
<tr>
<td>$M_{\infty} = 6$</td>
<td>28.51 22.90 18.94 83.27</td>
<td>76.88 74.30</td>
</tr>
</tbody>
</table>

The reduction in the wall shear stress directly translates to a decrease in the boundary layer skin friction. While the reduction in the kinetic energy of the flow delays the occurrence of the secondary instabilities leading to transition, consequently elongating the laminar region of the boundary layer. Ultimately, this results in a more stable and attached boundary layer.

The present findings indicate that a larger control surface results in a larger drop in the wall shear stress and the vortex kinetic energy of the Görtler vortices. On the other hand, increasing the control surface adversely affects the computational efficiency of the method as the simulation becomes more computationally expensive, although the increase in the computational cost is relatively insignificant.

Although the control streamwise locations $X_{c2}$ and $X_{c3}$ also cause a significant decrease in the wall shear stress and vortex kinetic energy compared to the uncontrolled case, for what follows, we exclusively show results for the $X_{c1}$ case.

4.4.2 Effect of the Optimal Control Iterations

Next, we consider the effect of increasing the optimal control iterations on the wall shear stress and the vortex kinetic energy, as shown in figures 4.3 and 4.4. As priorly mentioned, we apply a
total of ten control iterations. However, we reduce the number of curves and only plot six of the control iterations and the uncontrolled flow case to avoid clustered figures.

Figure 4.3

Effect of increasing optimal control iterations on the wall shear stress for $M_\infty = 3$ (a), $M_\infty = 4$ (b), $M_\infty = 5$ (c), and $M_\infty = 6$ (d)
Figure 4.4

Effect of increasing optimal control iterations on the vortex energy for \( M_\infty = 3 \) (a), \( M_\infty = 4 \) (b), \( M_\infty = 5 \) (c), and \( M_\infty = 6 \) (d)

As expected, more control iterations lower the wall shear stress and vortex kinetic energy of the boundary layer flow. We also observe the number of iterations required to achieve a maximum decrease in the shear stress and vortex kinetic energy scales with the Mach number. For instance, for the two high-supersonic cases, the method achieves maximum reduction after six iterations (figures 4.3 and 4.4 a and b). On the other hand, the method requires ten iterations for the two hypersonic cases (figures 4.3 and 4.4 c and d).

The streamwise velocity contours in figures 4.5 and 4.6 visually show the effect of the control approach on the centrifugal instabilities of the boundary layer. In the case of the uncontrolled
flow (first contour from the top), the Görtler vortices correspond to fully developed mushroom-like structures with alternating low and high-speed streaks in the spanwise direction. As we apply more control iterations, the shape of the instabilities, especially in the region closest to the wall, considerably changes. Developed instability structures gradually flatten as the number of control iterations increases due to the drop in the wall shear stress and vortex energy levels in the boundary layer, as discussed above.
Streamwise velocity contours in the YZ-plane for the uncontrolled flow (top), 3\textsuperscript{rd} control iteration (middle), and 10\textsuperscript{th} control iteration (bottom) for $M_\infty = 3$ (a), $M_\infty = 4$ (b)
Streamwise velocity contours in the YZ-plane for the uncontrolled flow (top), $3^{rd}$ control iteration (middle), and $10^{th}$ control iteration (bottom) for $M_\infty = 5$ (c), and $M_\infty = 6$ (d)

Contour plots in figures 4.7 and 4.8 show the streamwise velocity distribution in the ZX-plane for the uncontrolled (top contour) flow and the $10^{th}$ control iteration (bottom contour). We can see
the alternating high- and low-speed streaks for the uncontrolled flow case. However, the contours of the controlled flow show a more uniform spanwise distribution of the streamwise velocity, indicating the disappearance of the streamwise-oriented vortical structures.
Figure 4.7

Streamwise velocity contours in the ZX-plane for the uncontrolled flow (top) and 10th control iteration (bottom) for $M_\infty = 3$ (a), $M_\infty = 4$ (b)
Streamwise velocity contours in the ZX-plane for the uncontrolled flow (top) and $10^{th}$ control iteration (bottom) for $M_\infty = 5$ (c), and $M_\infty = 6$ (d)
Figures 4.9 and 4.10 illustrate streamwise contour plots of the transpiration velocity, $v_w$. The contours show how the distribution and intensity of the control velocity vary with control iterations.

**Figure 4.9**

Transpiration velocity contours in the ZX-plane for the 1st control iteration (top) and 10th control iteration (bottom) for $M_\infty = 3$ (a), $M_\infty = 4$ (b)
Figure 4.10

Transpiration velocity contours in the ZX-plane for the 1st control iteration (top) and 10th control iteration (bottom) for $M_{\infty} = 5$ (c), and $M_{\infty} = 6$ (d)
CHAPTER V
CONCLUSION

High-amplitude freestream turbulence and surface roughness elements can excite a laminar boundary-layer flow sufficiently enough to cause streamwise-oriented vortices to develop. These vortices resemble elongated streaks having alternate spanwise variations of the streamwise velocity. Following the transient growth phase, the fully developed vortex structures downstream undergo an inviscid secondary instability mechanism and, ultimately, transition to turbulence. This mechanism becomes much more complicated in high-speed boundary layer flows due to compressibility and thermal effects, which become more significant for higher Mach numbers. In this research, we formulated and tested an optimal control algorithm to suppress the growth rate of the aforementioned streamwise vortex system. The derivation of the optimal control algorithm followed two stages;

In the first stage, to optimize the computational cost of the analysis, we developed an efficient numerical algorithm based on the nonlinear boundary region equations (NBREs), a reduced form of the compressible Navier-Stokes equations in a high-Reynolds-number asymptotic framework. Validation of the NBREs algorithm against direct numerical simulation (DNS) results showed good agreement. The numerical simulations of the proposed numerical algorithm are substantially less computationally costly than a full DNS and have a more rigorous theoretical foundation than parabolized stability equation (PSE) based models. The substantial reduction in computational
time required to predict the full development of a Görtler vortex system in high-speed flows allows investigation into feedback control in reasonable total computational time, which is the focus of the second part of the study.

We investigated Görtler vortices in high-speed boundary layer flows over concave surfaces using the numerical solution to NCBREs. We studied the nonlinear development of the centrifugal instabilities developing in the boundary layer under different configurations of the spanwise separation of the upstream disturbances -which dictates the spanwise separation of the downstream Görtler vortices- and upstream inflow Mach number. We implemented steady blowing and suction at the wall to create low-intensity disturbances to trigger the vortex formation in the boundary layer.

We considered a wide range of spanwise separations and Mach numbers, covering both supersonic and hypersonic regimes. Contours of velocity and temperature at various crossflow planes showed the development of the vortex system in the form of mushroom-shaped structures evolving in the streamwise direction. We calculated the kinetic energy of the primary instabilities and plotted them against the streamwise coordinate. Examining the kinetic energy plots showed that as the upstream spanwise separation increases, the scaled kinetic energy maxima increase, and the streamwise location where the energy saturation set in moves further downstream. We also calculated the wall shear stresses and the wall heat fluxes and observed that the wall shear stress increases as the spanwise separation increases, and - as expected - the wall shear stress is lower for the adiabatic wall condition as a result of the significant increases in temperature in proximity to the walls. There were also jumps in the wall shear stress at approximately the streamwise location corresponding to the point of energy saturation initiation. For the isothermal wall condition, the wall heat flux shows a characteristic decrease as the spanwise separation increases; there is also a
decay in the wall heat flux at the streamwise location corresponding to the point where the energy saturation initiates.

We also studied the nonlinear development of these secondary instabilities at different wall temperatures (including heating and cooling) and freestream Mach number conditions. Contour plots of temperature at various crossflow planes showed the development of the vortex system in the form of mushroom-shaped structures evolving in the streamwise direction. These vortices grow as the wall temperature increases (heating) and shrink as the wall temperature decreases (cooling). However, they appear to start growing again as the temperature at the wall further reduces (i.e., at 100 K), suggesting closer examination. We observed that as the wall temperature increases -with respect to the adiabatic wall condition, i.e., heating- the scaled kinetic energy and growth rate maximums increase as well, while they decrease for the cooling wall temperature cases. We observed the same trend for the wall shear stress and the wall heat flux. The jump of the wall shear stress appears to take place approximately at the streamwise location corresponding to the point of energy saturation initiation. In the future, we intend to consider more wall-temperature conditions, especially in the cooling case, due to the peculiar pattern we noticed in the temperature contour plots.

We also used the efficient NCBRE model to investigate the nonlinear development of centrifugal instabilities in a compressible curved free shear layer flow. We wanted to understand the characteristics of these centrifugal instabilities, which exhibit similarities with the Görtler vortices in boundary layer flows over concave surfaces. The study encompasses variations in the free stream Mach number ($M_{\infty}$), the relative velocity difference between the two streams of the shear layer ($\Delta V$), the shear layer thickness ($\delta$), and the amplitude of the inflow disturbance ($A$).
Upon closer examination of the kinetic energy plots for the $M_\infty = 2$ case, we observe that $E(x)$ is directly proportional to $\Delta V$ and inversely proportional to $\delta$ across all the considered Mach numbers. However, the increase in shear layer thickness has an insignificant effect on energy reduction, with a mere 1% drop in $E$ resulting from a 100% increase in $\delta$. Furthermore, increasing the amplitude of the inflow disturbance ($A$) slightly boosts the kinetic energy, with less than 1% increase in $E$ observed for a 100% increase in $A$. Interestingly, a larger magnitude of $A$ hampers the influence of the relative velocity difference. The energy curves corresponding to different $\Delta V$ values exhibit a considerably reduced gap when comparing the $A = 0.02$ and $A = 0.04$ cases. We observed similar trends in the parametric study of centrifugal instability development for the $M_\infty = 4$ and $M_\infty = 6$ cases.

Examining the contour plots of crossflow velocity magnitude for $\Delta V = 30\%$, we find that increasing the disturbance amplitude leads to significant growth in the mushroom-like structure's amplitude and renders the secondary structures more visible, indicating increased mixing for all Mach numbers under consideration. When comparing different Mach numbers at the same crossflow plane location, we observed a slower development of mushroom-like structures as $M_\infty$ increases, suggesting that achieving the same mixing efficiency would require traveling further downstream for higher Mach numbers. Consequently, numerical simulations for higher Mach number cases would incur higher computational costs due to the need for larger grid sizes to maintain comparable mixing efficiency.

Overall, the framework of the NCBRE mathematical model proved robust, and the associated numerical algorithm provided results in a very short time compared to other mathematical models, such as DNS or PSE, making our NCBRE approach suitable for multiple parametric studies.
In the second part of this research, we conducted an optimal control study by formulating a novel algorithm that is capable of suppressing the growth of the centrifugal instabilities (i.e. Görtler vortices) developing in a compressible boundary layer flow over a curved surface. The method involved using the NCBREs algorithm and Lagrange multipliers within an optimal control variational problem. We implemented the control mechanism in the boundary layer flow using transpiration velocity (blowing and suction) applied at the wall in the wall-normal direction as control variables. We defined the wall shear stress as the cost-functional for the optimal control algorithm.

We applied the control strategy at different streamwise locations to determine the subsequent optimal streamwise location yielding the highest decrease in the wall shear stress and vortex kinetic energy. Our results demonstrate that the closer one moves the control point \( X_c \) toward the offset point in region III \( X_0 \), the higher the reduction in the wall shear stress, and commensurately the vortex kinetic energy because of the increase in the control surface area.

The optimal control approach induces a significant reduction in wall shear stress and vortex kinetic energy as a result of the blowing or suction associated with the transpiration velocity control variable. We reported 12.09\%, 11.61\%, 10.56\%, and 11.02\% reduction in the wall shear stress and 72.31\%, 72.38\%, 73.27\%, and 83.27\% drop in the vortex kinetic energy for respectively for the \( M_\infty = 3, M_\infty = 4, M_\infty = 5, \) and \( M_\infty = 6 \) cases when applying the control method at \( X_{c1} \) and comparing the 10\textsuperscript{th} control iteration to the uncontrolled flow case. Results showed that the method requires more control iterations to achieve maximum reduction as the Mach number increases. For instance, the two high-supersonic cases \( M_\infty = 3 \) and \( M_\infty = 4 \) required six control iterations, whereas the two hypersonic ones \( M_\infty = 5 \) and \( M_\infty = 5 \) required ten. The decrease in the wall
shear stress directly translates to a decrease in the boundary layer skin friction. The reduction in
the flow kinetic energy delays the development of the secondary instabilities, leading to transition,
consequently elongating the laminar region of the boundary layer. Ultimately, this results in a more
stable and attached boundary layer.

The crossflow contours of the streamwise primary instability visually captured the effect of
the control approach on the centrifugal instabilities in the boundary layer. The control iterations
gradually flatten the flow instabilities (especially near the wall). This finding goes hand in hand
with the decrease in the kinetic energy and wall shear stress levels in the boundary layer. Consistent
with these results, contour plots demonstrated how the optimal control approach results in a more
uniform streamwise velocity distribution parallel to the wall as opposed to the high and low-speed
streaks of the uncontrolled flow case.

To conclude, the results demonstrated that the optimal control algorithm based on the ACBRE
(adjoint compressible boundary region equations) suppresses the growth rate of the streamwise
vortex system of high supersonic and hypersonic boundary layer flows.
REFERENCES


